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ON GALOIS THEORY OF CENTRAL SEPARABLE ALGEBRAS OVER ARTINIAN RINGS

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Let A be a separable algebra over the center C of A and B a subring of A. Let G be a finite group of automorphisms of A and A/B an *outer G-Galois extension* in the sense of Miyashita [6]. In [4], we had the following result: If C is a separable algebra over the center R of B, then C is a *G^*-Galois extension* of R and $G\cong G^*$, where G^* is the group of automorphisms of C induced by G.

In this note, we shall show the following result: If C is an *artinian* ring, then C is a *G^*-Galois extension* of R and $G\cong G^*$.

Let A' be a ring such that the center of A' is C and A' is projective as a C-module. Let T be a subring of A. Since A is a central separable algebra, A is projective as a C-module ([1], Th. 2.1). Hence we may regard T as a subring of $A\otimes_C A'$ by the natural ring monomorphism.

LEMMA 1. *If* $V_A(T)$[1]$=C$, *then* $V_{A\otimes_C A'}(T)=A'$.

Proof. Since A' is projective as a C-module, there exists a C-free module F such that A' is inbedded in F by a C-monomorphism $f: A'\to F$. We have the exact sequence

$$0\longrightarrow A\otimes_C A'\xrightarrow{f^*} A\otimes_C F,$$

where $f^*=1\otimes f$. We can regard $A\otimes_C A'$ as a two-sided A-module and $A\otimes_C F$, too. Then f^* is a two-sided A-module monomorphism. Since A is a separable algebra over C, C is a direct summand of A as a C-module ([1], Th. 2.1). Hence we have $A=C\oplus D$, where D is a C-submodule of A. Then,

$$A\otimes_C A'=A'\oplus D\otimes_C A',\qquad A\otimes_C F=F\oplus D\otimes_C F$$

and $f^{*-1}(F)=A'$. We take any element z of $V_{A\otimes_C A'}(T)$ and we set $x=f^*(z)$. Let $[y_i]_{i\in I}$ be a base of F, then $x=\sum_i x_i\otimes y_i$, where $x_i\in A$ and $x_i=0$ for almost all i. Since $tz-zt=0$ for all $t\in T$ and f^* is a two-sided A-module monomorphism, we have

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1) We denote by $V_A(T)$ the commutor of T in A.

$$\sum_i (tx_i - x_i t) \otimes y_i = 0$$

for all $t \in T$. Hence x_i belongs to C for all i. Hence x belongs to F. Since $z = f^{*-1}(x)$, z belongs to A'. Clearly $A' \subset V_{A \otimes_C A'}(T)$. Thus we have the lemma.

We denote by A^0 the opposite algebra of A.

COROLLARY 1. *If* $V_A(T) = C$, *then* $V_{A \otimes_C A^0}(T) = A^0$.

Proof. Since A^0 is a central separable algebra, A^0 is projective as a C-module. Hence we have the corollary by Lemma 1.

Let $\sum_{\sigma \in G} \oplus A u_\sigma$ be the *trivial crossed product* of A with G. We set $\Delta = \sum_{\sigma \in G} \oplus A u_\sigma$. The following results are well known:

1. We define the map

$$j_1\colon \quad A \otimes_C A^0 \longrightarrow \mathrm{Hom}_C(A, A)$$

by

$$j_1(a \otimes b^0)x = axb \text{ for } a, b, x \in A.$$

Then j_1 is a ring isomorphism ([1], Th. 2.1).

2. We regard A as a right B-module and we define the map

$$j_2\colon \quad \Delta \longrightarrow \mathrm{Hom}_B(A, A)$$

by

$$j_2(au_\sigma)x = a\sigma(x) \text{ for } a, x \in A \quad \text{and} \quad \sigma \in G.$$

Then j_2 is a ring isomorphism ([3], Th. 1).

LEMMA 2. *Let* σ *be a non-unit element of* G. *If* a *is an element of* A *such that* $a(c - \sigma(c)) = 0$ *for all* c *of* C, *then* $a = 0$.

Proof. We regard A as a right BC-module,[2] then $\mathrm{Hom}_{BC}(A, A) \cong V_{A \otimes_C A^0}(B^0 C)$. Since A/B is an outer G-Galois extension, $V_A(B) = C$. Hence,

$$\mathrm{Hom}_{BC}(A, A) \cong V_{A \otimes_C A^0}(B^0 C) = A$$

by Corollary 1. On the other hand,

$$\mathrm{Hom}_{BC}(A, A) \cong V_\Delta(C).$$

Thus $V_\Delta(C) = A$.

If $a(c - \sigma(c)) = 0$, for all c of C, then $c(au_\sigma) = (au_\sigma)c$ for all c of C. Hence $au_\sigma \in V_\Delta(C)$. Since $V_\Delta(C) = A$, $a = 0$.

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2) We denote by BC the subring of A generated by B and C.

COROLLARY 2 ([4], Cor. 4). *G is isomorphic to G^*.*

Proof. Let H be the kernel of the natural epimorphism $G \to G^*$. If we give any element σ of H, $c-\sigma(c)=0$ for all c of C. Hence $H=\{1\}$ by Lemma 2.

We denote by $R(C)$ the *Jacobson* radical of C. $R(C)$ is the intersection of all maximal ideal of the ring C. If C is an *artinian* ring, $R(C)$ is a nilpotent ideal.

THEOREM. *If C is an artinian ring, then C is G^*-Galois extension of R and $G \cong G^*$.*

Proof. Let σ be a non-unit element of G. We suppose that there exists a maximal ideal $\mathfrak{P}$ of C that contains the set $\{c-\sigma(c);\ c \in C\}$.

If $R(C)$ is the zero ideal, $C=\mathfrak{P} \oplus \mathfrak{P}'$, where $\mathfrak{P}'$ is a non-zero ideal of C. Hence $a\mathfrak{P}=0$ for a non-zero element a of $\mathfrak{P}'$. This is impossible by Lemma 2.

Next, we assume that $R(C)$ is a non-zero ideal. We set $\bar{C}=C/R(C)$ and set $\bar{\mathfrak{P}}=(\mathfrak{P}+R(C))/R(C)$. Then $\bar{C} \neq \bar{\mathfrak{P}}$, since $\mathfrak{P} \supset R(C)$. Hence, there exists a non-zero idempotent element e such that $e\mathfrak{P}$ is a nilpotent ideal with the index of nilpotency n. Since $(e\mathfrak{P})^n=\mathfrak{P}e(\mathfrak{P})^{n-1}=0$ and $e(\mathfrak{P})^{n-1} \neq 0$, $a\mathfrak{P}=0$ for a non-zero element a of $e(\mathfrak{P})^{n-1}$. This is impossible by Lemma 2.

Thus, given $\sigma(\neq 1) \in G$ and any maximal ideal $\mathfrak{P}$ of C, there exists an element c of C such that $c-\sigma(c) \notin \mathfrak{P}$. The theorem follows easily from Theorem 1.3 of [2].

From the result of Theorem, if C is an *artinian* ring, A/B is a Galois extension in the sense of Kanzaki [5]. Hence B is separable over R and $A=BC \cong B \otimes_R C$.

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