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MYERS' THEOREM WITH DENSITY

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Abstract

We provide generalizations of theorems of Myers and others to Riemannian manifolds with density and provide a minor correction to Morgan [8].

1. Introduction

A *manifold with density* (see [8] and references therein) is a Riemannian manifold with a positive density function $\Psi(x)$ used to weight volume, area, and length. In terms of the underlying Riemannian volume dV_0, area dA_0, and length ds_0, the new, weighted volume, mD area, and length are given by

$$dV = \Psi\, dV_0, \quad dA = \Psi\, dA_0, \quad ds = \Psi\, ds_0.$$

Such a density is not equivalent to scaling the metric conformally by a factor Ψ, since in that case volume and area would scale by higher powers of Ψ. Previous treatments (see [8]) weight only volume and $(n-1)D$ area, but we will be especially interested in weighting length. Even when only length enters in, as for Myers' Theorem, so that the weighting is equivalent to a conformal change of metric, different and sometimes stronger estimates appear naturally.

Manifolds with density, the smooth case of Gromov's "mm spaces" [6] or the earlier "spaces of homogeneous type" (see [5, pp. 587, 591]), long have arisen on an *ad hoc* basis in mathematics. An example of much interest to probabilists is *Gauss space* G^n: Euclidean space with Gaussian probability density

$$\Psi = (2\pi)^{-n/2} e^{-x^2/2}$$

(see e.g. [7] or [13]).

The Theorem of (Bonnet and) Myers ([11] or [4, Thm. 2.12]) says that the diameter of a smooth, connected, complete, nD Riemannian manifold with Ricci curvature at least $a > 0$ is at most $\pi\sqrt{(n-1)/a}$. Our Section 3 provides the following generalization of Myers' Theorem:

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THEOREM (3.1). *Let M^n be a smooth, connected Riemannian manifold with smooth density $e^{\psi} \leq b$, complete in the weighted metric. If (in the unweighted metric)*

$$\mathrm{Ric} - \Delta\psi + \mathrm{Hess}\,\psi \geq a > 0,$$

then the diameter satisfies

$$\mathrm{diam} \leq d_0 = \pi b\sqrt{(n-1)/a}.$$

The proof is a minor modification of the standard proof of Myers' Theorem. The result is sometimes stronger than applying Myers' Theorem to the conformally altered metric, as shown in Section 3.2.

Section 4 provides a minor generalization of Bishop's Theorem and proof. Section 5 provides a minor generalization of a theorem of Morgan and Ritoré [10, Cor. 3.9] on isoperimetric regions in cones. Section 6 provides a minor correction to Morgan [8].

V. Bayle [1, E.2.1, p. 233] and Z. Qian [12, Thm. 5] provide other generalizations of the theorems of Myers and Bishop which depend on $|\nabla\psi|^2$ and hence do not apply to Gauss space for example.

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2. First and second variation

For use in the proof of our main Theorem 3.1, we state first and second variation formulas for (weighted) length in a manifold with density. The only change to the classical formulas [4, Sect. 2.4] are the new terms involving $d\psi/d\boldsymbol{N}$; cf. [1, Sect. 3.4.6] or [8, Prop. 7].

2.1. PROPOSITION (First and Second Variation Formulas). *Let M^n be a smooth Riemannian manifold with sectional curvature K and smooth density $\Psi = e^{\psi}$. Let γ be a smooth curve with classical unit tangent $\boldsymbol{T}$, let $\boldsymbol{N}$ be a classical parallel unit normal, and let κ denote the classical curvature of γ in that direction. Consider a smooth normal variation vectorfield $u\boldsymbol{N}$ supported on the interior of γ. The first variation of (weighted) length satisfies*

$$\delta^1(u) = -\int_\gamma u(\kappa - d\psi/d\boldsymbol{N})\,ds. \tag{1}$$

For a stationary curve γ, $\kappa = d\psi/d\boldsymbol{N}$ and the second variation satisfies

$$\delta^2(u) = \int_\gamma (du/ds_0)^2 - u^2\kappa^2 - u^2K(\boldsymbol{N},\boldsymbol{T}) + u^2(d^2\psi/d\boldsymbol{N}^2)\,ds. \tag{2}$$

3. Myers' Theorem with density

The following theorem, an extension of Myers' Theorem to manifolds with density, is the main result of this paper.

