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GLOBAL SOLUTIONS OF CERTAIN FOURTH ORDER DIFFERENTIAL EQUATIONS

Dedicated to Professor Yûsaku Komatu on his 60th birthday

BY TOSHIHIKO NISHIMOTO

§ 1. Introduction.

In this paper, we consider homogeneous and nonhomogeneous fourth order ordinary differential equations of the form

$$\frac{d^4u}{dz^4}-\left(z\frac{d^2u}{dz^2}+\lambda\frac{du}{dz}\right)=0, \tag{1.1}$$

$$\frac{d^4v}{dz^4}-\left(z\frac{d^2v}{dz^2}+\lambda\frac{dv}{dz}\right)=b(z), \tag{1.2}$$

where u and v are unknown functions of z, z is the complex independent variable, λ is a constant and $b(z)$ is a known function of z. These equations are related to the Orr-Sommerfeld and its adjoint equations, we call them together Orr-Sommerfeld type equations that play the fundamental role in the theory of hydrodynamic stability of viscous fluids. The Orr-Sommerfeld type equations are of the form

$$\varepsilon^2\frac{d^4\varphi}{dx^4}-\left\{p_3(x,\varepsilon)\frac{d^2\varphi}{dx^2}+p_2(x,\varepsilon)\frac{d\varphi}{dx}+p_1(x,\varepsilon)\varphi\right\}=0, \tag{1.3}$$

where ε is a small positive parameter and $p_i(x,\varepsilon)$ $(i=1,2,3)$ can be expanded asymptotically in power series of ε with holomorphic coefficients. Except for a small neighborhood of turning point x where $p_3(x,0)=0$, asymptotic solutions of (1.3) were obtained by the W-K-B type approximation, Nishimoto [1]. On the other hand, asymptotic expansions in the direct neighborhood of a turning point are constructed by either the related equation method or the matching procedure. If we apply the matching method to the equation (1.3) in the neighborhood of its simple turning point which is assumed to be at the origin, according to Nishimoto [2], page 238-239 with $n=4$, $m=2$ and $q=1$, it becomes necessary to study the equations

$$\frac{d^4u_0}{dx^4}-\left(ax\frac{d^2u_0}{dx^2}+b\frac{du_0}{dx}\right)=0,$$

$$\frac{d^4u_i}{dx^4}-\left(ax\frac{d^2u_i}{dx^2}+b\frac{du_i}{dx}\right)=\sum_{k=1}^{i}b_{1k}u_{i-k}+b_{2k}\frac{du_{i-k}}{dx}+b_{3k}\frac{d^2u_{i-k}}{dx^2}+b_{4k}\frac{d^3u_{i-k}}{dx^3}.$$

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in the large as the first and the higher terms of the inner asymptotic expansion. By putting $z=a^{1/3}x$, the above equations take the form (1.1) and (1.2) with $\lambda=b/a$. Then the results of this paper are intended to use for construction of the inner solution of (1.3), by which one of the lackness of the previous paper [1] will be covered.

Instead of the equation (1.1) we analyze the equation for $y=du/dz$

$$\frac{d^3y}{dz^3}-\left(z\frac{dy}{dz}+\lambda y\right)=0 \tag{1.4}$$

in the large by the Laplace integral method. The convergent expression of solutions in the neighborhood of $z=0$, the asymptotic expression at $z=\infty$ and their Stokes phenomenon are considered. Our method is quite standard and there are many contributions to the general theory of the Laplace integral method and its applications. Therefore the results may not be new in theory, but there is no complete representation applicable to the solutions of (1.4) for arbitrarily constant λ. Moreover a rigorous treatment of the matching method requires the asymptotic behavior of solutions of nonhomogeneous equation (1.2) for large value of z, which is studied in section 3.

§ 2. Solutions of the homogeneous equation.

2.1. The solutions of the equation (1.4) are, as is easily verified, expressed by the Laplace contour integrals

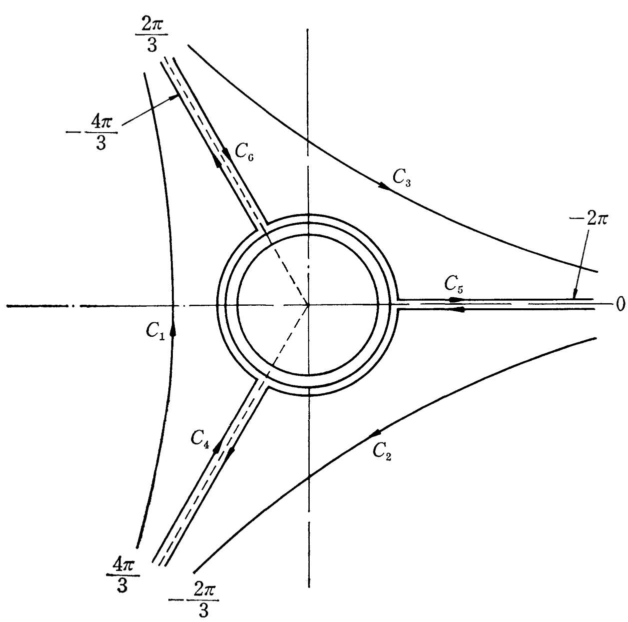

Fig. 1.

$$y_j(z)=\frac{1}{2\pi i}\int_{C_j} t^{\lambda-1}\exp\left(zt-\frac{1}{3}t^3\right)dt \qquad (j=1,2,3,4,5,6), \tag{2.1}$$

where the integral paths C_j are as indicated in Fig. 1. The angles written at the end of curves mean that the curves extend to infinity at these directions. The constant λ is assumed not to be an integer avoiding the complexity of the descriptions of the results obtained. The value of $t^{\lambda-1}$ is determined by cutting the complex t-plane along the half line of argument $(4/3)\pi$ for C_j $(j=1,2,3,4)$, 0 for C_5 and $(2/3)\pi$ for C_6 respectively.

The solutions $y_j(z)$ are connected by the identities

$$y_1(z)+y_2(z)+y_3(z)=y_4(z) \tag{2.2}$$

and for $\omega=\exp 2\pi i/3$,

$$y_1(z)=\omega^{\lambda}y_3(\omega z)=\omega^{2\lambda}y_2(\omega^2 z), \tag{2.3}$$

$$y_4(z)=\omega^{\lambda}y_6(\omega z)=\omega^{2\lambda}y_5(\omega^2 z). \tag{2.4}$$

It is easy to calculate from (2.1) the convergent power series for $y_j(z)$ about $z=0$, that is, by expanding $\exp zt$ into power series of zt, integrating term by term and evaluating the definite integrals in the coefficients of the powers of z.

From (2.1), we have for

$$y_1(z)=\sum_{k=0}^{\infty} a_k^{(1)} z^k, \tag{2.5}$$

where $a_k^{(1)}$ is given by

$$\begin{aligned} a_k^{(1)} &= \frac{1}{2\pi i}\int_{C_1}\frac{t^{k+\lambda-1}}{k!}e^{-\frac{1}{3}t^3}dt \\ &= \frac{1}{2\pi i\cdot k!}\left(e^{\frac{2}{3}(k+\lambda)\pi i}-e^{\frac{4}{3}(k+\lambda)\pi i}\right)\int_0^{\infty} r^{k+\lambda-1}e^{-\frac{1}{3}r^3}dr \\ &= \frac{1}{2\pi i\cdot k!}\left(e^{\frac{2}{3}(k+\lambda)\pi i}-e^{\frac{4}{3}(k+\lambda)\pi i}\right)3^{\frac{k+\lambda}{3}-1}\Gamma\left(\frac{k+\lambda}{3}\right), \end{aligned}$$

when $\mathrm{Re}(\lambda+k)>0$, and if $\mathrm{Re}(\lambda+k)<0$, the integration by parts gives the same formula. Similarly we have for $y_4(z)$

$$y_4(z)=\sum_{k=0}^{\infty} a_k^{(4)} z^k, \tag{2.6}$$

with

$$a_k^{(4)}=\frac{1}{2\pi i\cdot k!}\left(e^{-\frac{2}{3}(k+\lambda)\pi i}-e^{\frac{4}{3}(k+\lambda)\pi i}\right)3^{\frac{k+\lambda}{3}-1}\Gamma\left(\frac{k+\lambda}{3}\right).$$

The convergent power series expressions for other $y_j(z)$ are obtained by using (2.3) and (2.4). The above constructed power series converge for all finite z since the differential equation (1.4) has a singularity only at $z=$infinity. The problem is thus to determine the asymptotic expansions of $y_j(z)$ in the neighborhood of infinity.

