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CERTAIN COEFFICIENT INEQUALITIES FOR UNIVALENT FUNCTIONS

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1. Let S be the family of normalized regular functions $f(z)$ univalent in the unit circle $|z|<1$

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n.$$

In this paper we shall prove the following theorems.

THEOREM 1. *In* S

$$\left|a_5-2a_2a_4-\frac{3}{2}a_3^2+4a_2^2a_3-a_2^4\right|\leqq\frac{15}{2}.$$

Equality occurs only for $z/(1-e^{i\varepsilon}z)^2$, ε: *real.*

THEOREM 2. *In* S

$$\left|a_5-2a_2a_4-\frac{3}{2}a_3^2+4a_2^2a_3-\frac{23}{16}a_2^4\right|<\frac{1}{2}+\frac{3}{8}r_0+\frac{1}{4r_0}$$

where r_0 *is the root of* $2r^3-4r=1$ *satisfying* $1.5256<r_0<1.5257$.

Both theorems are proved by Bombieri's method [1] together with Schiffer's variational method. Theorem 2 does not assert its exactness. A similar inequality can be proved for

$$a_5-2a_2a_4-\frac{3}{2}a_3^2+4a_2^2a_3-Da_2^4,$$

when $1<D<23/16$. However it is not again best possible. For $D=79/54$ we have proved already

$$\left|a_5-2a_2a_4-\frac{3}{2}a_3^2+4a_2^2a_3-\frac{79}{54}a_2^4\right|\leqq\frac{1}{2},$$

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[9]. For $D=3/2$ Grunsky's inequality gives the result. For $1<D\leqq 23/16$ Koebe's function is not extremal.

THEOREM 3. *In S*

$$\left|a_5-2a_2a_4+\frac{1-\delta}{2}a_3^2+\delta a_2^2a_3+\frac{1-\delta}{2}a_2^4\right|\leqq\frac{1}{2}+e^{-2\delta/(4-\delta)}$$

for $0\leqq\delta\leqq 4$. *This is best possible.*

It is known as Garabedian-Schiffer's inequality [4] if $\delta=3$. For $\delta=0$ it is Górsky-Poole's inequality [6]. $\delta=4$ gives a Grunsky inequality. Jenkins [8] had given another proof in the case $\delta=3$ as a corollary of his coefficient theorem. Górsky-Poole stated a conjecture for general δ $(0<\delta<4)$. However Jenkins' theorem includes the result, although he did not state it explicitly. We state it explicitly in Theorem 3.

THEOREM 4. *In S*

$$\left|a_5-4a_2a_4-\frac{3}{2}a_3^2+9a_2^2a_3-\beta a_2^4\right|$$

$$\leqq\begin{cases}-(16\beta+67.5), & \beta\leqq-\dfrac{35}{8},\\[2mm] \dfrac{1}{6}(R^2+4R+3), & -\dfrac{35}{8}<\beta<\beta_*,\\[2mm] 16\beta+67.5, & \beta_*\leqq\beta,\end{cases}$$

where $3(9+2\beta)R^2=R+1$, $R>0$ *and* β_* *is defined by*

$$3(9+2\beta_*)R_*^2=R_*+1,$$

$$R_*^4+4R_*^3+30R_*^2-16R_*-16=0, \qquad 0.95<R_*<0.96.$$

These estimations are sharp. When $\beta\leqq-35/8$ *or* $\beta>\beta_*$, *equality occurs only for* $z/(1+e^{i\varepsilon}z)^2$, ε: *real. If* $\beta=\beta_*$, *an extremal function is* $z/(1+e^{i\varepsilon}z)^2$, ε: *real.*

A special case $|a_5-4a_2a_4-3a_3^2/2+9a_2^2a_3-9a_2^4/2|\leqq 9/2$ is an extension of $|a_4-3a_2a_3+2a_2^3|\leqq 2$. There are formal analogies between the corresponding Schiffer's differential equations. The method of proof of Theorem 4 has a wide range of applicability, although we do not know its general theory at all. We shall give examples for which the method of proof of Theorem 4 is applicable.

THEOREM 5. *For* $\delta\leqq-1$

$$|a_5-4a_2a_4+\delta a_3^2+8a_2^2a_3-4a_2^4|\leqq-5-9\delta.$$

Equality occurs only for $z/(1-e^{i\varepsilon}z)^2$, ε: *real.*

THEOREM 6.

$$|a_4-3a_2a_3+Ba_2^3|$$

$$\leqq\begin{cases} 8B-14 & \textit{for} \quad B\geqq\dfrac{23}{12}, \\[2ex] \dfrac{2}{3}+\dfrac{1}{18(2-B)} & \textit{for} \quad \dfrac{22-\sqrt{5}}{12}\leqq B<\dfrac{23}{12}, \\[2ex] 14-8B & \textit{for} \quad B<\dfrac{22-\sqrt{5}}{12}. \end{cases}$$

These estimations are best possible. Equality occurs only for $z/(1-e^{i\varepsilon}z)^2$, ε: *real when* $B\geqq 23/12$ *and* $B<(22-\sqrt{5})/12$. *For* $B=(22-\sqrt{5})/12$ *there is an extremal function other than* $z/(1-e^{i\varepsilon}z)^2$, ε: *real.*

We cannot give the extremal functions explicitly for $(22-\sqrt{5})/12\leqq B<23/12$, which involve a hyperelliptic integral and an elliptic integral with unknown coefficients. For $B=2$ we already proved it in [9].

THEOREM 7.

$$\Re\left\{a_4-3a_2a_3+Ba_2^3+2\beta\left(a_3-\frac{3}{4}a_2^2\right)+\beta^2a_2\right\}$$

$$\leqq -14+8B+2\beta^2$$

if $B\geqq(23-\beta)/12$ *and* $\beta\geqq 3$. *Equality occurs only for* $z/(1-z)^2$.

For $\beta<3$ we do not have any effective method excepting the use of Grunsky's inequality. We do not enter into this method.

THEOREM 8. *If* $\Re a_2\geqq 1.8$, *then*

$$\Re\left(a_5-a_2a_4-\frac{3}{2}a_3^2+2a_2^2a_3-\frac{5}{16}a_2^4\right)\leqq\frac{5}{2}.$$

Equality occurs only for $z/(1-z)^2$.

This inequality can be proved starting from the Garabedian inequality [3]. In view of the method employed here we can prove several other inequalities involving some parameters.

By the way we shall give a proof of

$$|a_4-2a_2a_3+a_2^3|\leqq\frac{2}{3},$$

which had been proved by Schiffer [10] and Golusin [5]. Jenkins [8] also had given another proof of it by his method. Recently Duren [2] has given its elementary proof.

