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SEMI-INVARIANT IMMERSIONS

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§ 1. Introduction.

Let $\tilde{M}$ be a differentiable manifold and let $\tilde{F}$ be a tensor field of type (1, 1) defined on $\tilde{M}$. If M is a submanifold immersed in $\tilde{M}$, M is said to be an invariant submanifold if the tangent space to M at each point of M is invariant under the endomorphism $\tilde{F}$. If $\tilde{M}$ is a complex manifold and $\tilde{F}$ is the almost complex structure on $\tilde{M}$ then the invariant submanifolds of $\tilde{M}$ are just the complex submanifolds. (See, Schouten and Yano [4]). If $\tilde{M}$ is a normal contact (or Sasakian) manifold with $(\tilde{F}, \tilde{\xi}, \tilde{\eta})$ as the almost contact structure on $\tilde{M}$, then there does not exist an invariant submanifold M with $\tilde{\xi}$ normal to M. (See § 4). Invariant submanifolds have been studied by many people. (See, Kubo [2], Schouten and Yano [4], Yano and Okumura [9], [10]).

The purpose of this paper is to study submanifolds M of $\tilde{M}$ for which there is a distribution D that is nowhere tangent to M and such that the subspace spanned by D and the tangent space to M is invariant under $\tilde{F}$ and $\tilde{F}D$ is tangent to M at each point of M. (See, Tashiro [6]).

§ 2. Preliminaries.

Let M be a differentiable manifold. A tensor field F of type (1, 1) on M defines an *almost complex* structure if

$$F^2=-I,$$

in which case M is of even dimension. This almost complex structure is *integrable*, i. e., M is complex, if $[F, F]=0$, where $[F, F]$ is the Nijenhuis tensor of F defined by

$$[F, F](X, Y)=[FX, FY]-F[FX, Y]-F[X, FY]+F^2[X, Y].$$

Here X and Y are vector fields on M. A Riemannian metric g is a Hermitian metric for F if

$$g(FX, FY)=g(X, Y).$$

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An *almost contact structure* on M is defined by

$$\phi^2=-I+\eta\otimes\xi, \quad \phi\xi=0, \quad \eta\circ\phi=0, \quad \eta(\xi)=1,$$

where ϕ is a tensor field of type (1, 1), ξ is a vector field and η is a 1-form on M. (See, Sasaki [3]). M is of odd dimension and this almost contact structure is said to be *normal* if $[\phi, \phi]+d\eta\otimes\xi=0$. A Riemannian metric g on M is associated with the almost contact structure on M if

$$g(\phi X, \phi Y)=g(X, Y)-\eta(X)\eta(Y), \quad \eta(X)=g(\xi, X).$$

In this case we say that M has an *almost contact metric* structure.

We say M has an (f, U, V, u, v, λ)-*structure* if

$$f^2=-I+u\otimes U+v\otimes V,$$

$$fU=-\lambda V, \quad fV=\lambda U, \quad u\circ f=\lambda v, \quad v\circ f=-\lambda u,$$

$$u(U)=v(V)=1-\lambda^2, \quad u(V)=v(U)=0,$$

where f is a tensor field of type (1, 1), U and V are vector fields, u and v are 1-forms and λ is a function on M. M is of even dimension and this (f, U, V, u, v, λ)-structure is *normal* if $[f, f]+du\otimes U+dv\otimes V=0$. A Riemannian metric g on M is associated with this structure if

$$g(fX, fY)=g(X, Y)-u(X)u(Y)-v(X)v(Y),$$

$$u(X)=g(U, X), \quad v(X)=g(V, X)$$

and we say that M has a (f, g, u, v, λ)-*structure*. (See, Blair, Ludden and Yano [1], Yano and Okumura [7], [8]).

§ 3. Semi-invariant submanifolds of almost complex manifolds.