2.2. At first, we obtain the asymptotic expansions of $y_1(z)$ for large positive values of z by using a simpler method of Olver [3] rather than the methods of steepest descent, and then consider the extension of regions of their validity in later.

By the transformation $t=\sqrt{z}\,s$, the integral expression (2.1) becomes

$$y_1(z)=\frac{1}{2\pi i}z^{\frac{\lambda}{2}}\int_{C_1'}s^{\lambda-1}\exp\left\{\frac{3}{2}\xi\left(s-\frac{1}{3}s^3\right)\right\}ds\,,\qquad \xi=\frac{2}{3}z^{\frac{3}{2}}\,, \tag{2.7}$$

where we can assume the integral path C_1' consists of two rays of arguments $(2/3)\pi$ and $(4/3)\pi$ starting from $s=-1$.

Let $J_\pm(\xi)$ be

$$J_+(\xi)=\int_{-1}^{-1+\infty e^{2\pi i/3}}s^{\lambda-1}\exp\frac{3}{2}\xi\left(s-\frac{1}{3}s^3\right)ds\,,$$

$$J_-(\xi)=\int_{-1}^{-1+\infty e^{4\pi i/3}}s^{\lambda-1}\exp\frac{3}{2}\xi\left(s-\frac{1}{3}s^3\right)ds\,,$$

then

$$y_1(z)=\frac{1}{2\pi i}z^{\frac{\lambda}{2}}\{J_+(\xi)-J_-(\xi)\}\,. \tag{2.8}$$

Following to [3],

$$J_+(\xi)=e^{\lambda\pi i}\int_{1}^{1+\infty e^{-\pi i/3}}s^{\lambda-1}\exp\left\{-\frac{3}{2}\xi\left(s-\frac{1}{3}s^3\right)\right\}ds$$

$$=e^{\lambda\pi i}e^{-\xi}\int_{1}^{1+\infty e^{-\pi i/3}}s^{\lambda-1}\exp\left\{\xi\left(\frac{3}{2}(s-1)^2+\frac{1}{2}(s-1)^3\right)\right\}ds\,.$$

Reverting the expression

$$v=-\frac{3}{2}(s-1)^2-\frac{1}{2}(s-1)^3\,,$$

and constructing $s^{\lambda-1}ds/dv$ in power series of v, we have

$$s^{\lambda-1}\frac{ds}{dv}=\sum_{k=0}^{\infty}b_k v^{\frac{k-1}{2}}\,,$$

$$b_0=\frac{1}{2}\left(-\frac{2}{3}\right)^{\frac{1}{2}},\ b_1=\frac{2}{3}\left(\frac{1}{6}-\frac{1}{2}(\lambda-1)\right),\ b_2=\left\{\frac{(\lambda-1)(\lambda-3)}{4}+\frac{5}{48}\right\}\left(-\frac{2}{3}\right)^{\frac{3}{2}},\ \text{etc.,} \tag{2.9}$$

in general b_k has a form $b_k'(-2/3)^{(k+1)/2}$ with constant b_k'. Here we use the branch $(-2/3)^{1/2}=(2/3)^{1/2}i$. Then we obtain the asymptotic expansion of $J_+(\xi)$ such that

$$J_+(\xi)\sim e^{\lambda\pi i}e^{-\xi}\sum_{k=0}^{\infty}\Gamma\left(\frac{k+1}{2}\right)b_k\xi^{-\frac{k+1}{2}}\,,\qquad \text{as}\ \ \xi\to+\infty\,.$$

By the same method, $J_-(\xi)$ can be expanded asymptotically in the form

$$J_{-}(\xi)\sim e^{\lambda\pi i}e^{-\xi}\sum_{k=0}^{\infty}\Gamma\Big(\frac{k+1}{2}\Big)\tilde{b}_k\xi^{-\frac{k+1}{2}}, \qquad \text{as} \quad \xi\to+\infty$$

with $\tilde{b}_k=(-1)^{k+1}b_k$.

On substituting these results in (2.8), we have

$$(2.10) \qquad y_1(z)\sim\frac{1}{\pi i}z^{\frac{\lambda}{2}}e^{\lambda\pi i}e^{-\xi}\sum_{k=0}^{\infty}\Gamma\Big(k+\frac{1}{2}\Big)b_{2k}\xi^{-(k+\frac{1}{2})}$$

$$=\sqrt{\frac{3}{2}}\frac{1}{\pi i}z^{\frac{\lambda}{2}-\frac{3}{4}}e^{\lambda\pi i}e^{-\xi}\sum_{k=0}^{\infty}\Gamma\Big(k+\frac{1}{2}\Big)b_{2k}\xi^{-k}, \qquad \text{as} \quad \xi\to+\infty.$$

2.3. Next, we calculate the asymptotic expansion of $y_4(z)$ as z tends to infinity. For positive z, putting $t=\sqrt{z}\,s$, $y_4(z)$ becomes

$$(2.11) \qquad y_4(z)=\frac{1}{2\pi i}z^{\frac{\lambda}{2}}\int_{C_4'}s^{\lambda-1}\exp\Big\{z^{\frac{3}{2}}\Big(s-\frac{1}{3}s^3\Big)\Big\}ds,$$

where we can suppose that the integral path C_4' consists of a circle $C^{(1)}$ of radius r $(0<r<1)$, and segments $C^{(2)}$ starting from the points P with coordinate $r\exp 4\pi i/3$ and P' with coordinate $r\exp(-2\pi i/3)$ and extending to infinity of directions $4\pi i/3$ and $-2\pi i/3$ respectively (Fig. 2). We consider the contributions to the integral (2.11) from the part $C^{(1)}$ and the part $C^{(2)}$ of C_4' respectively. The latter is estimated by

$$(2.12) \qquad \Big|z^{\frac{\lambda}{2}}\int_{C^{(2)}}s^{\lambda-1}\exp\Big\{z^{\frac{3}{2}}\Big(s-\frac{1}{3}s^3\Big)\Big\}ds\Big|$$

$$=\Big|z^{\frac{\lambda}{2}}(e^{-\frac{2}{3}\lambda\pi i}-e^{\frac{4}{3}\lambda\pi i})\int_r^{\infty}u^{\lambda-1}\exp\Big\{z^{\frac{3}{2}}\Big(ue^{-\frac{2}{3}\pi i}-\frac{1}{3}u^3\Big)\Big\}du\Big|$$

$$\leqq K_1|z^{\frac{\lambda}{2}}|\exp\Big(-\frac{z^{\frac{3}{2}}r}{2}\Big)\int_r^{\infty}u^{\mathrm{Re}\,\lambda-1}\exp\Big(-\frac{z^{\frac{3}{2}}u^3}{3}\Big)du$$

$$\leqq\begin{cases} K\exp\Big(-\dfrac{z^{\frac{3}{2}}r}{2}\Big) & (\mathrm{Re}\,\lambda>0), \\ Kz^{\frac{\mathrm{Re}\,\lambda-1}{2}}\exp\Big(-\dfrac{z^{\frac{3}{2}}r}{2}\Big) & (\mathrm{Re}\,\lambda\leqq 0), \end{cases}$$

for some constant K depending on r and λ.