3.1. THEOREM. *Let M^n be a smooth, connected Riemannian manifold with smooth density $e^\psi \le b$, complete in the weighted metric. If (in the unweighted metric)*

$$\mathrm{Ric} - \Delta\psi + \mathrm{Hess}\,\psi \ge a > 0,$$

then the diameter satisfies

$$\mathrm{diam} \le d_0 = \pi b\sqrt{(n-1)/a}.$$

Proof. Let γ be a shortest geodesic of length L between two points. Let $\boldsymbol{N}$ be a parallel unit normal to γ. For a smooth function u vanishing at the endpoints, the second variation of length 2.1(2) must be nonnegative:

$$0 \le \delta^2(u) = \int_\gamma [(du/ds_0)^2 - u^2K(\boldsymbol{N}, \boldsymbol{T}) + u^2(d^2\psi/d\boldsymbol{N}^2)]\, ds.$$

Averaging over all choices of $\boldsymbol{N}$ yields:

$$\begin{aligned} 0 &\le \int_\gamma \left[(du/ds_0)^2 - \frac{u^2}{n-1}(\mathrm{Ric}(\boldsymbol{T}, \boldsymbol{T}) - \Delta\psi + d^2\psi/d\boldsymbol{T}^2)\right] ds \\ &\le \int_\gamma \left[b^2(du/ds)^2 - \frac{u^2}{n-1}a\right] ds. \\ 0 &\le \int_\gamma [(du/ds)^2 - u^2\pi^2/d_0^2]\, ds. \end{aligned}$$

As in the standard proof of Myers' Theorem, let $u = \sin(\pi s/L)$ and conclude that $L \le d_0$.

3.2. Comparison with Myers' Theorem. Another upper bound d_1 on the diameter may be obtained by making the conformal change of metric to $ds = e^\psi\, ds_0$ and applying the classical Myers Theorem:

$$\mathrm{diam} \le d_1 = \sup \pi\sqrt{(n-1)/\widetilde{\mathrm{Ric}}(\tilde{\boldsymbol{T}}, \tilde{\boldsymbol{T}})}$$

where [2, Thm. 1.159]

$$\widetilde{\mathrm{Ric}}(\tilde{\boldsymbol{T}}, \tilde{\boldsymbol{T}}) = e^{-2\psi}[\mathrm{Ric}(\boldsymbol{T}, \boldsymbol{T}) - \Delta\psi + (n-2)(-d^2\psi/d\boldsymbol{T}^2 - |\nabla\psi|^2 + (d\psi/d\boldsymbol{T})^2)],$$

$\boldsymbol{T}$ is a unit vector under ds_0, and $\tilde{\boldsymbol{T}}$ is its positive multiple of unit length under ds. For $n = 2$, $\widetilde{\mathrm{Ric}}$ is just the Gauss curvature

$$\tilde{G} = e^{-2\psi}(G - \Delta\psi).$$

Even for $n = 2$, sometimes our $d_0 < d_1$ and sometimes $d_0 > d_1$. Consider the unit sphere in $\boldsymbol{R}^3$ with density e^ψ with $\psi = (z-1)/2$, varying from e^{-1} at the south pole to 1 at the north pole. Easy computation shows that $d^2\psi/d\boldsymbol{T}^2 = -z/2$. In particular, at the south pole $\tilde{G} = 0$ and $d_1 = \infty$. Another easy computation shows that our best $d_0 = \pi\sqrt{2} \approx 4.4$ (realized at the south pole). I think that the actual diameter is half a circle of longitude, about 2.

