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A REMARK ON ANALYTIC MAPPINGS BETWEEN TWO ULTRAHYPERELLIPTIC SURFACES

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1. Let R and S be two ultrahyperelliptic surfaces defined by two equations $y^2=G(z)$ and $u^2=g(w)$, respectively, where G and g are two entire functions each of which has no zero other than an infinite number of simple zeros. Let $\mathfrak{P}_R$ and $\mathfrak{P}_S$ be the projection maps: $(z, y)\to z$ and $(w, u)\to w$, respectively. Let φ be a non-trivial analytic mapping of R into S. Then $h(z)=\mathfrak{P}_S\circ\varphi\circ\mathfrak{P}_R^{-1}(z)$ is a single-valued regular function of z in $|z|<\infty$ [8]. This entire function $h(z)$ is called the projection of the analytic mapping φ. We denote by $\mathfrak{A}(R, S)$ the family of non-trivial analytic mappings of R into S and by $\mathfrak{H}(R, S)$ the family of projections of analytic mappings belonging to $\mathfrak{A}(R, S)$. Let $\mathfrak{H}_P(R, S)$ be the subfamily of $\mathfrak{H}(R, S)$ consisting of polynomials and $\mathfrak{H}_T(R, S)$ the subfamily of $\mathfrak{H}(R, S)$ consisting of transcendental entire functions.

Let $P(R)$ and $P(S)$ be the Picard constants of R and S, respectively (cf. Ozawa [6]).

In [5] one of the authors has obtained

THEOREM A. *Let R and S be two ultrahyperelliptic surfaces with $P(R)=P(S)=4$. If $\mathfrak{H}(R, S)\neq\emptyset$, then $\mathfrak{H}(R, S)=\mathfrak{H}_P(R, S)$ or $\mathfrak{H}(R, S)=\mathfrak{H}_T(R, S)$. Further if $\mathfrak{H}_P(R, S)\neq\emptyset$, then $\mathfrak{H}_P(R, S)$ consists of polynomials of the same degree and the same modulus of the leading coefficients.*

In this paper we shall consider the structure of $\mathfrak{H}_T(R, S)$ and $\mathfrak{H}_P(R, S)$. Our result is the following:

THEOREM. *Let R and S be two ultrahyperelliptic surfaces with $P(R)=P(S)=4$. Then the followings hold.*

(I) *If $\mathfrak{H}_T(R, S)\neq\emptyset$, then $\mathfrak{H}_T(R, S)$ consists of transcendental entire functions of the same order, the same type and the same class.*

(II) *If $\mathfrak{H}_P(R, S)\neq\emptyset$ and $p(z)$ and $q(z)$ are elements of $\mathfrak{H}_P(R, S)$, then either* (i) *there exist a root of unity μ and a constant k such that $p(z)=\mu q(z)+k$ or* (ii) *there exist constants k, l and m such that $q(z)=r(z)^2+k$ and $p(z)=(r(z)+l)^2+m$, where $r(z)$ is a polynomial.*

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2. The notions of order, type and class of a meromorphic function are found in Hayman [3, pp. 16–18]. We shall say that the category of a meromorphic function $f(z)$ is larger than that of a meromorphic function $g(z)$ if the order of $f(z)$ is larger than that of $g(z)$ or if the orders are equal and non-zero finite and further the type of $f(z)$ is larger than that of $g(z)$ or if the both are of minimal type and further $f(z)$ is of divergence class and $g(z)$ is of convergence class.

In the first place we shall prove the following:

LEMMA. *Let $f(z)$ and $g(z)$ be two entire functions. If the category of $f(z)$ is larger than that of $g(z)$, then for any non-constant entire function $h(z)$*

$$\varliminf_{r\to\infty}\frac{T(r, h\circ g)}{T(r, h\circ f)}=0. \tag{2.1}$$

Proof. It follows from Pólya [8] (cf. [3, p. 51]) that

$$\frac{T(r, h\circ g)}{T(r, h\circ f)}\leqq\frac{3\log M(M(r, g), h)}{\log M(CM(r/4, f), h)},$$

where C is a positive constant. We know from Hadamard's three-circle theorem that $\log M(r, h)$ is an increasing convex function of $\log r$, so that $\log M(r, h)/\log r$ is finally increasing. Hence (2.1) follows from

$$\varliminf_{r\to\infty}\frac{\log M(r, g)}{\log M(r/4, f)}=0. \tag{2.2}$$

Now, we shall prove (2.2) when the category of $f(z)$ is larger than that of $g(z)$.