Let $\tilde{M}$ be an almost complex manifold with almost complex structure $\tilde{F}$ and let $\tilde{U}$ be a non-vanishing vector field on $\tilde{M}$. Let $\iota: M\to\tilde{M}$ be an immersion such that

$$(3.1) \qquad \begin{cases} \tilde{F}\iota_*X=\iota_*fX-\eta(X)\tilde{U}, \\ \tilde{F}\tilde{U}=\iota_*\xi, \\ \tilde{U} \text{ is never tangent to } \iota(M) \end{cases}$$

for all vector fields X on M, where ι_* is the differential of the immersion. Then we say that M is a *semi-invariant submanifold of* $\tilde{M}$ *with respect to* $\tilde{U}$. Thus f, ξ, η are a tensor field of type (1, 1), a vector field and a 1-form on M respectively. A straightforward calculation shows that

$$(3.2) \qquad f^2=-I+\eta\otimes\xi, \quad f\xi=0, \quad \eta\circ f=0, \quad \eta(\xi)=1,$$

that is, (f, ξ, η) is an almost contact structure on M. Thus we have the following theorem.

THEOREM 3.1. *If $\tilde{M}$ is an almost complex manifold and $\tilde{U}$ is a non-vanishing vector field on $\tilde{M}$, then a semi-invariant submanifold M of $\tilde{M}$ with respect to $\tilde{U}$ possesses an induced almost contact structure.*

If $\tilde{M}$ is complex, then we have that

$$\begin{aligned}0&=[\tilde{F}, \tilde{F}](\iota_* X, \iota_* Y)\\&=\iota_*\{[f, f](X, Y)+d\eta(X, Y)\xi\}\\&\quad-\{(\mathcal{L}_{fX}\eta)(Y)-(\mathcal{L}_{fY}\eta)(X)\}\tilde{U}\\&\quad-\eta(X)(\mathcal{L}_{\tilde{U}}\tilde{F})\iota_* Y+\eta(Y)(\mathcal{L}_{\tilde{U}}\tilde{F})\iota_* X,\end{aligned}$$

where $\mathcal{L}$ denotes Lie differentiation and hence the following

THEOREM 3.2. *Under the hypothesis of Theorem* 3.1, *if in addition $\tilde{M}$ is complex and $\tilde{U}$ is analytic* (i. e. $\mathcal{L}_{\tilde{U}}\tilde{F}=0$), *then the almost contact structure on M is normal.*

If $\tilde{g}$ is a Hermitian metric on $\tilde{M}$ and $\tilde{U}$ is a unit normal to $\iota(M)$, then we have

$$\begin{aligned}g(X, Y)&=\tilde{g}(\iota_* X, \iota_* Y)\\&=\tilde{g}(\tilde{F}\iota_* X, \tilde{F}\iota_* Y)\\&=\tilde{g}(\iota_* fX-\eta(X)\tilde{U}, \iota_* fY-\eta(Y)\tilde{U})\\&=g(fX, fY)+\eta(X)\eta(Y).\end{aligned}$$

That is, the induced metric on M is an associated metric and thus M possesses an almost contact metric structure.

Example. The Calabi-Eckmann manifold, $S^{2p+1}\times S^{2q+1}$, is a complex manifold. If we denote by (ϕ, ξ, η) the Sasakian structure of S^{2p+1} and by (ϕ', ξ', η') that of S^{2q+1}, then the complex structure $\tilde{F}$ of $S^{2p+1}\times S^{2q+1}$ is given by

$$\tilde{F}(X, X')=(\varphi X+\eta'(X')\xi, \varphi' X'-\eta(X)\xi'),$$

where X and X' are arbitrary vector fields of S^{2p+1} and S^{2q+1} respectively.

Thus we have

$$\begin{aligned}\tilde{F}(X, 0)&=(\varphi X, -\eta(X)\xi')\\&=(\varphi X, 0)-\eta(X)(0, \xi')\end{aligned}$$

and

$$\tilde{F}(0, \xi')=(\xi, 0).$$

$\tilde{U}=(0,\xi')$ being never tangent to S^{2p+1}, we have that S^{2p+1} and S^{2q+1} are semi-invariant submanifolds with respect to the distinguished direction of the contact structure of the opposite factor.