2. Proof of Theorem 1. Recently Bombieri [1] has considered the following problem: Let $Q(\zeta)d\zeta^2$ be a quadratic differential on the ζ-sphere. Let be given a good subset T_0 of the set $\bar{T}$ of critical trajectories of $Qd\zeta^2$, a continuously differentiable Jordan arc J on the ζ-sphere. Under which conditions on J can we assert that $J\cap T_0$ is either empty or a single point? His answer is given in his Theorem 1 and its corollary and its Remark. We make use of his method. We consider the extremal problem

$$\max_{S} \Re F,$$

$$F=a_5-2a_2a_4-\frac{3}{2}a_3^2+4a_2^2a_3-a_2^4.$$

In this problem we may assume that $|\arg a_2|\leqq\pi/4$. By Schiffer's variational method the image of $|z|=1$ by any extremal function w satisfies

$$\left(\frac{dw}{dt}\right)^2\frac{1}{w^6}(2a_2w+1)(a_2^2w^2+1)+1=0$$

with a suitable parameter t. This implies

(1)
$$z^2\frac{w'^2}{w^6}(2a_2w+1)(a_2^2w^2+1)=\frac{1}{z^4}(1+Rz^3+4Fz^4+\bar{R}z^5+z^8),$$
$$R=2a_4-4a_2a_3+4a_2^3.$$

Let $Q^*(w)dw^2$ be the associated quadratic differential

$$-\frac{dw^2}{w^6}(2a_2w+1)(a_2^2w^2+1).$$

Let $Q(\zeta)d\zeta^2$ be $Q^*(1/\zeta)d(1/\zeta)^2$. Then

$$Q(\zeta)d\zeta^2=-(2a_2^3+a_2^2\zeta+2a_2\zeta^2+\zeta^3)\frac{d\zeta^2}{\zeta}.$$

Assume that $\Im a_2\neq 0$. We put $a_n=x_n+iy_n$. Let ζ be real. Then

$$\Im Q(\zeta)d\zeta^2=-(2(3x_2^2y_2-y_2^3)+2x_2y_2\zeta+2y_2\zeta^2)\frac{d\zeta^2}{\zeta}.$$

In our case the right hand side does not vanish, since $y_2^2\leqq x_2^2$. Now we can apply Bombieri's Theorem 1, Corollary. Then we have that the image Γ of $|z|=1$ by ζ and the real axis intersect only at the origin. Further $\zeta=0$ is a simple pole of $Q(\zeta)d\zeta^2$, since $a_2\neq 0$. Hence $\zeta=0$ is an end point of Γ. Hence Γ should lie in either the upper half-plane or the lower half-plane. Further Γ has the tangent vector with the argument $-\arg(-a_2^3)$ at the origin. Since $-\Im(-a_2^3)\Im a_2>0$, Γ lies in the same half-plane as a_2 does. However it is known that

$$\frac{1}{2\pi}\int_0^{2\pi}\zeta(e^{i\theta})d\theta=-a_2.$$

Hence the mean of Γ lies in the opposite half-plane as a_2 does. This contradiction gives that $y_2=0$ for extremal functions. Thus we may seek for the extremal functions among univalent functions with real a_2. Further then the critical trajectories of $Q(\zeta)d\zeta^2$ are symmetric with respect to the real axis.

Assume that $a_2=0$. Then by our earlier result in [9]

$$\left|a_5-2a_2a_4-\frac{3}{2}a_3^2+4a_2^2a_3-\frac{79}{54}a_2^4\right|\leqq\frac{1}{2}$$

or

$$\left|a_5-2a_2a_4-\frac{3}{2}a_3^2+4a_2^2a_3-\frac{3}{2}a_2^4\right|\leqq\frac{1}{2}$$

we have

$$|F|\leqq\frac{1}{2}<\frac{15}{2}.$$

Hence we may omit the case $a_2=0$.

Thus $a_2\neq 0$, which implies that the origin is a simple pole of $Q(\zeta)d\zeta^2$. Therefore Γ starts from the origin along the negative real axis. Since Γ has the mapping radius 1, Γ should meet to $-2a_2$, which is a simple zero of $Q(\zeta)d\zeta^2$. Then Γ should fork at $-2a_2$ into two curves, whose tangent vectors at $-2a_2$ have the arguments $2\pi/3$ and $4\pi/3$. If $a_2=2$, then Γ stops at -4. In every case we have the second representation of (1)

$$z^2\frac{w'^2}{w^6}(2a_2w+1)(a_2^2w^2+1)=\frac{1}{z^4}(z^2+Pz+Q)^2(z^4+Uz^3+Vz^2+Wz+M). \tag{2}$$

Originally the right hand side of (2) or of (1) satisfies

$$\overline{g\left(\frac{1}{\bar{z}}\right)}=g(z).$$

Hence $|Q|=|M|=1$, $Q\bar{P}=P$, $U=\bar{W}M$, $V=\bar{V}M$, $MQ^2=1$ and

$$2PQM+Q^2W=0, \tag{3}$$

$$(P^2+2Q)M+2PQW+Q^2V=0, \tag{4}$$

$$2PM+(P^2+2Q)W+2PQV+Q^2U=R, \tag{5}$$

$$M+2PW+(P^2+2Q)V+2PQU+Q^2=4F. \tag{6}$$

Let $Q=e^{i\phi}$, $M=e^{-2i\phi}$, $P=re^{i\alpha}$, $W=se^{i\beta}$, $V=te^{i\gamma}$. Then $e^{2i\alpha}=e^{i\phi}$, $e^{-2i\gamma}=e^{2i\phi}$. By (3)

$$s+2r\cos(\alpha-3\phi-\beta)=0,$$

$$r\sin(\alpha-3\phi-\beta)=0.$$

Assume $r=0$. Then $s=0$. By (4)

$$2e^{-i\phi}+te^{2i\phi+i\gamma}=0.$$

Hence $t=2$, $\cos(3\phi+\gamma)=-1$. Since $U=\overline{W}M$, $U=0$ in this case. By (6)

$$4F=e^{-2i\phi}+2te^{i\phi+i\gamma}+e^{2i\phi}$$

$$=2\cos 2\phi+2te^{i\phi+i\gamma}.$$

Hence

$$4F\leqq 2.$$

For the Koebe function $z/(1-z)^2$ $4F=30$. Hence we may omit this case. Next we assume that $r\neq 0$. Then $\sin(3\phi+\beta-\alpha)=0$. This implies that $s=2r$, $\cos(\alpha-3\phi-\beta)=-1$. Hence we have $e^{i\beta}=-e^{(\alpha-3\phi)i}=-e^{-5i\alpha}$. We now divide into two cases: i) $e^{i\gamma}=e^{-i\phi}$, ii) $e^{i\gamma}=-e^{-i\phi}$.

Case i). In this case we have

$$Q=e^{2i\alpha},\qquad P=re^{i\alpha},\qquad M=e^{-4i\alpha},\qquad W=-2re^{-5i\alpha},$$

$$V=te^{-2i\alpha},\qquad U=\overline{W}M=se^{-i\beta-4i\alpha}=-2re^{i\alpha}.$$

By (4) we have

$$(-3r^2+2)e^{-2i\alpha}+te^{2i\alpha}=0,$$

which implies

$$(-3r^2+2+t)\cos 2\alpha=0,$$

$$(-3r^2+2-t)\sin 2\alpha=0.$$

If $\cos 2\alpha=0$, then $t=2-3r^2$. By (6) we have

$$4F=2\cos 4\alpha-8r^2\cos 4\alpha+r^2t+2t$$

$$=2+4r^2-3r^4.$$

This shows that

$$4F\leqq\frac{10}{3}=\max_{0\leqq r}(2+4r^2-3r^4).$$

Here max is attained at $r^2=2/3$. Again $4F\leqq 10/3<30$. Hence we may omit this case. If $\sin 2\alpha=0$, then $t=3r^2-2$. In this case $4F=-2-4r^2+3r^4$. Since $P=-z_1-z_2$, $Q=z_1z_2$ with $|z_1|=|z_2|=1$, we have $r\leqq 2$. Here equality occurs only for $z_1=z_2$. In this case

$$4F\leqq 30=\max_{0\leqq r\leqq 2}(3r^4-4r^2-2).$$

Equality occurs only for $r=2$. Hence $z_1=z_2$. Returning to the image Γ of $|z|=1$ by $\zeta=1/w$, Γ has only two end points and Γ contains $-2a_2$. If $-2a_2$ is not an end point of Γ, two end points other than the origin, one of which may degenerate to $-2a_2$, appear. In any case $z_1\neq z_2$, which is a contradiction. Hence Γ reduces to a single segment $[-2a_2, 0]$, which must have the mapping radius 1. Hence $-2a_2=-4$. This implies that equality occurs only for the Koebe function $z/(1-z)^2$. If $t=3r^2-2$ and $t=2-3r^2$, then $t=0$, $r^2=2/3$. By (6)

$$4F=(2-8r^2)\cos 4\alpha\leqq\frac{10}{3}.$$

Hence this may be omitted.