Next, the contribution from $C^{(1)}$ is studied. To do so, we consider a mapping $u=s-s^3/3$. By this, a neighborhood of $s=0$ is transformed onto a neighborhood of $u=0$, in particular, the points P and P' to P_1 and P_1' respectively, and $C^{(1)}$ to some curve $\tilde{C}$ in the u-plane. We can suppose that the curve $\tilde{C}$ is a circle starting from P_1 and ending at P_1' clockwise, and accordingly the cut in the u-plane can be a segment connecting the origin and P_1' (Fig. 3). Thus we have

$$\frac{1}{2\pi i}z^{\frac{\lambda}{2}}\int_{C^{(1)}}s^{\lambda-1}\exp\Big\{z^{\frac{3}{2}}\Big(s-\frac{1}{3}s^3\Big)\Big\}ds=\frac{1}{2\pi i}z^{\frac{\lambda}{2}}\int_{\tilde{C}}s(u)^{\lambda-1}\exp(z^{\frac{3}{2}}u)\frac{ds}{du}du.$$

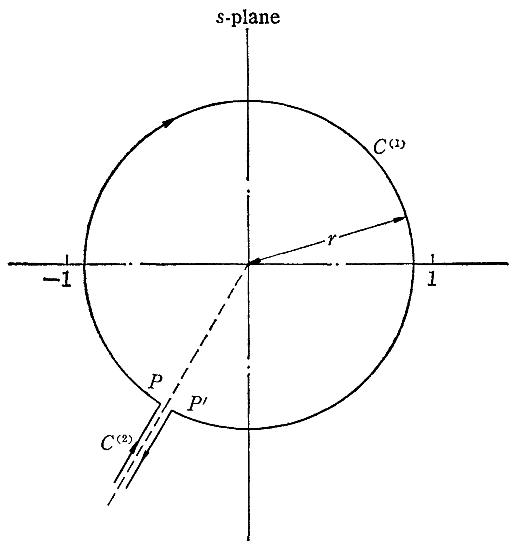

Fig. 2.

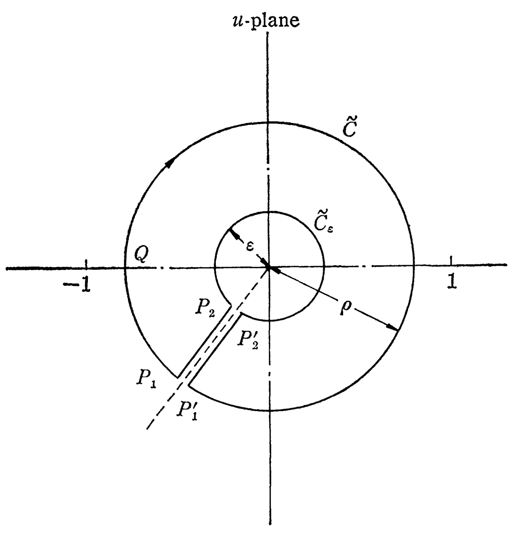

Fig. 3.

From the expression $u=s-s^3/3$, we can find the convergent power series of u for $s=s(u)$ and then for $s(u)^{\lambda-1}ds/du$ such that

$$s(u)^{\lambda-1}\frac{ds}{du}=u^{\lambda-1}\{\sum_{k=0}^{\infty} d_{2k}u^{2k}\}\,, \tag{2.13}$$

where d_{2k} are constants, in particular $d_0=1$, $d_2=(\lambda+2)/3$, $d_4=(\lambda+4)(\lambda+5)/15$, etc.

Since the coordinate of P_1 is $r\exp 4\pi i/3-r^3/3$, this can be written as $\rho\exp i\alpha$ with $0<\rho<1$, $\pi<\alpha<3\pi/2$. Let C_ε be a circle around the origin of radius ε, and P_2, P_2', and Q be points with coordinates $\varepsilon\exp i\alpha$, $\varepsilon\exp i(\alpha-2\pi)$ and $\rho\exp\pi i$ respectively (Fig. 3).

By the Cauchy's integral formula,

$$\int_{\tilde{C}} s(u)^{\lambda-1}\exp(z^{\frac{3}{2}}u)\frac{ds}{du}du \tag{2.14}$$
$$=\left\{\int_{P_1P_2}+\int_{\tilde{C}_\varepsilon}+\int_{P_2'P_1}\right\}s(u)^{\lambda-1}\exp(z^{\frac{3}{2}}u)\frac{ds}{du}du\,.$$

Let n be a positive integer such that $\mathrm{Re}(\lambda+2n+1)>-1$ and

$$s(u)^{\lambda-1}\frac{ds}{du}=u^{\lambda-1}\{\sum_{k=0}^{\infty}d_{2k}u^{2k}+O(u^{2n+2})\}\,. \tag{2.15}$$

Then we have

$$\int_{\tilde{C}} u^{\lambda+2n+1}\exp(z^{\frac{3}{2}}u)du$$
$$=\left\{(1-e^{-2\lambda\pi i})\int_{P_1P_2}+\int_{\tilde{C}_\varepsilon}\right\}u^{\lambda+2n+1}\exp(z^{\frac{3}{2}}u)du$$
$$=(1-e^{-2\lambda\pi i})\int_{P_1O}u^{\lambda+2n+1}\exp(z^{\frac{3}{2}}u)du\,,$$

since

$$\lim_{\varepsilon\to 0}\left|\int_{\tilde{C}_\varepsilon}u^{\lambda+2n+1}\exp(z^{\frac{3}{2}}u)du\right|$$
$$=\lim_{\varepsilon\to 0}\varepsilon^{\lambda+2n+2}\left|\int_0^{2\pi}\exp\{(\lambda+2n+1)i\theta+z^{\frac{3}{2}}\varepsilon e^{i\theta}\}\right|d\theta=0\,.$$

Furthermore we have

$$\left|\int_{P_1O}u^{\lambda+2n+1}\exp(z^{\frac{3}{2}}u)du\right|=\left|e^{(\lambda+2n+2)\alpha i}\int_0^{\rho}r^{\lambda+2n+1}\exp(z^{\frac{3}{2}}re^{i\alpha})dr\right| \tag{2.16}$$
$$\leqq K\int_0^{\infty}r^{\mathrm{Re}\,\lambda+2n+1}\exp(z^{\frac{3}{2}}r\cos\alpha)dr$$
$$=K\frac{\Gamma(\mathrm{Re}\,\lambda+2n+2)}{(\cos\alpha)^{\mathrm{Re}\,\lambda+2n+2}}z^{-\frac{3}{2}(\mathrm{Re}\,\lambda+2n+2)}.$$

Let m be the smallest positive integer such that $\mathrm{Re}\,\lambda+m>-1$. Then by integration by parts, we have for $2k<m$

$$\int_{\tilde{C}}u^{2k+\lambda-1}\exp(z^{\frac{3}{2}}u)du$$
$$=\left[\frac{1}{2k+\lambda}u^{2k+\lambda}\exp(z^{\frac{3}{2}}u)\right]_{u=\rho e^{i\alpha}}^{u=\rho e^{i(\alpha-2\pi)}}-\int_{\tilde{C}}\frac{1}{2k+\lambda}z^{\frac{3}{2}}u^{2k+\lambda}\exp(z^{\frac{3}{2}}u)du$$
$$=-\frac{1}{2k+\lambda}\int_{\tilde{C}}z^{\frac{3}{2}}u^{2k+\lambda}\exp(z^{\frac{3}{2}}u)du+O[\exp(z^{\frac{3}{2}}\rho\cos\alpha)]\,.$$