§4. Semi-invariant submanifolds of almost contact manifolds.

Let $\tilde{M}$ be an almost contact manifold with $(\tilde{F}, \tilde{\xi}, \tilde{\eta})$ as the almost contact structure. Let $\iota: M \to \tilde{M}$ be an immersion such that

$$\begin{aligned} &\tilde{F}\iota_* X = \iota_* fX + \omega(X)\tilde{\xi}\,, \\ &\tilde{\xi} \text{ is never tangent to } \iota(M)\,, \end{aligned} \tag{4.1}$$

for all vector fields X on M. Then we say that M is a *semi-invariant submanifold of* $\tilde{M}$.

Applying $\tilde{F}$ to the equation in (4.1) we see that $f^2=-I$ and $\omega(fX)=\tilde{\eta}(\iota_* X)$ for all X. Thus f is an almost complex structure on M. If the almost contact structure is normal, then

$$\begin{aligned} 0 &= [\tilde{F}, \tilde{F}](\iota_* X, \iota_* Y) + d\tilde{\eta}(\iota_* X, \iota_* Y)\tilde{\xi} \\ &= \iota_*[f, f](X, Y) + \{d\omega(fX, Y) + d\omega(X, fY)\}\,\tilde{\xi}\,, \end{aligned}$$

where we have used the fact that for a normal almost contact structure $\mathcal{L}_{\tilde{\xi}}\tilde{F}=0$. Thus we see that the almost complex structure f is integrable and $d\omega$ is of bidegree (1, 1), i. e., $d\omega(fX, Y)+d\omega(X, fY)=0$ for all X, Y. We state this as the following theorem.

THEOREM 4.1. *If $\tilde{M}$ is an almost contact manifold, then a semi-invariant submanifold M possesses an induced almost complex structure. If, in addition, the almost contact structure on $\tilde{M}$ is normal, then the almost complex structure on M is integrable.*

Now let $\tilde{g}$ be a metric on $\tilde{M}$ associated with the almost contact structure and g the induced metric on M. Then we have that

$$\tilde{g}(\tilde{F}\iota_* X, \tilde{F}\iota_* Y) = \tilde{g}(\iota_* X, \iota_* Y) - \tilde{\eta}(\iota_* X)\tilde{\eta}(\iota_* Y)$$

or

$$g(fX, fY) - \omega(X)\omega(Y) = g(X, Y) - \omega(fX)\omega(fY)\,.$$

Thus, if we define $\bar{g}$ by

$$\bar{g}(X, Y) = g(X, Y) - \omega(fX)\omega(fY)\,, \tag{4.2}$$

we see that $\bar{g}$ is a Hermitian metric for f. Here we have used the fact that $\tilde{\eta}(\iota_* X)=\omega(fX)$ and $\tilde{\xi}$ is not tangent to M.

Suppose that $\tilde{\xi}$ is normal to M, that is, $\tilde{g}(\iota_* X, \tilde{\xi})=0$ for all X. Then from (4.1) we see that $\omega(X)=0$ for all X. That is, M is an invariant submanifold of

$\tilde{M}$, which cannot happen if $\tilde{M}$ is Sasakian, since for an invariant submanifold, we have

$$\tilde{F}\iota_* X = \iota_* fX .$$

On the other hand, $\tilde{\xi}$ being a unit normal to the submanifold, we have

$$\tilde{\nabla}_{\iota_* X}\tilde{\xi} = -\iota_* HX + \nabla^{\perp}_X \tilde{\xi} , \tag{4.3}$$

where $\tilde{\nabla}$ is the operator of covariant differentiation with respect to $\tilde{g}$, H is the Weingarten map corresponding to $\tilde{\xi}$ and $\nabla^{\perp}$ is the connection in the normal bundle. However for a Sasakian manifold, we have $\tilde{\nabla}_{\iota_* X}\tilde{\xi} = \tilde{F}\iota_* X = \iota_* fX$ and consequently from (4.3) we have

$$fX = -HX ,$$

which is a contradiction since f is skew-symmetric and H is symmetric.