On the other hand, if you start with the metric $ds = e^{\psi}\, ds_0$ and consider density $e^{-\psi}$, then of course $d\tilde{s} = ds_0$, $\tilde{G} = 1$, and $d_1 = \pi$ (sharp), while an easy computation shows that $d_0 = \pi e/\sqrt{3/2} \approx 7$ (at the north pole).

3.3. Gauss space. The most important manifold with density is Gauss space G^n, defined as Euclidean space $\boldsymbol{R}^n$ with Gaussian probability density

$$e^{\psi} = (2\pi)^{-n/2} e^{-r^2/2}$$

(see [8]). Unfortunately Theorem 3.1 does not apply immediately to Gauss space, because it is not complete at infinity. The shortest path between two points may be two paths to infinity. The analysis applies to each component, however. Moreover, since the end at infinity is free, the variation $u = \sin(\pi s/2L)$ may be used to obtain the improved estimate $L \le d_0/2$ on each component. Therefore we still obtain the same bound d_0 on the diameter. Since $d^2\psi/d\boldsymbol{T}^2 = -1$, taking $a = 0 + n - 1$ yields

$$d_0 = \pi(2\pi)^{-n/2}.$$

The actual diameter, given by the following Proposition 3.4, is $(\pi/2)^{1/2}(2\pi)^{-n/2} \approx 1.3(2\pi)^{-n/2}$, about 40% of our upper bound d_0. (Similarly the standard Myers argument gives the useless bound $d_1 = +\infty$, because $\widetilde{\mathrm{Ric}}(\tilde{\boldsymbol{T}}, \tilde{\boldsymbol{T}})$ is not positive because $|\nabla\psi|$ is unbounded, except that for $n = 2$, $d_1 = 1/2\sqrt{2} \approx .35 < d_0 = .5$.)

The following proposition was proved by Williams undergraduate Ya Xu is response to a question at a faculty seminar October 21, 2005.

3.4. Proposition (Ya Xu). *Gauss space G^n has diameter $D_0 = (\pi/2)^{1/2}(2\pi)^{-n/2}$.*

Proof. The diameter is at least D_0, the distance from the origin to infinity. But for any two points, the distance between them is at most half the length of the straight line they lie on (because points near infinity are close together). By rotational symmetry, we may assume that they lie in G^2 on the horizontal line $\{y = a\}$, which has weighted length element

$$(2\pi)^{-n/2} e^{-x^2/2} e^{-a^2/2}\, dx$$

and weighted length $(2\pi)^{-n/2}(2\pi)^{1/2} e^{-a^2/2} \le 2D_0$. Therefore the diameter must equal D_0.

4. Bishop's Theorem with density

The following easy theorem extends Bishop's Theorem to manifolds with density. Let M_δ denote the model space of constant curvature δ, Ricci curvature $(n-1)\delta$, and balls of radius t of volume $V_\delta(t)$ and surface area $n\alpha_n s_\delta^{n-1}(t)$.

4.1. THEOREM. *Let M^n be a smooth Riemannian manifold with smooth density Ψ, with classical Ricci curvature bounded below by $(n-1)\delta$ and* $\operatorname{Hess}\Psi \le \gamma$. *Then the weighted volume $V(p,r)$ of a ball about a point p of unweighted radius r satisfies*:

$$V(p,r) \le n\alpha_n \int_0^r s_\delta^{n-1}(t)\left(\Psi(p) + \frac{1}{2}\gamma t^2\right) dt, \tag{1}$$

with equality if and only if the ball is isometric to a ball in M_δ and $\Psi(\exp x) = \Psi(p) + D\Psi_p(x) + \gamma|x|^2$. *If* $\operatorname{Hess}\Psi \le 0$, *then*

$$V(p,r) \le \Psi(p)V_\delta(r);$$

in particular,

$$V(M) \le \Psi(p)V(M_\delta),$$

with equality if and only if $M = M_\delta$ and $\Psi(x)$ is constant.