Repeating this process for $m-2k+1$ times

$$\int_{\tilde{C}} u^{2k+\lambda-1}\exp(z^{\frac{3}{2}}u)du$$
$$=(-1)^{m-2k+1}\frac{z^{\frac{3}{2}(m-2k+1)}}{(2k+\lambda)(2k+\lambda+1)\cdots(m+\lambda)}\int_{\tilde{C}} u^{m+\lambda}\exp(z^{\frac{3}{2}}u)du$$
$$+O[z^{\frac{3}{2}(m-2k)}\exp(z^{\frac{3}{2}}\rho\cos\alpha)].$$

Since $\operatorname{Re}\lambda+m>-1$, the above integral becomes as before

$$\int_{\tilde{C}} u^{m+\lambda}\exp(z^{\frac{3}{2}}u)du=(1-e^{-2\lambda\pi i})\int_{P_1O} u^{m+\lambda}\exp(z^{\frac{3}{2}}u)du$$
$$=(1-e^{-2\lambda\pi i})\int_{QO} u^{m+\lambda}\exp(z^{\frac{3}{2}}u)du+O[\exp(z^{\frac{3}{2}}\rho\cos\alpha)],$$

$$\int_{QO} u^{m+\lambda}\exp(z^{\frac{3}{2}}u)du=e^{(m+\lambda)\pi i}\int_0^{\rho} r^{m+\lambda}\exp(-z^{\frac{3}{2}}r)dr \qquad (u=re^{\pi i})$$
$$=e^{(m+\lambda)\pi i}\left\{\int_0^{\infty} r^{m+\lambda}\exp(-z^{\frac{3}{2}}r)dr-\int_{\rho}^{\infty} r^{m+\lambda}\exp(-z^{\frac{3}{2}}r)dr\right\}$$
$$=e^{(m+\lambda)\pi i}\Gamma(m+\lambda+1)z^{-\frac{3}{2}(m+\lambda+1)}+O[z^{-\frac{3}{2}}\exp(-z^{\frac{3}{2}}\rho)],$$

by using the asymptotic property of incomplete Gamma function. Thus

$$\int_{\tilde{C}} u^{2k+\lambda-1}\exp(z^{\frac{3}{2}}u)du=e^{(\lambda+1)\pi i}(1-e^{-2\lambda\pi i})\Gamma(2k+\lambda)z^{-\frac{3}{2}(2k+\lambda)}$$
$$+O[z^{\frac{3}{2}(m-2k+1)}\exp(z^{\frac{3}{2}}\rho\cos\alpha)]. \tag{2.17}$$

Combining the results from (2.11) to (2.17) we obtain

$$y_4(z)=\frac{1}{2\pi i}z^{\frac{\lambda}{2}}\{\sum_{k=0}^{n} d_{2k}e^{(\lambda+1)\pi i}(1-e^{-2\lambda\pi i})\Gamma(2k+\lambda)z^{-\frac{3}{2}(2k+\lambda)}$$
$$+O[z^{\frac{3}{2}(m+1)}\exp(z^{\frac{3}{2}}\rho\cos\alpha)]+O[z^{-\frac{3}{2}(\operatorname{Re}\lambda+2n+2)}]\}, \tag{2.18}$$

with $\pi<\alpha<3\pi/2$.

2.4. We consider here z as the complex variable and extend regions of validity of the asymptotic expansions obtained in the previous paragraph. Firstly, the solution $y_1(z)$ has the asymptotic expansion (2.10) in the sector $|\arg z|<\pi$ as z tends to infinity, by the same reasonings for the Airy functions, and so limit our considerations to the solution $y_4(z)$.

We consider the integral representation (2.11). If $-\pi/2<\arg z^{3/2}<\pi/6$, the integral converges and the contribution from the part $C^{(2)}$ is exponentially small, then for $-\pi/3<\arg z<\pi/9$, the asymptotic expansion (2.18) is valid. To extend the region of validity still further, the integral path C_4' in Fig. 2 is deformed to C_4'' by rotating it around the origin through an angle $-\beta/3$, as indicated in Fig. 4.

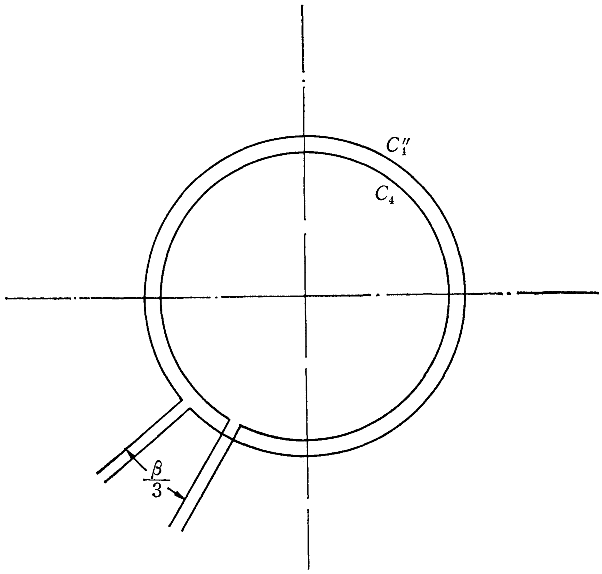

Fig. 4.

The branch of $s^{\lambda-1}$ in the integrand is to be determined by analytic continuation in an obvious way along the integral path. By the Cauchy's theorem we can prove that the integral

$$\frac{1}{2\pi i}z^{\frac{\lambda}{2}}\int_{C_4''}s^{\lambda-1}\exp z^{\frac{3}{2}}\left(s-\frac{1}{3}s^3\right)ds \tag{2.19}$$

converges uniformly when

$$-\frac{\pi}{2}+\beta+\delta\leqq\frac{3}{2}\arg z\leqq\frac{\pi}{2}+\beta-\delta\,, \tag{2.20}$$

where δ is an arbitrarily small positive number, and represents the analytic continuation of $y_4(z)$ in this sector.

The analysis in the preceding paragraph shows that the function defined by (2.19) has the same asymptotic expansion as (2.18) if $\operatorname{Re} z^{3/2}(s-s^3/3)<0$ along the straight segment of C_4'', that is if (2.20) and

$$-\frac{5}{6}\pi+\frac{\beta}{3}+\delta\leqq\frac{3}{2}\arg z\leqq\frac{\pi}{6}+\frac{\beta}{3}-\delta \tag{2.21}$$

are both satisfied.