Case ii). $e^{i\gamma}=-e^{-i\varphi}$. Similarly by (4) we have

$$(-3r^2+2-t)\cos 2\alpha=0,$$

$$(-3r^2+2+t)\sin 2\alpha=0.$$

When $\cos 2\alpha=0$, then $t=3r^2-2$ and $4F\leqq 10/3$. When $\sin 2\alpha=0$, then $t=2-3r^2$ and $4F\leqq 30$. Here equality occurs only for $z/(1-z)^2$. When $t=3r^2-2$ and $t=2-3r^2$, then $4F\leqq 10/3$, which may be omitted. Therefore the proof of Theorem 1 has been completed.

3. Proof of Theorem 2. We consider the extremal problem

$$\max_S \Re F,$$

$$F=a_5-2a_2a_4-\frac{3}{2}a_3^2+4a_2^2a_3-\frac{23}{16}a_2^4.$$

By Schiffer's variational method every extremal function satisfies

$$z^2\frac{w'^2}{w^6}\left(\frac{1}{4}a_2^3w^3+a_2^2w^2+2a_2w+1\right)=\frac{1}{z^4}(1+2Rz^3+4Fz^4+2\bar{R}z^5+z^8),$$

$$R=a_4-2a_2a_3+\frac{9}{8}a_2^3.$$

Let $Q(\zeta)d\zeta^2$ be

$$-\frac{d\zeta^2}{\zeta}\left(\frac{1}{4}a_2^3+a_2^2\zeta+2a_2\zeta^2+\zeta^3\right).$$

Here we may assume that $x_2^2 \geqq y_2^2$, $a_2 = x_2 + iy_2$. In this case we can make use of Bombieri's theorem 1, Corollary and its Remark, assuming $y_2 \neq 0$. ζ real. Then similarly we have a contradiction. Hence a_2 must be real non-negative. If $a_2 = 0$, then we can again use our earlier result as in Theorem 1. We, then, have $\Re F \leqq 1/2$, which may be omitted, if we can show the existence of functions in S satisfying $\Re F > 1/2$. If $a_2 > 0$, then the image Γ of $|z| = 1$ by $\zeta = 1/w$ starts from the origin along the negative real axis. Then Γ should meet to a simple zero $-x_0$ of $Q(\zeta)d\zeta^2$, since $x_0 < 3a_2/2$. And then Γ forks into two curves at $-x_0$. Further there is only one zero of $a_2^3/4 + a_2^2\zeta + 2a_2\zeta^2 + \zeta^3$ in the open interval $(-4, 0)$. Hence we have the second expression

$$z^2 \frac{w'^2}{w^6}\left(\frac{1}{4} a_2^3 w^3 + a_2^2 w^2 + 2a_2 w + 1\right)$$

$$= \frac{1}{z^4}(z^2 + Pz + Q)^2 (z^4 + Uz^3 + Vz^2 + Wz + M),$$

$$|Q| = |M| = 1,\ Q\bar{P} = P,\ U = \bar{W}M,\ V = \bar{V}M,\ MQ^2 = 1.$$

Hence we have (3), (4), (5), (6) in the proof of Theorem 1. Now we can use the process in the proof of Theorem 1. In what follows we shall make use of the same notations as in the proof of Theorem 1.

Firstly the case $r = 0$ appears. Then $t = 2$ and $s = 0$, $e^{3i\phi + i\tau} = -1$. Hence

$$4F = 2\cos 2\phi + 2te^{i\phi + i\tau}$$

$$= 2\cos 2\phi - 2t\cos 2\phi = -2\cos 2\phi \leqq 2.$$

Thus $F \leqq 1/2$.

Secondly we have the case $r \neq 0$, $\cos(\alpha - 3\phi - \beta) = -1$, $s = 2r$ and $r = \phi + 2p\pi$. If $\cos 2\alpha = 0$, then $t = 2 - 3r^2$. By (6) we have $4F = 2 + 4r^2 - 3r^4$. In this case by (5)

$$2Re^{-i\alpha} = 8r - 4r^3.$$

By Grunsky's inequality

$$\left|a_4 - 2a_2a_3 + \frac{13}{12}a_2^3\right| \leqq \frac{2}{3}.$$

Hence

$$|R| = \left|a_4 - 2a_2a_3 + \frac{9}{8}a_2^3\right|$$

$$\leqq \left|a_4 - 2a_2a_3 + \frac{13}{12}a_2^3\right| + \frac{1}{24}|a_2^3| \leqq 1.$$

Here equality occurs only for $z/(1 - e^{i\phi}z)^2$. By (5)

$$|8r-4r^3|=2|Re^{-i\alpha}|\leqq 2.$$

This gives two admissible intervals of r

$$[0, r_2],\ [r_1, r_0],$$

where r_2 and r_1 are two roots of $2r(2-r^2)=1$ satisfying $0.258<r_2<0.259$, $1.26<r_1<1.27$ and r_0 has been defined already. Since $4F=2+4x-3x^2$, $x=r^2$ is symmetric with respect to $x=2/3$ and is monotone increasing for $x<2/3$, $4F$ takes the maximum value at $x=r_2^2$ by $r_1^2+r_2^2>1.26^2+0.256^2>4/3$ and its value is $2+4r_2^2-3r_2^4$.

If $\sin 2\alpha=0$, then $t=3r^2-2$ and $4F=-2-4r^2+3r^4$. By (5) we have $2Re^{-i\alpha}=4r^3-8r$. Hence $|2r^3-4r|\leqq 1$. In this case again we have two admissible intervals $[0, r_2]$, $[r_1, r_0]$. However $4F=-2-4x+3x^2$, $x=r^2$ is negative for $[0, 4/3]$ and is monotone increasing for $x>2/3$. Therefore $4F\leqq -2-4r_0^2+3r_0^4$. Since $2r_0^3-4r_0=1$, we have

$$4F\leqq 2+\frac{3}{2}r_0+\frac{1}{r_0}.$$

If $t=2-3r^2$ and $t=3r^2-2$, then $t=0$, $r^2=2/3$. Since $|R|\leqq 1$, $|8r-4r^3|\leqq 2$. Hence $r^2=2/3$ should be excluded.

The above results hold in both cases i) and ii). Now we shall compare the results. Since

$$2+\frac{3}{2}r_0+\frac{1}{r_0}>2+4r_2^2-3r_2^4>2,$$

we have

$$F\leqq\frac{1}{2}+\frac{3}{8}r_0+\frac{1}{4r_0}.$$

Assume that equality occurs. Then $R=1$ and $2r^3-4r=1$, which leads us to the Koebe function $z/(1-e^{i\varepsilon}z)^2$, ε: real. Then $|F|=1/2$, which is a contradiction.

If $a_2>0$, then Γ should fork at $-x_0$, $x_0<3a_2/2\leqq 3$. Hence the extremal function in this case does not coincide with the Koebe function. Moreover we can say that $\max\Re F>1/2$. If not, then the Koebe function should satisfy the corresponding Schiffer's differential equation, which has been just excluded.

Therefore the proof of Theorem 2 has been completed.

Since the same reasoning as in the above goes through for

$$a_5-2a_2a_4-\frac{3}{2}a_3^2+4a_2^2a_3-Da_2^4$$

with $1<D<23/16$, the Koebe function is not an extremal function.