In the case that the order of $f(z)$ is larger than that of $g(z)$ (2.1) follows from Theorem 5 in Gross-Yang [2].

Suppose that the orders are equal to λ $(0<\lambda<+\infty)$ and $f(z)$ is of maximal type and $g(z)$ is of mean type or of minimal type, that is,

$$\varlimsup_{r\to\infty}\frac{\log M(r, f)}{r^{\lambda}}=\infty \quad\text{and}\quad \varlimsup_{r\to\infty}\frac{\log M(r, g)}{r^{\lambda}}<A \quad (A<+\infty).$$

Then for arbitrary large $K(>A)$, there is a sequence $\{r_n\}$ of positive, increasing and unbounded numbers such that

$$\frac{\log M(r_n/4, f)}{r_n^{\lambda}}>K \quad\text{and}\quad \frac{\log M(r_n, g)}{r_n^{\lambda}}<A$$

and so

$$\varliminf_{r\to\infty}\frac{\log M(r, g)}{\log M(r/4, f)}\leqq\varliminf_{r\to\infty}\frac{\log M(r_n, g)}{\log M(r_n/4, f)}\leqq\frac{A}{K}.$$

Hence we obtain (2.2) since K is arbitrary.

A similar argument shows that (2.2) is true when $f(z)$ is of mean type and

$g(z)$ is of minimal type.

Next suppose that $f(z)$ is of divergence class and $g(z)$ is of convergence class, that is,

$$\int_{r_0}^{\infty}\frac{\log M(r, f)}{r^{\lambda+1}}dr=\infty \quad \text{and} \quad \int_{r_0}^{\infty}\frac{\log M(r, g)}{r^{\lambda+1}}dr<+\infty .$$

Then we have

$$\varliminf_{r\to\infty}\frac{\log M(r, g)}{\log M(r/4, f)}\leqq\varliminf_{r\to\infty}\frac{\displaystyle\int_{r_0}^{r}\frac{\log M(s, g)}{s^{\lambda+1}}ds}{\displaystyle\int_{r_0}^{r}\frac{\log M(s/4, f)}{s^{\lambda+1}}ds}=0,$$

which gives (2.2).

Thus the proof of our Lemma is complete.

3. *Proof of Theorem.* Let R and S be two ultrahyperelliptic surfaces with $P(R)=P(S)=4$ defined by the equations $y^2=G(z)$ and $u^2=g(w)$, respectively. Then by a result in Ozawa [7], we get

$$F(z)^2G(z)=(e^{H(z)}-\alpha)(e^{H(z)}-\beta), \qquad \alpha\beta(\alpha-\beta)\neq 0, \quad H(0)=0,$$

where $F(z)$ is a suitable entire function and $H(z)$ is a non-constant entire function and

$$f(w)^2g(w)=(e^{L(w)}-\gamma)(e^{L(w)}-\delta), \qquad \gamma\delta(\gamma-\delta)\neq 0, \quad L(0)=0,$$

where $f(w)$ is a suitable entire function and $L(w)$ is a non-constant entire function.

In the first place we shall prove (I). Now suppose, to the contrary, that there are two entire functions $h_1(z)$ and $h_2(z)$ belonging to $\mathfrak{H}_T(R, S)$ and the category of $h_2(z)$ is larger than that of $h_1(z)$. Then it follows from our Lemma that

$$\varliminf_{r\to\infty}\frac{T(r, L\circ h_1)}{T(r, L\circ h_2)}=0. \tag{3.1}$$

On the other hand Hiromi-Ozawa [4] implies that for each $h_i(z)$ belonging to $\mathfrak{H}_T(R, S)$, one of two equations

$$H(z)=L\circ h_i(z)-L\circ h_i(0) \quad \text{and} \quad H(z)=-L\circ h_i(z)+L\circ h_i(0) \tag{3.2}$$

is valid. Hence we get

$$T(r, L\circ h_1)=T(r, L\circ h_2)+O(1),$$

which contradicts (3.1). Hence all entire functions belonging to $\mathfrak{H}_T(R, S)$ are of the same category. This completes the proof of (I).

Next we shall prove (II). By Theorem A, for certain μ with $|\mu|=1$, a, b_j, c_j, we have

$$q(z)=az^n+\sum_{j=0}^{n-1} b_j z^j, \tag{3.3}$$

$$p(z)=\mu a z^n+\sum_{j=0}^{n-1} c_j z^j. \tag{3.4}$$

The statement (II) of Theorem holds if $L(z)$ is a linear polynomial. So in the following lines we may assume that $L(z)$ is not a linear polynomial.