The range of values of $\arg z$ for which β can be chosen to fulfil the conditions (2.20) and (2.21) is thus obtained by eliminating β from these inequalities. This gives

$$-\pi+\frac{4}{3}\delta\leqq\arg z\leqq\frac{\pi}{3}-\frac{4}{3}\delta \qquad (\delta>0)\,. \tag{2.22}$$

Combining the results obtained in the paragraphs 2.1-2.4, we have established the following theorem

THEOREM. *The solutions $y_j(z)$ $(j=1, 4)$ of the differential equation* (1.4) *defined by the integral* (2.1) *are expressed by convergent power series in the neighborhood of the origin such that*

$$y_j(z)=\sum_{k=0}^{\infty} a_k^{(j)} z^k, \qquad a_k^{(j)}=\frac{1}{2\pi i}\int_{C_j} \frac{t^{k+\lambda-1}}{k!} \exp\left(-\frac{1}{3}t^3\right)dt.$$

For large absolute values of z, $y_j(z)$ can be expanded asymptotically in the form

$$y_1(z)\sim\sqrt{\frac{3}{2}}\,\frac{e^{\lambda\pi i}}{\pi i} z^{\frac{\lambda}{2}-\frac{3}{4}} e^{-\xi}\left\{\sum_{k=0}^{\infty} \Gamma\left(k+\frac{1}{2}\right) b_{2k}\xi^{-k}\right\}, \qquad \xi=\frac{2}{3}z^{\frac{3}{2}},$$

$$for \qquad |\arg z|<\pi, \quad |z|\geqq R>0,$$

$$y_4(z)\sim\frac{e^{-\lambda\pi i}-e^{\lambda\pi i}}{2\pi i} z^{-\lambda}\left\{\sum_{k=0}^{\infty} \Gamma(2k+\lambda) d_{2k} z^{-3k}\right\},$$

$$for \qquad -\pi<\arg z<\frac{\pi}{3}, \quad |z|\geqq R>0$$

where R is a large positive number, and the coefficients b_{2k} and d_{2k} are determined by the equations (2.9) *and* (2.13) *respectively.*

2.5. From the relations (2.3) and (2.4), we can deduce the formulas of the functions $y_2(z)$, $y_3(z)$, $y_5(z)$ and $y_6(z)$. Since we have

$$y_2(z)=\omega^{-2\lambda}y_1(\omega^{-2}z), \qquad y_3(z)=\omega^{-\lambda}y_1(\omega^{-1}z),$$

$$y_5(z)=\omega^{-2\lambda}y_4(\omega^{-2}z), \qquad y_6(z)=\omega^{-\lambda}y_4(\omega^{-1}z),$$

then the asymptotic expansions of these functions are as follows.

$$y_2(z)\sim-\sqrt{\frac{3}{2}}\,\frac{e^{-\lambda\pi i}}{\pi i} z^{\frac{\lambda}{2}-\frac{3}{4}} e^{-\xi}\left\{\sum_{k=0}^{\infty} \Gamma\left(k+\frac{1}{2}\right) b_{2k}\xi^{-k}\right\}, \qquad \xi=\frac{2}{3}z^{\frac{3}{2}},$$

$$\text{for} \qquad \frac{\pi}{3}<\arg z<\frac{7\pi}{3}, \qquad |z|\geqq R>0,$$

$$y_3(z)\sim\sqrt{\frac{3}{2}}\,\frac{1}{\pi} z^{\frac{\lambda}{2}-\frac{3}{4}} e^{-\xi_1}\left\{\sum_{k=0}^{\infty} \Gamma\left(k+\frac{1}{2}\right) b_{2k}\xi_1^{-k}\right\}, \qquad \xi_1=-\frac{2}{3}z^{\frac{3}{2}},$$

$$\text{for} \qquad -\frac{\pi}{3}<\arg z<\frac{5\pi}{3}, \quad |z|\geqq R>0,$$

$$y_5(z)\sim\frac{e^{\lambda\pi i}}{2\pi i}(e^{-2\lambda\pi i}-1) z^{-\lambda}\left\{\sum_{k=0}^{\infty} \Gamma(2k+\lambda) d_{2k} z^{-3k}\right\},$$

$$\text{for} \qquad \frac{\pi}{3}<\arg z<\frac{5\pi}{3}, \qquad |z|\geqq R>0,$$

$$y_6(z)\sim\frac{e^{\lambda\pi i}}{2\pi i}(e^{-2\lambda\pi i}-1)z^{-\lambda}\Big\{\sum_{k=0}^{\infty}\Gamma(2k+\lambda)d_{2k}z^{-3k}\Big\},$$

$$\text{for}\qquad -\frac{\pi}{3}<\arg z<\pi\,,\qquad |z|\geqq R>0\,.$$

2.6. From the functions constructed in the preceding paragraphs, we can obtain the fundamental systems of solutions of the equation (1.1).

Let $u_j(z)$ be a function defined by

$$u_j(z)=C+\int_0^z y_j(z)dz$$

for some constant C, then $u_j(z)$ is a solution of (1.1). Since

$$\frac{1}{2\pi i}\int_0^z\int_{C_j}t^{\lambda-1}\exp\Big(zt-\frac{1}{3}t^3\Big)dt\,dz$$
$$=\frac{1}{2\pi i}\int_{C_j}t^{\lambda-2}\exp\Big(zt-\frac{1}{3}t^3\Big)dt-\frac{1}{2\pi i}\int_{C_j}t^{\lambda-2}e^{-\frac{t^3}{3}}dt\,,$$

then by choosing the constant C as

$$\frac{1}{2\pi i}\int_{C_j}t^{\lambda-2}e^{-\frac{t^3}{3}}dt\,,$$

we have

$$u_j(z)=\frac{1}{2\pi i}\int_{C_j}t^{\lambda-2}\exp\Big(zt-\frac{1}{3}t^3\Big)dt\,.$$

The convergent expression, asymptotic expansion and its Stokes phenomenon of $u_j(z)$ can be analyzed by the same way as in the preceding paragraphs with $\lambda-1$ in place of λ. Also, the derivatives $y_j^{(k)}(z)$ $(k=1, 2)$ are defined by

$$y_j^{(k)}(z)=\frac{1}{2\pi i}\int_{C_j}t^{\lambda-1+k}\exp\Big(zt-\frac{1}{3}t^3\Big)dt\,.$$

Thus we have obtained the solutions of (1.1) and their derivatives. The fundamental system of solutions is given, for example, by $[1, u_6(z), u_1(z), u_3(z)]$ since its Wronskian becomes

$$\det\begin{bmatrix}1 & u_6(z) & u_1(z) & u_3(z)\\ 0 & u_6'(z) & u_1'(z) & u_3'(z)\\ 0 & u_6''(z) & u_1''(z) & u_3''(z)\\ 0 & u_6'''(z) & u_1'''(z) & u_3'''(z)\end{bmatrix}=\det\begin{bmatrix}u_6'(0) & u_1'(0) & u_3'(0)\\ u_6''(0) & u_1''(0) & u_3''(0)\\ u_6'''(0) & u_1'''(0) & u_3'''(0)\end{bmatrix}$$
$$=\frac{3^{\lambda-\frac{1}{2}}}{8\pi^3}(e^{2\lambda\pi i}-1)\Gamma\Big(\frac{\lambda}{3}\Big)\Gamma\Big(\frac{\lambda+1}{3}\Big)\Gamma\Big(\frac{\lambda+2}{3}\Big)\,.$$

The system $[1, u_6(z), u_3(z), u_1(z)]$ is one of the fundamental systems of solutions of (1.1) whose asymptotic behavior as z tends to infinity is known in the range of argument $-\frac{\pi}{3}<\arg z<\pi$. Another fundamental systems are $[1, u_5(z), u_2(z), u_3(z)]$ for $\pi/3<\arg z<5\pi/3$, and $[1, u_4(z), u_1(z), u_2(z)]$ for $-\pi<\arg z<\pi/3$.