4. Proof of Theorem 3. Let Σ be the family of functions univalent for $|z|>1$, regular apart from a simple pole at the point at infinity and having expansion at that point

$$g(z)=z+c_0+\sum_{n=1}^{\infty}\frac{c_n}{z^n}.$$

Jenkins [8] proved the following result:

Let g belong to Σ. Then for ϕ real and $0\leqq\sigma\leqq 2$

$$\Re\left\{e^{-4i\phi}\left(c_3+\frac{1}{2}c_1^2-\sigma^2e^{2i\phi}c_1\right)\right\}\geqq-\frac{1}{2}-\frac{3}{16}\sigma^4+\frac{1}{8}\sigma^4\log\frac{\sigma^2}{4}.$$

This is best possible.

Jenkins, using this result, proved Garabedian-Schiffer's inequality $|c_3|\leqq 1/2+e^{-6}$. Evidently we have

$$c_0=-a_2,\ c_1=a_2^2-a_3,\ c_2=-a_4+2a_2a_3-a_2^3,$$

$$c_3=-a_5+2a_2a_4+a_3^2-3a_2^2a_3+a_2^4.$$

Jenkins' method does work for

$$c_3+\frac{c}{2}c_1^2, \qquad -3\leqq c\leqq 1.$$

Firstly we put

$$\Re\left\{e^{-4i\phi}\left(c_3+\frac{c}{2}c_1^2\right)\right\}=-\left|c_3+\frac{c}{2}c_1^2\right|.$$

Then

$$\left|c_3+\frac{c}{2}c_1^2\right|\leqq\frac{1}{2}+\frac{3}{16}\sigma^4-\frac{\sigma^4}{8}-\log\frac{\sigma^2}{4}+\frac{1-c}{2}\Re(e^{-4i\phi}c_1^2)-\sigma^2\Re(e^{-2i\phi}c_1)$$

$$\leqq\frac{1}{2}+\frac{3}{16}\sigma^4-\frac{\sigma^4}{8}\log\frac{\sigma^2}{4}+\frac{1-c}{2}(\Re e^{-2i\phi}c_1)^2-\sigma^2\Re(e^{-2i\phi}c_1).$$

We may assume that $\Re(e^{-2i\phi}c_1)>0$. Then there is a σ in $(0, 2]$ satisfying

$$\Re(e^{-2i\phi}c_1)=\frac{1}{4}\sigma^2\left(1-\log\frac{\sigma^2}{4}\right).$$

If $\Re(e^{-2i\phi}c_1)=0$, then we put $\sigma=0$. Using this σ,

$$\left|c_3+\frac{c}{2}c_1^2\right|$$

$$\leqq\frac{1}{2}+\frac{3}{16}\sigma^4-\frac{\sigma^4}{8}\log\frac{\sigma^2}{4}+\frac{1-c}{2}\frac{\sigma^4}{16}\left(1-\log\frac{\sigma^2}{4}\right)^2-\frac{\sigma^4}{4}\left(1-\log\frac{\sigma^2}{4}\right)$$

$$=\frac{1}{2}-\frac{1+c}{32}\sigma^4+\frac{1+c}{16}\sigma^4\log\frac{\sigma^2}{4}+\frac{1-c}{32}\sigma^4\left(\log\frac{\sigma^2}{4}\right)\equiv\Theta(\sigma, c).$$

$\Theta(\sigma, c)$ is increasing for $0<\sigma<\sigma_0$ and decreasing for $\sigma_0<\sigma$, where $\sigma_0 = 2\exp(-(3+c)/(2-2c))$. Let $\Theta(\sigma_0, c)$ be max $\Theta(\sigma, c)$. Then with

$$\Theta(\sigma_0, c)=\frac{1}{2}+e^{-2(3+c)/(1-c)}$$

$$\left|c_3+\frac{c}{2}c_1^2\right|\leqq\Theta(\sigma_0, c).$$

Let δ be $3+c$ and c_3, c_1 be represented by a_2, a_3, a_4, a_5, then we have the desired result. The equality statement is similar as in Jenkins'.

5. Proof of Theorem 4. Let us consider the extremal problem

$$\max_S \Re T,$$

$$T=a_5-4a_2a_4-\frac{3}{2}a_3^2+9a_2^2a_3+\tilde{\beta}a_2^4.$$

Then any extremal functions satisfy the differential equation

$$z^2\frac{w'^2}{w^6}[\{-2a_4+6a_2a_3+(14+4\tilde{\beta})a_2^3\}w^3+1]$$

$$=\frac{1}{z^4}\{1-2a_2z+a_2^2z^2+(18+4\tilde{\beta})a_2^3z^3+4Tz^4$$

$$+(18+4\bar{\beta})\bar{a}_2{}^3z^5+\bar{a}_2{}^2z^6-2\bar{a}_2z^7+z^8\}.$$

We have the second expression of the right hand side

$$\frac{1}{z^4}(z^2+2Pz+P^2)(z^6+Az^5+Bz^4+Cz^3+Dz^2+Ez+F),$$

$$|P|=|F|=1,\ \bar{A}=\bar{F}E,\ \bar{B}=\bar{F}D,\ \bar{C}=\bar{F}C.$$

Hence we have

$$P^2F=1,$$

$$P^2E+2PF=-2a_2,$$

$$P^2D+2PE+F=a_2^2,$$

$$P^2C+2PD+E=(18+4\tilde{\beta})a_2^3,$$

$$P^2B+2PC+D=4T.$$

Let $P=e^{i\theta}$, $E=te^{i\tau}$, $D=se^{i\beta}$, $C=re^{i\alpha}$. Then $e^{-2i\alpha}=e^{2i\theta}$ and

$$te^{i\gamma-i\theta}+2e^{-4i\theta}=-2a_2e^{-3i\theta},$$

$$se^{i\beta}+2te^{i\gamma-i\theta}+e^{-4i\theta}=a_2^2e^{-2i\theta},$$

$$re^{i\theta+i\alpha}+2se^{i\beta}+te^{i\gamma-i\theta}=(18+4\tilde{\beta})a_2^3e^{-i\theta},$$

$$4T=2s\cos\beta+2re^{i\theta+i\alpha}.$$

Hence

$$4T=\Re[4(9+2\tilde{\beta})a_2^3e^{-i\theta}-2a_2^2e^{-2i\theta}-4a_2e^{-3i\theta}-2e^{-4i\theta}].$$

First we consider the case $9+2\tilde{\beta}\leqq 0$. Then

$$4T\leqq -2(18+4\tilde{\beta})|a_2^3|+2|a_2^2|+4|a_2|+2$$

$$\leqq -270-64\tilde{\beta}.$$

However for the function $z/(1+e^{i\varepsilon}z)^2$, ε: real

$$4|T|=-270-64\tilde{\beta}.$$

Thus $\max\Re T=-67.5-16\tilde{\beta}$. Equality occurs only for $z/(1-e^{i\pi/4}z)^2$, $z/(1-e^{i3\pi/4}z)^2$, $z/(1-e^{-i\pi/4}z)^2$, $z/(1-e^{-i3\pi/4}z)^2$. Secondly we consider the case $9+2\tilde{\beta}>0$. In this case

$$4T=2(18+4\tilde{\beta})|a_2|^3\cos(3\varphi-\theta)-2|a_2|^2\cos(2\varphi-2\theta)$$

$$-4|a_2|\cos(\varphi-3\theta)-2\cos(-4\theta),$$

$$a_2=|a_2|e^{i\varphi}.$$

For simplicity's sake we put $|a_2|=R$ and

$$F(R,\theta,\varphi)=2(18+4\tilde{\beta})R^3\cos(3\varphi-\theta)-2R^2\cos(2\varphi-2\theta)$$

$$-4R\cos(\varphi-3\theta)-2\cos(-4\theta).$$

We now seek for the maximum value of $F(R,\theta,\varphi)$ in $0\leqq R\leqq 2$, $0\leqq\theta\leqq 2\pi$, $0\leqq\varphi\leqq 2\pi$. For the inner maximum we have

$$\frac{\partial F}{\partial R}=0,\quad \frac{\partial F}{\partial\varphi}=0,\quad \frac{\partial F}{\partial\theta}=0.$$

