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THE DIAGONAL DISTRIBUTION OF THE BIVARIATE POISSON DISTRIBUTION

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0. Summary.

In this paper we consider a limiting distribution of a diagonal distribution of a bivariate binomial distribution and its property.

1. The derivation of the diagonal distribution of a bivariate binomial distribution.

In this section we shall derive the diagonal distribution of a bivariate binomial distribution. We assume that the bivariate random variable (X, Y) has a bivariate binomial law as follows:

$$P(X=0, Y=0)=p_{00}, \qquad P(X=1, Y=0)=p_{10},$$
$$(1.1)$$
$$P(X=0, Y=1)=p_{01} \quad \text{and} \quad P(X=1, Y=1)=p_{11};$$

where we assumed p_{00}, p_{10}, p_{01} and p_{11} are non-negative values and $p_{00}+p_{10}+p_{01}+p_{11}=1$. We put Z as the difference value of the two values X, Y:

$$Z=X-Y. \tag{1.2}$$

Then the distribution of the value Z is given by

$$P(Z=-1)=p_{01}, P(Z=0)=p_{00}+p_{11} \quad \text{and} \quad P(Z=1)=p_{10}. \tag{1.3}$$

If we have n mutually independent values of the distribution

$$Z_1, Z_2, \cdots, Z_n$$

then the sum U of the n values

$$U=Z_1+Z_2+\cdots+Z_n \tag{1.4}$$

have the distribution law given in the following theorem.

THEOREM 1. *For given mutually independent n random variables Z_i $(i=1, 2, \cdots, n)$*

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having the distribution law (1.3) *the sum value* $U=Z_1+Z_2+\cdots+Z_n$ *have the distribution law*

$$P(U=m)=\sum_{k-l=m}\sum_{\delta=\max(k+l-n,0)}^{\min(k,l)}\frac{n!}{(n-(k+l)+\delta)!(k-\delta)!(l-\delta)!\delta!}$$

$$\cdot p_{00}{}^{n-(k+l)+\delta}p_{10}{}^{k-\delta}p_{01}{}^{l-\delta}p_{11}{}^{\delta} \tag{1.5}$$

$$(m=0,\ \pm1,\ \pm2,\cdots).$$

Proof. Given n mutually independent bivariate random variables

$$(X_1, Y_1), (X_2, Y_2), \cdots, (X_n, Y_n)$$

having the bivariate Bernoulli law (1.1). We put

$$Z_i=X_i-Y_i \qquad (i=1, 2, \cdots, n),$$

then Z_i $(i=1, 2, \cdots, n)$ are mutually independent and have the distribution law (1.3). The sum U defined in (1.4) is expressed by the difference of the sums $\sum_{i=1}^n X_i$ and $\sum_{i=1}^n Y_i$

$$U=\sum_{i=1}^n(X_i-Y_i)=\sum_{i=1}^n X_i-\sum_{i=1}^n Y_i. \tag{1.6}$$

So we can get the distribution of the sum U by the joint distribution of the sums $\sum_{i=1}^n X_i$ and $\sum_{i=1}^n Y_i$. That is the distribution of U is given by the sum of the joint probabilities $P(\sum_{i=1}^n X_i=k, \sum_{i=1}^n Y_i=l)$ as follows:

$$P(U=m)=\sum_{k-l=m}P\left(\sum_{i=1}^n X_i=k,\ \sum_{i=1}^n Y_i=l\right). \tag{1.7}$$

Along n independent variables $(X_1, Y_1), (X_2, Y_2), \cdots, (X_n, Y_n)$ we put α, β, γ and δ as the numbers of (0, 0), (1, 0), (0, 1) and (1, 1). Then we have

$$\alpha+\beta+\gamma+\delta=n.$$

For fixed k, l the event

$$\left(\sum_{i=1}^n X_i=k,\ \sum_{i=1}^n Y_i=l\right) \tag{1.8}$$

is expressed by the union of $n!/\alpha!\,\beta!\,\gamma!\,\delta!$ mutually exclusive events

$$(\overbrace{(0,0),\cdots,(0,0)}^{\alpha}, \overbrace{(1,0),\cdots,(1,0)}^{\beta}, \overbrace{(0,1),\cdots,(0,1)}^{\gamma}, \overbrace{(1,1),\cdots,(1,1)}^{\delta}), \cdots$$

where $\beta+\delta=k$ and $\gamma+\delta=l$. The probabilities of the $n!/\alpha!\,\beta!\,\gamma!\,\delta!$ events equals to the same

$$p_{00}{}^{\alpha}p_{10}{}^{\beta}p_{01}{}^{\gamma}p_{11}{}^{\delta}.$$

Then we have the probability of the event (1.8)

$$P\Big(\sum_{i=1}^{n} X_i=k,\ \sum_{i=1}^{n} Y_i=l\Big)=\sum_{\substack{\beta+\delta=k\\ \gamma+\delta=l\\ \alpha+\beta+\gamma+\delta=n}} \frac{n!}{\alpha!\,\beta!\,\gamma!\,\delta!}\, p_{00}{}^{\alpha}p_{10}{}^{\beta}p_{01}{}^{\gamma}p_{11}{}^{\delta}. \tag{1.9}$$