Let

$$[1, u_6(z), u_3(z), u_1(z)]=[1, u_5(z), u_2(z), u_3(z)]\Pi_1,$$

$$[1, u_5(z), u_2(z), u_3(z)]=[1, u_4(z), u_1(z), u_2(z)]\Pi_2,$$

$$[1, u_4(z), u_1(z), u_2(z)]=[1, u_6(z), u_3(z), u_1(z)]\Pi_3,$$

then we have

$$\Pi_1=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & \omega^{3\lambda}\\ 0 & 0 & 0 & -\omega^{3\lambda}\\ 0 & 1-\omega^{-3\lambda} & 1 & -1\end{bmatrix}, \quad \Pi_2=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \omega^{-3\lambda} & 0 & 1\\ 0 & 0 & 0 & -1\\ 0 & 1-\omega^{-3\lambda} & 1 & -1\end{bmatrix},$$

$$\Pi_3=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 0 & -1\\ 0 & 1-\omega^{-3\lambda} & 1 & -\omega^{-3\lambda}\end{bmatrix},$$

where $\omega=\exp 2\pi i/3$.

From these relations we can get precise asymptotic expansions of solutions in the whole z-plane.

§ 3. Solutions of nonhomogeneous equations.

3.1. To solve the nonhomogeneous equation (1.2) it is convenient to write it by an equivalent vectorial form

$$\frac{dv}{dz}=Av+B,$$

(3.1)

$$A=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & \lambda & z & 0\end{bmatrix}, \quad B=\begin{bmatrix}0\\ 0\\ 0\\ b(z)\end{bmatrix}.$$

Here v denotes the fourth column vector. By the method of variations of constant, the solutions of (3.1) are expressed by

$$v(z)=W(z)\left\{C+\int^{z} W(s)^{-1}B(s)ds\right\}, \tag{3.2}$$

where C is some constant vector and $W(z)$ is a fundamental system of solutions of homogeneous equation

$$\frac{dw}{dz}=Aw\,.$$

$W(z)$ can be a matrix of the form

$$W(z)=\begin{bmatrix} 1 & w_1(z) & w_2(z) & w_3(z) \\ 0 & w_1'(z) & w_2'(z) & w_3'(z) \\ 0 & w_1''(z) & w_2''(z) & w_3''(z) \\ 0 & w_1'''(z) & w_2'''(z) & w_3'''(z) \end{bmatrix}$$

such that each $w_j(z)$ is equal to some $u_k(z)$ constructed in section 2 and the functions $[1, w_1(z), w_2(z), w_3(z)]$ constitute a fundamental system of solutions of (1.1). From the formula of inverse matrix, we can write

$$W(s)^{-1}B(s)=b(s)\begin{bmatrix} W_{41}(s) \\ W_{42}(s) \\ W_{43}(s) \\ W_{44}(s) \end{bmatrix}, \tag{3.3}$$

where

$$W_{41}(s)=-W_0^{-1}\det\begin{bmatrix} w_1 & w_2 & w_3 \\ w_1' & w_2' & w_3' \\ w_1'' & w_2'' & w_3'' \end{bmatrix}, \qquad W_{42}(s)=W_0^{-1}\det\begin{bmatrix} 1 & w_2 & w_3 \\ 0 & w_2' & w_3' \\ 0 & w_2'' & w_3'' \end{bmatrix},$$

$$W_{43}(s)=-W_0^{-1}\det\begin{bmatrix} 1 & w_1 & w_3 \\ 0 & w_1' & w_3' \\ 0 & w_1'' & w_3'' \end{bmatrix}, \qquad W_{44}(s)=W_0^{-1}\det\begin{bmatrix} 1 & w_1 & w_2 \\ 0 & w_1' & w_2' \\ 0 & w_1'' & w_2'' \end{bmatrix}.$$

Here W_0 denotes the constant $\det W(z)$. Since $w_j(z)$ is a solution of (1.1) the determinant $W_{41}(z)$ satisfies the following third order differential equation and initial condition:

$$\frac{d^3w}{dz^3}-\left\{z\frac{dw}{dz}+(1-\lambda)w\right\}=-1\,, \tag{3.4}$$

$$w(0)=-W_0^{-1}\det\begin{bmatrix} w_1 & w_2 & w_3 \\ w_1' & w_2' & w_3' \\ w_1'' & w_2'' & w_3'' \end{bmatrix}_{z=0}, \quad w'(0)=-W_0^{-1}\det\begin{bmatrix} w_1 & w_2 & w_3 \\ w_1' & w_2' & w_3' \\ w_1''' & w_2''' & w_3''' \end{bmatrix}_{z=0},$$

$$w''(0)=-W_0^{-1}\det\begin{bmatrix} w_1 & w_2 & w_3 \\ w_1'' & w_2'' & w_3'' \\ w_1''' & w_2''' & w_3''' \end{bmatrix}_{z=0},$$

and $W_{4j}(z)$ $(j=2, 3, 4)$ satisfy the homogeneous equation

$$\frac{d^3w}{dz^3}-\left\{z\frac{dw}{dz}+(1-\lambda)w\right\}=0\,, \tag{3.5}$$

with the anologous initial conditions.

3.2. The differential equation (3.5) has the same form as (1.4) and then it can be solved globally as in section 2. The solution of (3.5) with prescribed initial values is written as a linear conbination of the functions $y_j(z)$ constructed in section 2 with $(1-\lambda)$ in place of λ. Furthermore, since a particular solution of (3.4) is $w(z)=1/(1-\lambda)$, then the solution of (3.4) satisfying the initial conditions is also determined.

For example, assume that

$$w_1(z)=u_6(z)\,,\qquad w_2(z)=u_3(z)\,,\qquad w_3(z)=u_1(z)\,. \tag{3.6}$$

Then the initial values of $W_{4i}(z)$ $(i=1, 2, 3, 4)$ at $z=0$ are

$$\begin{aligned}
&W_{41}(0)=\frac{1}{1-\lambda}\,,\qquad W_{41}'(0)=W_{41}''(0)=0\,,\\
&W_{42}(0)=W_0^{-1}\frac{1}{(2\pi i)^2}3^{(2\lambda+1)/3-2}(\omega-1)(1-\omega^\lambda)(1-\omega^{\lambda+1})\omega^\lambda\Gamma\left(\frac{\lambda}{3}\right)\Gamma\left(\frac{\lambda+1}{3}\right),\\
&W_{42}'(0)=W_0^{-1}\frac{1}{(2\pi i)^2}3^{(2\lambda+2)/3-2}(\omega^2-1)(1-\omega^\lambda)(1-\omega^{\lambda+2})w^\lambda\Gamma\left(\frac{\lambda}{3}\right)\Gamma\left(\frac{\lambda+2}{3}\right),\\
&W_{42}''(0)=W_0^{-1}\frac{1}{(2\pi i)^2}3^{2\lambda/3-1}(\omega^2-\omega)(1-\omega^{\lambda+1})(1-\omega^{\lambda+2})\omega^\lambda\Gamma\left(\frac{\lambda+1}{3}\right)\Gamma\left(\frac{\lambda+2}{3}\right),\\
&W_{43}(0)=-W_0^{-1}\frac{1}{(2\pi i)^2}3^{(2\lambda+1)/3-2}(\omega-\omega^2)(1-\omega^{3\lambda})\Gamma\left(\frac{\lambda}{3}\right)\Gamma\left(\frac{\lambda+1}{3}\right),\\
&W_{43}'(0)=-W_0^{-1}\frac{1}{(2\pi i)^2}3^{(2\lambda+2)/3-2}(\omega^2-\omega)(1-\omega^{3\lambda})\Gamma\left(\frac{\lambda}{3}\right)\Gamma\left(\frac{\lambda+2}{3}\right),\\
&W_{43}''(0)=-W_0^{-1}\frac{1}{(2\pi i)^2}3^{2\lambda/3-1}(\omega-\omega^2)(1-\omega^{3\lambda})\Gamma\left(\frac{\lambda+1}{3}\right)\Gamma\left(\frac{\lambda+2}{3}\right),\\
&W_{44}(0)=W_0^{-1}\frac{1}{(2\pi i)^2}3^{(2\lambda+1)/3-2}(1-\omega)(1-\omega^{3\lambda})\omega^{-2\lambda}\Gamma\left(\frac{\lambda}{3}\right)\Gamma\left(\frac{\lambda+1}{3}\right),
\end{aligned} \tag{3.7}$$

$$W'_{44}(0)=W_0^{-1}\frac{1}{(2\pi i)^2}3^{(2\lambda+2)/3-2}(1-\omega^2)(1-\omega^{3\lambda})\omega^{-2\lambda}\Gamma\Big(\frac{\lambda}{3}\Big)\Gamma\Big(\frac{\lambda+2}{3}\Big),$$

$$W''_{44}(0)=W_0^{-1}\frac{1}{(2\pi i)^2}3^{2\lambda/3-1}(\omega-\omega^2)(1-\omega^{3\lambda})\omega^{-2\lambda}\Gamma\Big(\frac{\lambda+1}{3}\Big)\Gamma\Big(\frac{\lambda+2}{3}\Big).$$