Remark. We can consider $S^{2p}\times S^{2q+1}$ as a hypersurface in $S^{2p+1}\times S^{2q+1}$. Now $S^{2p+1}\times S^{2q+1}$ carries an almost complex structure and hence, by a result of Tashiro [5], $S^{2p}\times S^{2q+1}$ possesses an almost contact structure. If S^{2p} is a semi-invariant submanifold of $S^{2p}\times S^{2q+1}$ then S^{2p} would be almost complex, which is not the case unless $p=1$ or 3. Of course, S^{2q+1} cannot be a semi-invariant submanifold.

§ 5. Semi-invariant submanifolds of manifolds with $(\tilde{F}, \tilde{U}, \tilde{V}, \tilde{u}, \tilde{v}, \lambda)$-structure.

Let $\tilde{M}$ possess an $(\tilde{F}, \tilde{U}, \tilde{V}, \tilde{u}, \tilde{v}, \lambda)$-structure with $\{\tilde{F}, \tilde{U}, \tilde{V}, \tilde{u}, \tilde{v}, \lambda\}$ being the tensor fields of the structure. Let $\iota: M \to \tilde{M}$ be an immersion such that $\lambda(1-\lambda^2) \neq 0$ on $\iota(M)$ and

$$\begin{aligned} &\tilde{F}\iota_* X = \iota_* fX + \omega(X)\tilde{U} , \\ &\tilde{V} = \iota_* \xi , \\ &\tilde{u}(\iota_* X) = 0 , \end{aligned} \tag{5.1}$$

for all vector fields X on M. Here f, ω, ξ are respectively a tensor field of type (1, 1), a 1-form and a vector field on M. Then we say that M is a *semi-invariant submanifold of $\tilde{M}$ with respect to $\tilde{U}$.*

If we apply $\tilde{u}$ to the first equation in (5.1) we have

$$\lambda \tilde{v}(\iota_* X) = (1-\lambda^2)\omega(X) . \tag{5.2}$$

At a point p in M suppose that $\tilde{U} = \iota_* U$. Then we have $1-\lambda^2 = \tilde{u}(\tilde{U}) = \tilde{u}(\iota_* U) = 0$. Thus the condition $1-\lambda^2 \neq 0$ on M implies that U is nowhere tangent to M.

Applying $\tilde{F}$ to the first equation in (5.1) we obtain

$$-\iota_* X + \tilde{v}(\iota_* X)\tilde{V} = \iota_* f^2 X + \omega(fX)\tilde{U} - \lambda\omega(X)\tilde{V} .$$

Using (5.2) and the second equation in (5.1) this becomes

$$
\begin{aligned}
& f^2X=-X+\frac{1}{\lambda}\omega(X)\xi\,, \\
(5.3) \qquad & \\
& \omega(fX)=0\,.
\end{aligned}
$$

Letting $X=\xi$ in the first equation of (5.1) we obtain

$$\tilde{F}\iota_*\xi=\iota_* f\xi+\omega(\xi)\tilde{U},$$

that is,

$$
\begin{aligned}
& f\xi=0\,, \\
(5.4) \qquad & \\
& \frac{1}{\lambda}\omega(\xi)=1\,.
\end{aligned}
$$

If we let $\eta=\frac{1}{\lambda}\omega$ then (5.3) and (5.4) prove the following theorem.