By $\partial F/\partial R=0$ and $\partial F/\partial\varphi=0$,

$$6(18+4\tilde{\beta})R^2\cos(3\varphi-\theta)=4R\cos(2\varphi-2\theta)+4\cos(\varphi-3\theta),$$

$$6(18+4\tilde{\beta})R^3\sin(3\varphi-\theta)=4R^2\sin(2\varphi-2\theta)+4R\sin(\varphi-3\theta).$$

If $R=0$, then $F(0,\theta,\varphi)=-2\cos(-4\theta)\leqq 2$. If $R\neq 0$, then

$$6(18+4\tilde{\beta})R^2e^{i(3\varphi-\theta)}=4Re^{i(2\varphi-2\theta)}+4e^{i(\varphi-3\theta)}.$$

Hence

$$6(18+4\tilde{\beta})R^2e^{i(2\varphi+2\theta)}=4Re^{i(\varphi+\theta)}+4,$$

which implies

$$6(18+4\tilde{\beta})R^2\cos(2\varphi+2\theta)=4R\cos(\varphi+\theta)+4,$$

$$6(18+4\tilde{\beta})R^2\sin(2\varphi+2\theta)=4R\sin(\varphi+\theta).$$

If $6(18+4\tilde{\beta})R\cos(\varphi+\theta)=2$, then

$$4R\cos(\varphi+\theta)-6(18+4\tilde{\beta})R^2=4R\cos(\varphi+\theta)+4,$$

which is a contradiction. Hence $6(18+4\tilde{\beta})R\cos(\varphi+\theta)\neq 2$. Thus we have $\sin(\varphi+\theta)=0$. Thus $e^{i(\varphi+\theta)}=\pm 1$ and

$$6(18+4\tilde{\beta})R^2=\pm 4R+4.$$

We divide into two cases: i) $e^{i\varphi}=e^{-i\theta}$, ii) $e^{i\varphi}=-e^{-i\theta}$.

Case i). $3(9+2\tilde{\beta})R^2-R-1=0$. If this has no solution in $(0, 2)$, we may omit this case. This implies that $\tilde{\beta}\leqq -35/8$. So we may assume that $\tilde{\beta}>-35/8$. In this case by $\partial F/\partial\theta=0$

$$\Im\{2(18+4\tilde{\beta})R^3e^{i(3\varphi-\theta)}-4R^2e^{i(2\varphi-2\theta)}-12Re^{i(\varphi-3\theta)}-8e^{-i4\theta}\}=0.$$

Then

$$-\frac{8}{3}R^2-\frac{32}{3}R-8\neq 0 \qquad \text{for} \quad 0\leqq R\leqq 2$$

shows that $\sin 4\theta=0$. Hence $\cos 4\theta=\pm 1$. When $\cos 4\theta=1$,

$$\begin{aligned} F(R;\theta,\varphi)&=2(18+4\tilde{\beta})R^3-2R^2-4R-2\\ &=-\frac{2}{3}R^2-\frac{8}{3}R-2<0, \end{aligned}$$

which may be omitted. When $\cos 4\theta=-1$,

$$\begin{aligned} F(R,\theta,\varphi)&=2R^2+4R+2-2(18+4\tilde{\beta})R^3\\ &=\frac{2}{3}(R^2+4R+3). \end{aligned}$$

Case ii). Then $3(9+2\tilde{\beta})R^2+R-1=0$. By $\partial F/\partial\varphi=0$,

$$\Im\{-2(18+4\tilde{\beta})R^3e^{-4i\theta}+4R^2e^{-4i\theta}+12Re^{-4i\theta}-8e^{-4i\theta}\}=0.$$

Hence by $2R^2-4R+3\neq 0$ $\sin 4\theta=0$. When $\cos 4\theta=1$,

$$F(R,\theta,\varphi)=-\frac{2}{3}(R^2-4R+3).$$

If $0\leqq R\leqq 1$, $F(R,\theta,\varphi)\leqq 0$, which may be omitted. Hence $1<R\leqq 2$. In this case $F(R,\theta,\varphi)$ attains its maximum at $R=2$ and $F(2,\theta,\varphi)=2/3$, which may be omitted, since we have already $\max F\geqq 2$. When $\cos 4\theta=-1$,

$$F(R,\theta,\varphi)=\frac{2}{3}(R-1)(R-3).$$

In this case we have $0\leqq R<1$. Hence

$$R=\frac{\sqrt{1+12(9+2\tilde{\beta})}-1}{6(9+2\tilde{\beta})}<1.$$

This implies

$$(2\tilde{\beta}+9)^2<0,$$

which is a contradiction. Hence case ii) may be omitted. Further we must consider two end points of (0, 2). If $R=0$, then $4T=F=-2\cos 4\theta\leqq 2$. If $R=2$, then

$$\Re 4T\leqq 4|67.5+16\tilde{\beta}|.$$

Summing up the results, we have for $-9/2<\tilde{\beta}\leqq -35/8$

$$\max \Re 4T=\max(2,4|67.5+16\tilde{\beta}|)$$

and for $\tilde{\beta}>-35/8$ with $3(9+2\tilde{\beta})R^2-R-1=0$

$$\max \Re 4T=\max\left(\frac{2}{3}(R^2+4R+3),\ 2,\ 4|67.5+16\tilde{\beta}|\right)$$

Hence for $-9/2<\tilde{\beta}\leqq -35/8$

$$\Re 4T\leqq -4(16\tilde{\beta}+67.5).$$

Equality occurs only for

$$\frac{z}{(1-e^{\pm i\pi/4}z)^2},\quad \frac{z}{(1-e^{\pm i3\pi/4}z)^2}.$$

We now consider the case $\tilde{\beta}>-35/8$. First the solution $R(\tilde{\beta})$ of $3(9+2\tilde{\beta})R^2-R-1=0$ is monotone decreasing for increasing $\tilde{\beta}$. Further $2(R^2+4R+3)/3>2$. If $\tilde{\beta}$ lies in $[-34/8, -33.5/8]$, then $4|67.5+16\tilde{\beta}|\leqq 2$. Hence in this case

$$4\Re T\leqq \frac{2}{3}(R^2+4R+3).$$

This is exact. Thus

$$\max 4\Re T=\begin{cases}\max\left\{\dfrac{2}{3}(R^2+4R+3),\ -4(67.5+16\tilde{\beta})\right\}, & -\dfrac{35}{8}<\tilde{\beta}<-\dfrac{34}{8},\\ \max\left\{\dfrac{2}{3}(R^2+4R+3),\ 4(67.5+16\tilde{\beta})\right\}, & -\dfrac{33.5}{8}<\tilde{\beta}.\end{cases}$$

By $3(9+2\tilde{\beta})R^2-R-1=0$ we compare

$$\frac{2}{3}(R^2+4R+3),\ \pm 4(67.5+16\tilde{\beta}).$$

If $-35/8<\tilde{\beta}<-34/8$, then $(1+\sqrt{7})/3<R<2$. In this case put

$$\begin{aligned}\psi(R)&=\frac{3R^2}{2}\left\{\frac{2}{3}(R^2+4R+3)+4(67.5+16\tilde{\beta})\right\}\\ &=R^4+4R^3-24R^2+16R+16.\end{aligned}$$

It is very easy to prove $\psi(R)>0$ for $(1+\sqrt{7})/3<R<2$. Hence

$$\max 4\Re T=\frac{2}{3}(R^2+4R+3).$$

This is of course sharp. If $\tilde{\beta}>-33.5/8$, then $0\leqq R<(4+2\sqrt{34})/15$. Put

$$\begin{aligned}\psi(R)&=\frac{3R^2}{2}\left\{\frac{2}{3}(R^2+4R+3)-4(67.5+16\tilde{\beta})\right\}\\ &=R^4+4R^3+30R^2-16R-16.\end{aligned}$$