Corresponding to (3.6) we take as the fundamental system of solutions of the equation (3.5) the system $[v_6(z), v_3(z), v_1(z)]$, where $v_j(z)$ $(j=6, 3, 1)$ are the solutions of (3.5) obtained by replacing λ with $(1-\lambda)$ in the definition of $y_j(z)$. By this choice, both of the systems have the same regions of validity of asymptotic expansions. The solutions of the equations (3.4) and (3.5) with initial conditions (3.7) are then expressed as follow;

$$\begin{aligned} &W_{41}(z)=(1-\lambda)^{-1},\\ &W_{42}(z)=2\pi i(1-\omega^{3\lambda})^{-1}\omega^{3\lambda/2}v_6(z),\\ &W_{43}(z)=2\pi i\,\omega^{3\lambda/2}v_1(z),\\ &W_{44}(z)=-2\pi i\,\omega^{-3\lambda/2}v_3(z).\end{aligned} \tag{3.8}$$

3.3. Suppose that the function $b(z)$ in the equation (3.1) has the form

$$b(z)=b_1(z)u_j(z)+b_2(z)u'_j(z)+b_3(z)u''_j(z)+b_4(z)u'''_j(z), \qquad (j=0, 1, 3, 6),$$

where $b_j(z)$ are polynomials of z and $u_0(z)=1$.

Then from (3.2), (3.3) and (3.8), the solution of (3.1) are expressed by linear combinations of the integrals of the following form of functions

$$b(s)u_j^{(i)}(s)v_k(s), \qquad \begin{pmatrix} i=0, 1, 2, 3 \\ j, k=0, 1, 3, 6 \end{pmatrix} \tag{3.9}$$

with coefficients $cu_j^{(i)}(z)$.

For applications, we want to obtain particular solutions whose asymptotic expansions at $z=\infty$ do not begin with constant terms. When $j=0$ or $k=0$, the problem is simple since if $k=0$ for example

$$\begin{aligned}\int_0^z b(s)u_j^{(i)}(s)ds&=\int_0^z b(s)\int_{C_j} t^{\lambda-2+i}e^{st-\frac{1}{3}t^3}dt\\ &=\int_{C_j}\Big\{\int_0^z b(s)e^{st}ds\Big\}t^{\lambda-2+i}e^{-\frac{1}{3}t^3}dt.\end{aligned}$$

If the integral of the inner bracket of the above expression is performed, then the above function of z is clearly expressed by the sum

$$\sum_l c_l z^l y_j(z, \lambda_k),$$

where c_l are constants and $y_j(z, \lambda_k)$ are functions defined in the section 2 with appropriate λ_k in place of λ. Thus in this case all of the global properties of this function become apparent.

3.4. Next, we consider the asymptotic behavior of an integral

$$P(z)=C+\int_0^z b(s)u_1(s)v_1(s)ds\,. \tag{3.10}$$

From the results of the section 2 the functions $u_1(s)$ and $v_1(s)$ have asymptotic expansions as s tends to infinity in the sector $-\pi<\arg s<\pi$

$$u_1(s)\sim\frac{e^{(\lambda-1)\pi i}}{2\sqrt{\pi}}s^{\frac{\lambda-1}{2}-\frac{3}{4}}\exp\left(-\frac{2}{3}s^{\frac{3}{2}}\right),$$

$$v_1(s)\sim\frac{e^{(1-\lambda)\pi i}}{2\sqrt{\pi}}s^{\frac{1-\lambda}{2}-\frac{3}{4}}\exp\left(-\frac{2}{3}s^{\frac{3}{2}}\right).$$

Then we have

$$b(s)u_1(s)v_1(s)\sim\frac{1}{4\pi}b(s)s^{-\frac{3}{2}}\exp\left(-\frac{4}{3}s^{\frac{3}{2}}\right). \tag{3.11}$$

Thus we can rewrite the integral (3.10) as

$$P(z)=C+\int_0^{\infty e^{i\alpha}}b(s)u_1(s)v_1(s)ds-\int_z^{\infty e^{i\alpha}}b(s)u_1(s)v_1(s)ds$$

with $-\pi/3<\alpha<\pi/3$, where the integral sign $\int_x^{\infty e^{i\alpha}}$ means that the integral is taken along a path starting x and tending to the infinity of the direction $e^{i\alpha}$. Clearly the first integral of the above expression exists from (3.11), and also the integration by parts gives

$$\begin{aligned}\int_z^\infty b(s)u_1(s)v_1(s)ds&\sim\int_z^\infty\frac{1}{4\pi}s^{-\frac{3}{2}}b(s)\exp\left(-\frac{4}{3}s^{\frac{3}{2}}\right)ds\\&=\int_z^\infty\frac{1}{4\pi}s^{-\frac{3}{2}}b(s)(-2s^{\frac{1}{2}})^{-1}(-2s^{\frac{1}{2}})\exp\left(-\frac{4}{3}s^{\frac{3}{2}}\right)ds\\&=\frac{1}{8\pi}z^{-2}b(z)\exp\left(-\frac{4}{3}z^{\frac{3}{2}}\right)\left\{1+O\left(\frac{1}{z}\right)\right\}.\end{aligned}$$

Here if we choose the constant C in (3.10) as

$$C=-\int_0^\infty b(s)u_1(s)v_1(s)ds\,,$$

we have

$$P(z)\sim-\frac{z^{-2}}{8\pi}b(z)\exp\left(-\frac{4}{3}z^{3/2}\right),\qquad -\pi<\arg z<\pi\,. \tag{3.12}$$

3.5. The asymptotic behavior of the function defined above when z tends to infinity on the negative real axis will be considered in this paragraph. From the paragraph 2.6, the following identities are valid:

$$\begin{aligned}u_1(s)&=\omega^{3\lambda}u_5(s)-\omega^{3\lambda}u_2(s)-u_3(s)\,,\\v_1(s)&=\omega^{-3\lambda}v_5(s)-\omega^{-3\lambda}v_2(s)-v_3(s)\,.\end{aligned} \tag{3.13}$$