THEOREM 5.1. *If $\tilde{M}$ is a manifold with an $(\tilde{F}, \tilde{U}, \tilde{V}, \tilde{u}, \tilde{v}, \lambda)$-structure, then a semi-invariant submanifold of $\tilde{M}$ with respect to $\tilde{U}$ possesses an induced almost contact structure.*

If the $(\tilde{F}, \tilde{U}, \tilde{V}, \tilde{u}, \tilde{v}, \lambda)$-structure is normal, then after some calculation similar to that in the previous section, we have that if λ=constant on M,

$$
\begin{aligned}
0&=[\tilde{F}, \tilde{F}](\iota_*X, \iota_*Y)+d\tilde{u}(\iota_*X, \iota_*Y)\tilde{U}+d\tilde{v}(\iota_*X, \iota_*Y)\tilde{V} \\
&=\iota_*([f, f](X, Y)+d\eta(X, Y)\xi) \\
&\quad +\omega(X)(\mathcal{L}_{\tilde{V}}\tilde{F})\iota_*Y-\omega(Y)(\mathcal{L}_{\tilde{V}}\tilde{F})\iota_*X \\
&\quad +\{(\mathcal{L}_{fX}\omega)(Y)-(\mathcal{L}_{fY}\omega)(X)\}\,\tilde{U}\,.
\end{aligned}
$$

Hence we have the following theorem.

THEOREM 5.2. *If $\tilde{M}$ is a manifold with a normal $(\tilde{F}, \tilde{U}, \tilde{V}, \tilde{u}, \tilde{v}, \lambda)$-structure satisfying $\mathcal{L}_{\tilde{V}}\tilde{F}=0$, then the almost contact structure induced on a semi-invariant submanifold with respect to $\tilde{U}$ on which λ=constant is normal.*

Let $\tilde{g}$ be a metric on $\tilde{M}$ associated with the $(\tilde{F}, \tilde{U}, \tilde{V}, \tilde{u}, \tilde{v}, \lambda)$-structure. If g is the metric induced on the semi-invariant submanifold M then using (5.1), (5.2) and the fact that $\frac{1}{\lambda}\omega=\eta$, we have

$$
\begin{aligned}
g(fX, fY)&=\tilde{g}(\iota_*fX, \iota_*fY) \\
&=\tilde{g}(\tilde{F}\iota_*X-\omega(X)\tilde{U}, \tilde{F}\iota_*Y-\omega(Y)\tilde{U}) \\
&=\tilde{g}(\tilde{F}\iota_*X, \tilde{F}\iota_*Y)-\omega(X)\tilde{g}(\tilde{F}\iota_*Y, \tilde{U}) \\
&\quad -\omega(Y)\tilde{g}(\tilde{F}\iota_*X, \tilde{U})+(1-\lambda^2)\omega(X)\omega(Y)
\end{aligned}
$$

$$=\tilde{g}(\iota_*X, \iota_*Y)-\tilde{u}(\iota_*X)\tilde{u}(\iota_*Y)-\tilde{v}^2(\iota_*X)\tilde{v}(\iota_*Y)$$
$$-\lambda\omega(X)\tilde{v}(\iota_*Y)-\lambda\omega(Y)\tilde{v}(\iota_*X)+(1-\lambda^2)\omega(X)\omega(Y)$$
$$=g(X, Y)-(1-\lambda^2)\eta(X)\eta(Y).$$

If we let $\bar{g}=\frac{1}{1-\lambda^2}g$ then we see that $\bar{g}$ is a metric associated with the almost contact structure on M.

Remark 1: In the metric case $\tilde{u}(\iota_*X)=0$ implies that $\tilde{U}$ is normal to M.

Remark 2: We can consider $S^{2p}\times S^{2q}$ as a submanifold of condimension 2 in $E^{2p+2q+2}$. Now $E^{2p+2q+2}$ is a Kaehler manifold and so $S^{2p}\times S^{2q}$ carries a $(\tilde{F}, \tilde{U}, \tilde{V}, \tilde{u}, \tilde{v}, \lambda)$-structure [1]. However S^{2p} or S^{2q} cannot be semi-invariant submanifolds since they are even-dimensional.

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