Proof. Since $\operatorname{Hess}\Psi \le \gamma$, therefore $\Psi(\exp x) \le \Psi(p) + D\Psi_p(x) + \gamma|x|^2$. The result now follows immediately from the classical Bishop theorem (the case $\Psi = 1$; [3, p. 256, Cor. 4], [4, Thm. 3.8]), since the contributions of $D\Psi_p(x)$ and $D\Psi_p(-x)$ to the estimate cancel out.

5. Isoperimetric regions in cones with density

A theorem of Morgan and Ritoré [10, Cor. 3.9] on isoperimetric regions in cones has the following generalization to manifolds with density:

THEOREM 5.1. *Let M^n $(n \ge 2)$ be a smooth, connected submanifold of the sphere $\boldsymbol{S}^N$ with smooth density $e^\psi \ge a > 0$ and* (*weighted*) *volume satisfying*

$$\operatorname{vol} M < a \operatorname{vol} \boldsymbol{S}^n. \tag{1}$$

Suppose that the Ricci curvature satisfies

$$\operatorname{Ric} - \operatorname{Hess}\psi > n-1. \tag{2}$$

Then in the cone C over M with the inherited density, *geodesic spheres about the vertex uniquely minimize perimeter for given volume.*

Remarks. The quantity in (2) is the generalized Ricci curvature (see [8]). If the opposite inequality holds in (1), then a ball about a point near infinity of density nearly a has smaller perimeter than the ball about the vertex.

Proof. The proof is a relatively minor generalization of the proof in [10] and employs the standard, unweighted metric. The generalization of [10, Thm.

2.1], an isoperimetric inequality after Bérard and Meyer, assumes that the density is uniformly continuous and bounded below by $a > 0$. Then in the conclusion [10, (2.1)] there is an additional factor of a on the right-hand side, because the local application of the Euclidean isoperimetric inequality has an additional factor of a.

The generalization of [10, Thm. 2.2] on the existence of an isoperimetric region U assumes that the density is smooth and bounded below by $a > 0$ and that (1) holds. [10] omits the proof that U is bounded, which follows for the classical case of density 1 by monotonicity because the mean curvature H is constant and hence bounded in $\boldsymbol{R}^N$. For general density, the generalized mean curvature $H - (1/n)\, d\psi/d\boldsymbol{n}$ is constant, and hence H is bounded as you go to infinity, so the monotonicity argument still applies. Generalization of the standard regularity to manifolds with density was observed by [9, 3.10].

The Minkowski formulae [10, 3.4] still hold with mean curvature replaced by the generalized mean curvature and Ricci curvature replaced by the generalized Ricci curvature $\mathrm{Ric} - \mathrm{Hess}\,\psi$.

The rest of the proof follows the proof of [10, Thm. 3.6]. The generalized Ricci curvature of the minimizer in the normal direction vanishes, which means that the normal direction is always radial, which means that the isoperimetric surface is a geodesic sphere.

6. A minor correction of Morgan

Theorem 2 of Morgan [8] gives a generalization of the Heinze-Karcher volume bound to manifolds with density. The final statement of Theorem 2 of Morgan [8] should be:

If equality holds, then S is totally geodesic, the region is a metric product, and inside the region, along geodesics normal to S, $-d^2\psi/dt^2 = \gamma$.

Proof. If equality holds, the parallel hypersurfaces are disjoint and totally geodesic ($\mathrm{II}^2 = 0$), so that the normal geodesics stay equidistant, which means that the region is a metric product.

The following Remark should say that "Theorem 2 is sharp for hyperplanes in Gauss space for example."

In the last line of Theorem 5, "$(n-1)D$ Euclidean space" should be "manifold."

In the last line of Corollary 9, "flat" should be "totally umbilic."

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