By (3.3) and (3.4), we get, about infinity,

$$\varphi=p(q^{-1}(z))=\mu z+\sum_{k=1}^{\infty} A_k z^{(n-k)/n}\equiv \mu z+S(z).$$

By (3.2) we get

$$L(p(z))=\varepsilon L(q(z))+C, \tag{3.5}$$

where C and ε are constants satisfying $\varepsilon=1$ or -1.

As in the proof of the Lemma 2 in [1] we have

$$\varphi=\mu z+e+\sum_{k=n+1}^{\infty} A_k z^{(n-k)/n} \tag{3.6}$$

or

$$\varphi=\mu z+\nu z^{1/2}+e+dz^{-1/2}+\sum_{k>3n/2} A_k z^{(n-k)/n}. \tag{3.7}$$

We shall prove (3.6) and (3.7). (3.6) and (3.7) are certainly true if $n\leqq 2$, so we assume that $n>2$.

Suppose that $A_{k'}$ is the first non-zero coefficient in $S(z)$ and that $k'<n$ and $k'\neq n/2$. Then we can see that for large R and any z with $|z|=R$, one of the determinations of $\varphi(z)$ satisfies $|\varphi(z)|\leqq R-\kappa R^{1/n}$, where κ $(0<\kappa<1)$ is a constant, which is independent of R, since $n>2$. We take a large R. Suppose that z_1 is a point where $|L(z)|$ takes its maximum on $|z|=R$. Let $z_2=\varphi(z_1)$ be a point such that $|z_2|\leqq R-\kappa R^{1/n}$. Then

$$\begin{aligned} M(R-\kappa R^{1/n}, L) &\geqq |L(z_2)|=|L(\varphi(z_1))| \\ &\geqq |L(z_1)|-|C|=M(R, L)-|C|. \end{aligned} \tag{3.8}$$

Hence $M(R, L)\leqq K_1R+K_2$, where K_1 and K_2 are constants. By this fact and (3.8) we can see that $L(z)$ must be a constant. It contradicts our assumption.

It remains to discuss the case when $n=2k$ is even, $A_k=\nu\neq 0$ and

$$\varphi=\mu z+\nu z^{1/2}+\sum_{j=k+1}^{\infty} A_j z^{(2k-j)/2k}. \tag{3.9}$$

We have to show that $A_j=0$ for $k<j<2k$ and $2k<j<3k$. Suppose that $\tilde{\alpha} z^{s/2k}$ is the first non-zero and non-constant term in the sum in the right-hand side of (3.9). Then as in [1, pp. 73-74], for large R and any z with $|z|=R$ there exists a point z_1 such that $\varphi(z_1)=\varphi(z)$ and $|z_1|\leqq R-\tilde{\kappa}R^{-(k-1)/2k}$, where $\tilde{\kappa}$ $(0<\tilde{\kappa}<1)$ is a constant, which is independent of R. By this fact

$$|L(z_1)|+|C|\geqq|L(\varphi(z_1))|=|L(\varphi(z))|\geqq|L(z)|-|C|\,.$$

Thus, as above, we have $M(R, L)\leqq\tilde{K}_1R^{1+(k-1)/2k}+\tilde{K}_2$, where $\tilde{K}_1$ and $\tilde{K}_2$ are constants. Hence $L(z)$ must be a linear polynomial. It contradicts our assumption.

If (3.6) holds, then we have the case (i) by (3.5), as in [1, p. 74]. If (3.7) holds, then we get

$$\varphi=\mu z+\nu z^{1/2}+e+dz^{-1/2}+\cdots,$$

where μ, ν, e and d are constants. Since $L(z)$ is not linear, there is an unbounded increasing sequence $\{R_n\}$ such that $M(R_n-1, L)+K\leqq M(R_n, L)$ for any constant K. Hence we have $\mu=1$ by (3.5), as in [1, pp. 72-73]. Put $p(z)-q(z)-e=\nu r(z)$. Then $r(z)$ is a polynomial and satisfies $r(q^{-1}(z))=z^{1/2}+(d/\nu)z^{-1/2}+\cdots$. Thus $r(q^{-1}(z))^2=z+2(d/\nu)+O(z^{-1/n})$. Hence $r(z)^2=q(z)+2(d/\nu)$. So we get $q(z)=r(z)^2+k$ and $p(z)=(r(z)+l)^2+m$, where k, l and m are constants. This completes the proof of (II).

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