There is only one solution R_* of $\psi(R)=0$ for $[0, (4+2\sqrt{34})/15)$ and $\psi(R)<0$ for $R_*>R$ and $\psi(R)>0$ for $R>R_*$. Further we have $0.95<R^*<0.96$. We put

$$6\beta_*=\frac{R_*+1}{R_*^2}-27.$$

Then we have

$$\max 4\Re T=\begin{cases}\dfrac{2}{3}(R^2+4R+3) & \text{for}\quad -\dfrac{33.5}{8}<\tilde{\beta}<\beta_*,\\ 4(67.5+16\tilde{\beta}) & \text{for}\quad \tilde{\beta}\geqq\beta_*,\end{cases}$$

where $\tilde{\beta}$ and R satisfy $3(9+2\tilde{\beta})R^2=R+1$. These estimations are sharp. If $\tilde{\beta}>\beta_*$, equality occurs only for

$$\frac{z}{(1\pm z)^2},\quad \frac{z}{(1\pm iz)^2}.$$

If $\tilde{\beta}=\beta_*$, equality occurs only for the above four functions and a function satisfying the differential equation. Together with this case $-35/8<\tilde{\beta}<\beta_*$ leads to the extremal function, for which we cannot give any explicit expression, because the corresponding differential equation still involves a hyperelliptic integral with unknown coefficients and an elliptic integral with unknown coefficients. However our process together with Schiffer's variational method gives the exactness of our estimation.

6. Proof of Theorem 5. The proof of this theorem depends upon the process of section 5. However we only need its simplified version. Let us consider

$$\operatorname*{Max}_{S} \Re(a_5-4a_2a_4+\delta a_3^2+8a_2^2a_3-4a_2^4)$$

for $-3/2\leqq\delta\leqq-1$. The extremal functions satisfy the differential equation

$$\begin{aligned}
& z^2\frac{w'^2}{w^6}[\{-2a_4+(10-4\delta)a_2a_3-4a_2^3\}w^3\\
&\qquad +\{(3+2\delta)a_3-a_2^2\}w^2+1]\\
&=\frac{1}{z^4}[1-2a_2z+(3+2\delta)a_3z^2+(4+4\delta)a_2a_3z^3\\
&\qquad +4(a_5-4a_2a_4+\delta a_3^2+8a_2^2a_3-4a_2^4)z^4\\
&\qquad +(4+4\delta)\bar{a}_2\bar{a}_3z^5+(3+2\delta)\bar{a}_3z^6-2\bar{a}_2z^7+z^8].
\end{aligned}$$

We have the second expression of the right hand side

$$\frac{1}{z^4}(z^2+2Pz+P^2)(z^6+Az^5+Bz^4+Cz^3+Dz^2+Ez+F).$$

We now make use of the same notations as in the proof of Theorem 4. Then

$$\begin{aligned}
& 4(a_5-4a_2a_4+\delta a_3^2+8a_2^2a_3-4a_2^4)\\
&=2s\cos\beta+2re^{i\theta+i\alpha}\\
&=-2(3+2\delta)|a_3|\cos(\psi-2\theta)+2(4+4\delta)|a_2||a_3|\cos(\varphi+\psi-\theta)\\
&\qquad -4|a_2|\cos(\varphi-3\theta)-2\cos(-4\theta),
\end{aligned}$$

where $a_2=|a_2|e^{i\varphi}$, $a_3=|a_3|e^{i\psi}$. Hence

$$\begin{aligned}
& 4(a_5-4a_2a_4+\delta a_3^2+8a_2^2a_3-4a_2^4)\\
&\leqq 2(3+2\delta)|a_3|-2(4+4\delta)|a_2||a_3|+4|a_2|+2\\
&\leqq -20-36\delta.
\end{aligned}$$

Here equality occurs only for

$$\frac{z}{(1+e^{\pm\pi i/4}z)^2}, \quad \frac{z}{(1+e^{\pm 3\pi i/4}z)^2}.$$

Hence in general

$$\Re(a_5-4a_2a_4+\delta a_3^2+8a_2^2a_3-4a_2^4)\leqq -5-9\delta.$$

For $\delta<-3/2$ an easy algebraic consideration leads to the result immediately. For $\delta>-1$ we do not have any effective method.

7. Proof of Theorem 6. Let us consider the problem

$$\max_S \Re(a_4-3a_2a_3+Ba_2^3).$$

This gives the differential equation satisfied by any extremal functions

$$z^2\frac{w'^2}{w^5}\{(-a_3+(3B-5)a_2^2)w^2+1\}$$

$$=\frac{1}{z^3}\{1-a_2z+(3B-6)a_2^2z^2+(3a_4-9a_2a_3+3Ba_2^3)z^3$$

$$+(3B-6)\bar{a}_2^2z^4-\bar{a}_2z^5+z^6\}.$$

This allows the second expression

$$\frac{1}{z^3}(z^2+2Pz+P^2)(z^4+Qz^3+Rz^2+Sz+T),$$

$$|T|=|P|=1, \quad \bar{Q}=\bar{T}S, \quad \bar{R}=\bar{T}R.$$

Hence we have

$$P^2T=1,$$

$$P^2S+2PT=-a_2,$$

$$P^2R+2PS+T=3(B-2)a_2^2,$$

$$P^2Q+2PR+S=3a_4-9a_2a_3+3Ba_2^3.$$

Putting $P=e^{i\theta}$, $T=e^{-2i\theta}$, $S=se^{i\alpha}$, $R=re^{i\beta}$,

$$e^{-2i\beta}=e^{2i\theta},$$

$$se^{i\alpha}+2e^{-3i\theta}=-a_2e^{-2i\theta},$$

$$re^{i\theta+i\beta}+2se^{i\alpha}+e^{-3i\theta}=3(B-2)a_2^2e^{-i\theta},$$

$$2s\cos\alpha+2re^{i\theta+i\beta}=3(a_4-3a_2a_3+Ba_2^3).$$

Hence

$$3(a_4-3a_2a_3+Ba_2^3)$$
$$=6(B-2)|a_2|^2\cos(2\varphi-\theta)+2|a_2|\cos(\varphi-2\theta)+2\cos 3\theta.$$

If $B\geqq 2$,

$$3(a_4-3a_2a_3+Ba_2^3)\leqq 6(B-2)|a_2|^2+2|a_2|+2$$
$$\leqq 24B-42.$$

Equality occurs only for $z/(1-e^{2\pi ki/3}z)^2$, $k=0, 1, 2$. We put

$$F(R, \theta, \varphi)=6(B-2)R^2\cos(2\varphi-\theta)+2R\cos(\varphi-2\theta)+2\cos 3\theta.$$

Assume that $B<2$ and $0<R<2$. We now consider $\max F$.

$$0=\frac{\partial}{\partial R}F(R, \theta, \varphi)=12(B-2)R\cos(2\varphi-\theta)+2\cos(\varphi-2\theta),$$

$$0=\frac{\partial}{\partial\varphi}F(R, \theta, \varphi)=-12(B-2)R^2\sin(2\varphi-\theta)-2R\sin(\varphi-2\theta).$$

Hence

$$6(B-2)Re^{(2\varphi-\theta)i}+e^{(\varphi-2\theta)i}=0,$$

which implies

$$6(B-2)R=-1, \qquad e^{\varphi i}=e^{-\theta i}.$$

If $B\geqq 23/12$, then $R\geqq 2$. Hence if $B\geqq 23/12$, there is no maximum in $0<R<2$. Hence we may compare $F(0, \theta, \varphi)$ and $F(2, \theta, \varphi)$, for which

$$F(0, \theta, \varphi)=2\cos 3\theta\leqq 2,$$
$$F(2, \theta, \varphi)=3(8B-14)\Re e^{3i\varphi}=3(8B-14)\cos 3\varphi.$$