We know the asymptotic expansions of the functions appeared on the right sides of the above identities on the range of the argument $\pi/3<\arg s<5\pi/3$. Then by using (3.13), $P(z)$ is written as

$$\begin{aligned}P(z)=C+\int_0^z b(s)\{&u_5(s)v_5(s)-u_5(s)v_2(s)-\omega^{3\lambda}u_5(s)v_3(s)\\&-u_2(s)v_5(s)+u_2(s)v_2(s)+\omega^{3\lambda}u_2(s)v_3(s)\\&-\omega^{-3\lambda}u_3(s)v_5(s)+\omega^{-3\lambda}u_3(s)v_2(s)+u_3(s)v_3(s)\}\,ds\,,\end{aligned}$$

and if we evaluate each integral by the same method as in the previous paragraph, then the asymptotic behavior of $P(z)$ for large absolute values of z, $\pi/3<\arg z<5\pi/3$, becomes

$$\begin{aligned}P(z)\sim&-\int_0^{\infty e^{i\alpha_1}}b(s)u_1(s)v_1(s)ds+\frac{(\cos 2\lambda\pi-1)\Gamma(\lambda-1)\Gamma(1-\lambda)}{2\pi^2}\int_0^z b(s)ds\\&-\int_0^{\infty e^{i\alpha_2}}b(s)u_5(s)v_2(s)ds+\frac{(1-e^{-2\lambda\pi i})\Gamma(\lambda-1)}{4\pi\sqrt{\pi}\,i}b(z)z^{-\frac{3}{2}\lambda+\frac{1}{4}}\exp\Big(-\frac{2}{3}z^{\frac{3}{2}}\Big)\\&-\int_0^{\infty e^{i\alpha_3}}\omega^{3\lambda}b(s)u_5(s)v_3(s)ds-\frac{e^{\lambda\pi i}(e^{2\lambda\pi i}-1)\Gamma(\lambda-1)}{4\pi\sqrt{\pi}}b(z)z^{-\frac{3}{2}\lambda+\frac{1}{4}}\exp\Big(\frac{2}{3}z^{\frac{3}{2}}\Big)\\&-\int_0^{\infty e^{i\alpha_2}}b(s)u_2(s)v_5(s)ds+\frac{(e^{-2\lambda\pi i}-1)\Gamma(1-\lambda)}{4\pi\sqrt{\pi}\,i}b(z)z^{\frac{3}{2}\lambda-\frac{11}{4}}\exp\Big(-\frac{2}{3}z^{\frac{3}{2}}\Big)\\&+\int_0^{\infty e^{i\alpha_2}}b(s)u_2(s)v_2(s)ds-\frac{1}{8\pi}z^{-2}b(z)\exp\Big(-\frac{4}{3}z^{\frac{3}{2}}\Big)\\&+\int_0^z\frac{e^{2\lambda\pi i}}{4\pi i}s^{-\frac{3}{2}}b_0(s)ds\\&-\int_0^{\infty e^{i\alpha_3}}\omega^{-3\lambda}b(s)u_3(s)v_5(s)ds-\frac{e^{-\lambda\pi i}(e^{-2\lambda\pi i}-1)\Gamma(1-\lambda)}{4\pi\sqrt{\pi}}b(z)z^{\frac{3}{2}\lambda-\frac{11}{4}}\exp\Big(\frac{2}{3}z^{\frac{3}{2}}\Big)\\&+\int_0^z\frac{e^{-2\lambda\pi i}}{4\pi i}s^{-\frac{3}{2}}b_0(s)ds\\&+\int_0^{\infty e^{i\alpha_3}}b(s)u_3(s)v_3(s)ds-\frac{1}{8\pi}z^{-2}b(z)\exp\Big(\frac{4}{3}z^{\frac{3}{2}}\Big)\,,\end{aligned}$$

where $b_0(s)$ is a polynomial whose terms come from $b(s)$ so that the integral $\int_0^z s^{-3/2}b_0(s)ds$ exists, and the quantities α_1, α_2, α_3 satisfy

$$-\frac{\pi}{3}<\alpha_1<\frac{\pi}{3}\,,\qquad \pi<\alpha_2<\frac{5}{3}\pi\,,\qquad \frac{\pi}{3}<\alpha_3<\pi\,.$$

Thus we have obtained the asymptotic expansion of $P(z)$ on the negative real axis:

$$P(z)\sim\frac{(\cos 2\lambda\pi-1)\Gamma(\lambda-1)\Gamma(1-\lambda)}{2\pi^2}\int_0^z b(s)ds$$

$$+\frac{1-e^{-2\lambda\pi i}}{4\pi\sqrt{\pi}\, i}\{\Gamma(\lambda-1)z^{-\frac{3}{2}\lambda+\frac{1}{4}}-\Gamma(1-\lambda)z^{\frac{3}{2}\lambda-\frac{11}{4}}\}b(z)\exp\left(-\frac{2}{3}z^{\frac{2}{3}}\right) \tag{3.14}$$

$$+\frac{1-e^{-2\lambda\pi i}}{4\pi\sqrt{\pi}}\{e^{-\lambda\pi i}\Gamma(1-\lambda)z^{\frac{3}{2}\lambda-\frac{11}{4}}-e^{3\lambda\pi i}\Gamma(\lambda-1)z^{-\frac{3}{2}\lambda+\frac{1}{4}}\}b(z)\exp\left(\frac{2}{3}z^{\frac{3}{2}}\right).$$

If we compare this asymptotic form with that of (3.12), we can understand the Stokes phenomenon of the function $P(z)$. We take up one more example from the terms in (3.9).

Let $Q(z)$ be a function of the form

$$Q(z)=C+\int_0^z b(s)u_6(s)v_6(s)ds\,.$$

By choosing a constant C appropriately, the asymptotic expansion of $Q(z)$ becomes

$$Q(z)\sim\frac{(\cos 2\lambda\pi-1)\Gamma(\lambda-1)\Gamma(1-\lambda)}{2\pi^2}\int_0^z b(s)ds$$

for large absolute values of z and $-\pi/3<\arg z<\pi$. The same procedure as for $P(z)$ gives us

$$Q(z)\sim\frac{(\cos 2\lambda\pi-1)\Gamma(\lambda-1)\Gamma(1-\lambda)}{2\pi^2}\int_0^z b(s)ds$$

$$+\frac{(\cos 2\lambda\pi-1)}{2\pi\sqrt{\pi}\, i}\{\Gamma(\lambda-1)z^{-\frac{3}{2}\lambda+\frac{1}{4}}+\Gamma(1-\lambda)z^{\frac{3}{2}\lambda-\frac{11}{4}}b(z)\exp\left(-\frac{2}{3}z^{\frac{3}{2}}\right)$$

$$+\frac{1-\cos 2\lambda\pi}{4\pi}b(z)z^{-2}\exp\left(-\frac{4}{3}z^{\frac{3}{2}}\right),$$

for $-\pi<\arg z\leqq-\pi/3$, and

$$Q(z)\sim\frac{(\cos 2\lambda\pi-1)\Gamma(\lambda-1)\Gamma(1-\lambda)}{2\pi^2}\int_0^z b(s)ds$$

$$+\frac{(1-\cos 2\lambda\pi)}{2\pi\sqrt{\pi}}\{e^{\lambda\pi i}\Gamma(\lambda-1)z^{-\frac{3}{2}\lambda+\frac{1}{4}}+e^{-\lambda\pi i}\Gamma(1-\lambda)z^{\frac{3}{2}\lambda-\frac{11}{4}}\}b(z)\exp\left(\frac{2}{3}z^{\frac{3}{2}}\right),$$

for $\arg z=\pi$.

The integral of the other terms in (3.9) can be analyzed analogously and from these the asymptotic behavior of the solution $v(z)$ of the nonhomogeneous differential equation can be obtained.

REFERENCES

[1] NISHIMOTO, T., On the Orr-Sommerfeld type equations, 1. Kodai Math. Sem. Rep., **24** (1970), 281-307.
[2] NISHIMOTO, T., A turning point problem of an n-th order differential equation of hydrodynamic type, Kodai Math. Sem. Rep., **20** (1968), 218-256.
[3] OLVER, F. W. J., Why steepest descent?. SIAM Review, **12** (1970), 228-247.

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