The sum bivariate random variable $(\sum_{i=1}^{n}X_i,\ \sum_{i=1}^{n}Y_i)$ has a bivariate distribution law

$$P\Big(\sum_{i=1}^{n} X_i=k,\ \sum_{i=1}^{n} Y_i=l\Big)=\sum_{\delta=\max(k+l-n,0)}^{\min(k,l)} \frac{n!}{(n-(k+l)+\delta)!\,(k-\delta)!\,(l-\delta)!\,\delta!} \cdot p_{00}{}^{n-(k+l)+\delta}p_{10}{}^{k-\delta}p_{01}{}^{l-\delta}p_{11}{}^{\delta}. \tag{1.10}$$

See Kawamura [1]. Therefore we have the distribution of U is gven by (1.7) and (1.10)

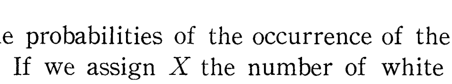
$$P(U=m)=\sum_{k-l=m}\ \sum_{\delta=\max(k+l-n,0)}^{\min(k,l)} \frac{n!}{(n-(k+l)+\delta)!\,(k-\delta)!\,(l-\delta)!\,\delta!} \cdot p_{00}{}^{n-(k+l)+\delta}p_{10}{}^{k-\delta}p_{01}{}^{l-\delta}p_{11}{}^{\delta} \qquad (m=0,\ \pm1,\ \pm2,\ \cdots)$$

as to be proved. ||

EXAMPLE (white ball and black ball model). We consider the experiment in which the next four events occur

a) neither white ball nor black ball exists,
b) one white ball and no black ball exists,
c) one black ball and no white ball exists,
d) both white and black ball exists

in a preassigned unit space. The given four events have the four patterns.



We shall give the probabilities of the occurrence of the four events p_{00}, p_{10}, p_{01} and p_{11} respectively. If we assign X the number of white ball 0 or 1 and Y the number of black ball 0 or 1 then the bivariate (X, Y) has the joint distribution (1.1).

The random variable $Z=X-Y$ is considered as the value of the difference of the two; non-negative gain X of the existence of white ball and non-negative gain Y of the existence of black ball.

pattern	▭	○	●	○ ●
(X, Y)	(0, 0)	(1, 0)	(0, 1)	(1, 1)
Z	0	1	-1	0
probability	p_{00}	p_{10}	p_{01}	p_{11}

If we have n mutually independent samples $(X_1, Y_1), (X_2, Y_2), \cdots, (X_n, Y_n)$ from the bivariate population then the sum vector

$$\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} Y_i\right)$$

has the distribution law (1.10). The random variable U defined in (1.6) becomes the difference of the numbers of the white balls and the black balls; the difference of the gains brought by the white balls and the black balls in the first n independent bivariate samples.

2. The limiting diagonal distribution.

We shall discuss the limiting property of the diagonal distribution of a bivariate binomial distribution given in the preceding section (1.5).

We assume for fixed non-negative λ_{10}, λ_{01} and λ_{11}, $np_{10}=\lambda_{10}$, $np_{01}=\lambda_{01}$ and $np_{11}=\lambda_{11}$ and n increases to infinitive then we have the limiting distribution of the distribution (1.5) of U as given in the following theorem.

THEOREM 2. *For fixed non-negative real values λ_{10}, λ_{01} and λ_{11} we assume the condition C: $np_{10}=\lambda_{10}$, $np_{01}=\lambda_{01}$ and $np_{11}=\lambda_{11}$. Then we have the fact that the distribution given in* (1.5) *converges as $n\to\infty$ to the distribution*

$$P(U=m)=\sum_{k-l=m}\sum_{\delta=0}^{\min(k,l)}\frac{\lambda_{10}{}^{k-\delta}\lambda_{01}{}^{l-\delta}\lambda_{11}{}^{\delta}}{(k-\delta)!\,(l-\delta)!\,\delta!}e^{-\lambda_{10}-\lambda_{01}-\lambda_{11}} \tag{2.1}$$

$$(m=0, \pm 1, \pm 2, \cdots).$$

Proof. Under the conditions $p_{10}=\lambda_{10}/n$, $p_{01}=\lambda_{01}/n$ and $p_{11}=\lambda_{11}/n$ the term of the right hand side of (1.5) becomes

$$\frac{n!}{(n-(k+l)+\delta)!\,(k-\delta)!\,(l-\delta)!\,\delta!}\left(1-\frac{\lambda_{10}+\lambda_{01}+\lambda_{11}}{n}\right)^{n-(k+l)+\delta} \tag{2.2}$$

$$\cdot\left(\frac{\lambda_{10}}{n}\right)^{k-\delta}\left(\frac{\lambda_{01}}{n}\right)^{l-\delta}\left(\frac{\lambda_{11}}{n}\right)^{\delta}.$$