Hence in general for $23/12\leqq B<2$

$$\Re(a_4-3a_2a_3+Ba_2^3)\leqq 8B-14.$$

If $12B<23$, then

$$R=\frac{1}{6(2-B)}<2, \qquad e^{\varphi i}=e^{-\theta i}$$

at the points giving the maximum of $F(R, \theta, \varphi)$. Then

$$F(R, \theta, \varphi)=\{6(B-2)R^2+2R+2\}\cos 3\varphi$$
$$=(R+2)\cos 3\varphi$$

at the maximum points. Therefore

$$\max F(R, \theta, \varphi) = R+2.$$

On the other hand

$$F(0, \theta, \varphi) = 2\cos 3\theta \leqq 2,$$

$$F(2, \theta, \varphi) = 3(8B-14)\cos 3\varphi.$$

Hence

$$\begin{aligned}&\max 3\Re(a_4 - 3a_2a_3 + Ba_2^3)\\ &= \max(\max F(R, \theta, \varphi),\ \max F(0, \theta, \varphi),\ \max F(2, \theta, \varphi))\\ &= \max(R+2,\ 3|8B-14|).\end{aligned}$$

If $4B \geqq 7$, then $R+2 \geqq 3(8B-14)$. Here equality does not occur, since $12B < 23$. If $4B < 7$, then $R+2 = -3(8B-14)$ has the solution B_* in $B < 7/4$. The value of B_* is $(22-\sqrt{5})/12$. This implies that if $B_* \leqq B < 7/4$ the maximum is $R+2$. If $B < B_*$ the maximum is $-24B+42$. Equality occurs in every case. Especially equality occurs only for $z/(1+e^{2\pi ki/3}z)^2$, $k=0,1,2$, if $B < B_*$. If $B \leqq B_*$, the functions mentioned above are extremal.

8. Proof of Theorem 7. We need a lemma.

LEMMA. *For $\beta \geqq 3$*

$$\Re\left(a_3 - \frac{3}{4}a_2^2 + \beta a_2\right) \leqq 2\beta.$$

Equality occurs only for $z/(1-z)^2$.

Proof. Let us consider the problem

$$\max \Re\left(a_3 - \frac{3}{4}a_2^2 + \beta a_2\right),$$

which leads to the differential equation satisfied by any extremal functions

$$\begin{aligned}&z^2\frac{w'^2}{w^4}\left\{\left(\frac{1}{2}a_2+\beta\right)w+1\right\}\\ &= \frac{1}{z^2}\left\{1+\left(\frac{1}{2}a_2+\beta\right)z+\left(2a_3-\frac{3}{2}a_2^2+\beta a_2\right)z^2+\left(\frac{1}{2}\bar{a}_2+\beta\right)z^3+z^4\right\}.\end{aligned}$$

Let $Q(\zeta)d\zeta^2$ be

$$-\frac{d\zeta^2}{\zeta}\left(\frac{1}{2}a_2+\beta+\zeta\right),$$

for which we are able to use Bombieri's method as in the second section. We shall omit its detail. Then we may assume that a_2 is real positive or zero. If $a_2=0$, then $|a_3|\leqq 1$. If a_2 is real positive, then the image of $|z|=1$ by $\zeta=1/w$ starts from the origin along the negative real axis. If $a_2/2+\beta\leqq 4$, then the image of $|z|=1$ by ζ forks at $\zeta=-a_2/2-\beta$ into two curves. In this case we have the second representation

$$\frac{1}{z^2}(z^2+Pz+Q)^2,\qquad |Q|=1,\qquad \bar{P}=\bar{Q}P.$$

Thus $Q^2=1$, $2PQ=a_2/2+\beta$,

$$2Q+P^2=2a_3-\frac{3}{2}a_2^2+\beta a_2.$$

Hence

$$\Re 2\left(a_3-\frac{3}{4}a_2^2+\beta a_2\right)=\pm 2+\Re\left(\frac{a_2}{4}+\frac{\beta}{2}\right)^2+\Re\beta a_2.$$

This implies that for the extremal functions with coefficients a_2, a_3

$$2\Re\left(a_3^*-\frac{3}{4}a_2^{*2}+\beta a_2^*\right)$$

$$\leqq 2+\frac{1}{16}\Re a_2^2+\frac{\beta}{4}\Re a_2+\frac{\beta^2}{4}+\beta\Re a_2,$$

for every g with coefficients a_2^*, a_3^*. If $\beta=3$, then the above phenomena occurs always. Therefore

$$2\Re\left(a_3-\frac{3}{4}a_2^2+\beta a_2\right)\leqq 12.$$

Equality occurs only for $z/(1-z)^2$. This gives the desired fact for the general $\beta\geqq 3$.

Now we return to the proof of Theorem 7. Let us consider the extremal problem

$$\max\Re\left\{a_4-3a_2a_3+\frac{23-\beta}{12}a_2^3+2\beta\left(a_3-\frac{3}{4}a_2^2\right)+\beta^2a_2\right\}.$$

Then the extremal functions satisfy the differential equation

$$z^2\frac{w'^2}{w^5}[\{-a_3+(3B-5)a_2^2+\beta a_2+\beta^2\}w^2+2\beta w+1]$$

$$=\frac{1}{z^3}[1+(2\beta-a_2)z+\{3(B-2)a_2^2+\beta a_2+\beta^2\}z^2$$

$$+(3a_4-9a_2a_3+3Ba_2^3+4\beta a_3-3\beta a_2^2+\beta^2 a_2)z^3$$
$$+\{3(B-2)\bar{a}_2^2+\beta\bar{a}_2+\beta^2\}z^4+(2\beta-\bar{a}_2)z^5+z^6]$$

with $B=(23-\beta)/12$. This has the second expression

$$\frac{1}{z^3}(z^2+2Pz+P^2)(z^4+Qz^3+Rz^2+Sz+T),$$
$$|P|=|T|=1, \quad \bar{Q}=\bar{T}S, \quad \bar{R}=\bar{T}R.$$

Thus we have

$$P^2T=1,$$
$$P^2S+2PT=2\beta-a_2,$$
$$P^2R+2PS+T=3(B-2)a_2^2+\beta a_2+\beta^2,$$
$$P^2Q+2PR+S=3a_4-9a_2a_3+3Ba_2^3+4\beta a_3-3\beta a_2^2+\beta^2 a_2\equiv A.$$

We put $P=e^{i\theta}$, $S=se^{i\alpha}$, $R=re^{i\delta}$. Then $T=e^{-2i\theta}$, $e^{-2i\alpha}=e^{2i\delta}$,

$$se^{i\alpha}+2e^{-3i\theta}=-a_2e^{-2i\theta}+2\beta e^{-2i\theta},$$
$$re^{i\theta+i\delta}+2se^{i\alpha}+e^{-3i\theta}=3(B-2)a_2^2e^{-i\theta}+\beta a_2e^{-i\theta}+\beta^2e^{-i\theta},$$
$$A=2s\cos\alpha+2re^{i\theta+i\delta}$$
$$=\Re\{6(B-2)a_2^2e^{-i\theta}+2a_2e^{-2i\theta}+2\beta a_2e^{-i\theta}+2\beta^2e^{-i\theta}-4\beta e^{-2i\theta}+2e^{-3i\theta}\}$$
$$=6(B-2)|a_2|^2\cos(2\varphi-\theta)+2|a_2|\cos(\varphi-2\theta)+2\beta|a_2|\cos(\varphi-\theta)$$
$$+2\beta^2\cos\theta-4\beta\cos 2\theta+2\cos 3\theta.$$