The value of (2.2) converges to

$$\frac{\lambda_{10}{}^{k-\delta}\lambda_{01}{}^{l-\delta}\lambda_{11}{}^{\delta}}{(k-\delta)!\,(l-\delta)!\,\delta!}\,e^{-(\lambda_{10}-\lambda_{01}-\lambda_{11})}$$

as n increases to infinitive. See Kendall and Stuart [2].

Therefore we have the limiting distribution of (1. 5) under the conditions denoted by C as follows:

$$\lim_{\substack{n\to\infty\\ \mathrm{C}}}\sum_{k-l=m}\sum_{\delta=\max(k+l-n,0)}^{\min(k,l)}\frac{n!}{(n-(k+l)+\delta)!\,(k-\delta)!\,(l-\delta)!\,\delta!}\cdot p_{00}{}^{n-(k+l)+\delta}p_{10}{}^{k-\delta}p_{01}{}^{l-\delta}p_{11}{}^{\delta}$$

$$=\sum_{k-l=m}\sum_{\delta=0}^{\min(k,l)}\frac{\lambda_{10}{}^{k-\delta}\lambda_{01}{}^{l-\delta}\lambda_{11}{}^{\delta}}{(k-\delta)!\,(l-\delta)!\,\delta!}\,e^{-\lambda_1-\lambda_{01}-\lambda_{11}}\qquad(m=0,\ \pm1,\ \pm2,\cdots).\ ||$$

Moreover we can verify the limiting distribution (2. 1) to a rather simplified form as given in the following theorem.

THEOREM 3. *Under the condition assumed in the preceding theorem we have another form of the limiting distribution* (2. 1) *in a simplified form*:

$$P(U=m)=\sum_{\beta-\gamma=\delta}\frac{\lambda_{10}{}^{\beta}}{\beta!}\,\frac{\lambda_{01}{}^{\gamma}}{\gamma!}\,e^{-\lambda_{10}-\lambda_{01}}\qquad(m=0,\ \pm1,\ \pm2,\cdots).\tag{2. 3}$$

Proof.

$$\begin{aligned}P(U=m)&=\sum_{k-l=m}\sum_{\delta=0}^{\min(k,l)}\frac{\lambda_{10}{}^{k-\delta}\lambda_{01}{}^{l-\delta}\lambda_{11}{}^{\delta}}{(k-\delta)!\,(l-\delta)!\,\delta!}\,e^{-\lambda_{10}-\lambda_{01}-\lambda_{11}}\\&=\sum_{k-l=m}\sum_{\substack{\beta+\delta=k\\ \gamma+\delta=l}}\frac{\lambda_{10}{}^{\beta}\lambda_{01}{}^{\gamma}\lambda_{11}{}^{\delta}}{\beta!\,\gamma!\,\delta!}\,e^{-\lambda_{10}-\lambda_{01}-\lambda_{11}}\\&=\sum_{\beta-\gamma=m}\sum_{\delta=0}^{\infty}\frac{\lambda_{10}{}^{\beta}\lambda_{01}{}^{\gamma}\lambda_{11}{}^{\delta}}{\beta!\,\gamma!\,\delta!}\,e^{-\lambda_{10}-\lambda_{01}-\lambda_{11}}\\&=\sum_{\beta-\gamma=m}\frac{\lambda_{10}{}^{\beta}\lambda_{01}{}^{\gamma}}{\beta!\,\gamma!}\,e^{-\lambda_{10}-\lambda_{01}}.\ ||\end{aligned}\tag{2. 4}$$

It is stated in the theorem 2 that the distribution of U in (2. 1) becomes to the main diagonal distribution of a general bivariate Poisson distribution. See Kawamura [1]. It is also stated that the distribution (2. 1) becomes to the distribution of a independent type bivariate Poisson distribution. Therefore we have the next theorem.

THEOREM 4. *If a random vector* (X, Y) *is distributed by a bivariate general Poisson distribution then the diagonal distribution* (*the distribution of the difference* $U=X-Y$) *becomes the diagonal distribution of a bivariate independent type Poisson distribution*:

$$P(U=m)=\sum_{k-l=m}\frac{\lambda_{10}{}^{k}\lambda_{01}{}^{l}}{k!\,l!}e^{-\lambda_{10}-\lambda_{01}} \tag{2.5}$$

$$(m=0,\ \pm 1,\ \pm 2,\ \cdots).$$

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