Our problem is to seek for the maximum of

$$3F=\Re A+2\beta\Re\left(a_3-\frac{3}{4}a_2^2\right)+2\beta^2\Re a_2.$$

By Lemma for $\beta\geqq 3$

$$\Re\left\{a_3-\frac{3}{4}a_2^2+\beta a_2\right\}\leqq 2\beta.$$

Equality occurs only for $z/(1-z)^2$. Hence if $\max\Re A$ is given by $z/(1-z)^2$, then $\max F$ is given by $z/(1-z)^2$. So we shall consider $\Re A$ for the extremal functions. In this case A is real. Let $|a_2|$ be R. For $0<R<2$

$$0=\frac{\partial A}{\partial R}=12(B-2)R\cos(2\varphi-\theta)+2\cos(\varphi-2\theta)+2\beta\cos(\varphi-\theta),$$

$$0=\frac{\partial A}{\partial\varphi}=-12(B-2)R^2\sin(2\varphi-\theta)-2R\sin(\varphi-2\theta)-2\beta R\sin(\varphi-\theta).$$

Thus at the points at which the maximum of A is attained

$$6(B-2)Re^{i(2\varphi-\theta)}+e^{i(\varphi-2\theta)}+\beta e^{i(\varphi-\theta)}=0,$$

which implies

$$6(B-2)R+\cos(\varphi+\theta)+\beta\cos\varphi=0,$$
$$\sin(\varphi+\theta)+\beta\sin\varphi=0,$$

or

$$6(B-2)R\cos\varphi+\cos\theta+\beta=0,$$
$$6(B-2)R\sin\varphi=\sin\theta.$$

Further

$$0=\frac{\partial A}{\partial\theta}=6(B-2)R^2\sin(2\varphi-\theta)+4R\sin(\varphi-2\theta)+2\beta R\sin(\varphi-\theta)$$
$$-2\beta^2\sin\theta+8\beta\sin 2\theta-6\sin 3\theta$$
$$=3R\sin(\varphi-2\theta)+\beta R\sin(\varphi-\theta)-2\beta^2\sin\theta+8\beta\sin 2\theta-6\sin 3\theta$$
$$=3R\sin\varphi\cos 2\theta-3R\cos\varphi\sin 2\theta+\beta R\sin\varphi\cos\theta-\beta R\cos\varphi\sin\theta$$
$$-2\beta^2\sin\theta+8\beta\sin 2\theta-6\sin 3\theta$$
$$=\frac{\sin\theta}{2(B-2)}\cos 2\theta-6R\cos\varphi\cos\theta\sin\theta+\frac{\beta\sin\theta}{6(B-2)}\cos\theta-\beta R\cos\varphi\sin\theta$$
$$-2\beta^2\sin\theta+16\beta\cos\theta\sin\theta-24\sin\theta\cos^2\theta+6\sin\theta.$$

Hence $\sin\theta=0$ or

$$\frac{\cos 2\theta}{2(B-2)}-6R\cos\varphi\cos\theta+\frac{\beta\cos\theta}{6(B-2)}-\beta R\cos\varphi$$
$$-2\beta^2+16\beta\cos\theta-24\cos^2\theta+6=0.$$

If the second alternative occurs, then eliminating $R\cos\varphi$ by

$$6(B-2)R\cos\varphi+\cos\theta+\beta=0$$

we have

$$\cos^2\theta(300-144B)+\beta\cos\theta(96B-184)+(\beta^2-3)(25-12B)=0,$$

Since $B=(23-\beta)/12$, $\beta\geqq 3$, we have

$$4(-2\beta^4-12\beta^3-3\beta^2+36\beta+36)<0 \qquad \text{for} \quad \beta\geqq 3,$$

which implies that

$$\cos^2\theta(300-144B)+\beta\cos\theta(96B-184)+(\beta^2-3)(25-12\beta)\neq 0.$$

Hence $\sin\theta=0$, which implies $\sin\varphi=0$ and $\cos\varphi=1$, since

$$6(2-B)R\cos\varphi=\beta+\cos\theta>0.$$

Evidently $\cos\theta=\pm 1$. Hence

$$6(2-B)R=\beta+1 \quad \text{or} \quad \beta-1.$$

If $\cos\theta=-1$, then

$$A=6(2-B)R^2+2R-2\beta R-2\beta^2-4\beta-2$$
$$=R-\beta R-2\beta^2-4\beta-2<0.$$

Hence this does not give the maximum. If $\cos\theta=1$, then

$$R=\frac{\beta+1}{6(2-B)}=2,$$

which contradicts $0<R<2$. Hence we may consider the values of A at $R=0$ and $R=2$. For $R=2$ we have simply for the extremal functions

$$A=\{(4-2\beta)e^{3i\varphi}+2\beta^2 e^{i\varphi}\}$$
$$=(4-2\beta)\cos 3\varphi+2\beta^2\cos\varphi.$$

We now consider $dA/d\varphi=0$. Then either $\sin\varphi=0$ or

$$12(2-\beta)\cos^2\varphi+\beta^2+3\beta-6=0.$$

If $\sin\varphi=0$, then $\cos\varphi=\pm 1$. Hence

$$A=\pm(2\beta^2-2\beta+4).$$

In this case

$$3F\leqq 2\beta^2-2\beta+4+4\beta^2=6\beta^2-2\beta+4.$$

Equality occurs only for $z/(1-z)^2$. If the second alternative occurs, then $\cos^2\varphi\leqq 1$ implies $3\leqq\beta\leqq 6$. Hence for $\beta>6$ the second case does not hold. If $\beta=6$ or $\beta=3$, then $\cos^2\varphi=1$, which gives

$$A=\pm(2\beta^2-2\beta+4),$$

$$3F \leqq 6\beta^2 - 2\beta + 4.$$

If $3 < \beta < 6$ holds, then

$$A = 8(2-\beta)\cos^3\varphi + 2(\beta^2 + 3\beta - 6)\cos\varphi$$

$$= \frac{4}{3}(\beta^2 + 3\beta - 6)\cos\varphi.$$

In this case $4a_3 - 3a_2^2 = 0$ implies

$$3F \leqq \frac{4}{3}(\beta^2 + 3\beta - 6)\cos\varphi + 4\beta^2\cos\varphi$$

$$= \frac{4}{3}(4\beta^2 + 3\beta - 6)\cos\varphi.$$

We now compare this with $6\beta^2 - 2\beta + 4$. We consider

$$\frac{16}{9}(4\beta^2 + 3\beta - 6)^2\cos^2\varphi - (6\beta^2 - 2\beta + 4)^2$$

$$= \frac{4}{27}\frac{\beta^2}{\beta - 2}(16\beta^4 - 171\beta^3 + 585\beta^2 - 1072\beta + 972).$$

It is very easy to show that the above expression is surely negative for $3 < \beta < 6$. Therefore for $R = 2$, $\beta \geqq 3$

$$3F \leqq 6\beta^2 - 2\beta + 4.$$

For $R = 0$ we have $a_2 = 0$. Thus

$$|a_4| \leqq \frac{2}{3}, \quad |a_3| \leqq 1.$$

Hence

$$A \leqq 2 + 4\beta,$$

$$3F \leqq 2 + 6\beta.$$

Evidently for $\beta \geqq 3$

$$6\beta^2 - 2\beta + 4 > 2 + 6\beta.$$

This completes the proof of Theorem 7.

Our Lemma can be completed for $0 \leqq \beta < 3$. This is implicitly included in Jenkin's results [8].

9. Proof of Theorem 8. Let $f(z)$ be in S. We put

$$\log \frac{f(z)-f(\zeta)}{z-\zeta} = \sum_{k,l=0}^{\infty} c_{kl} z^k \zeta^l,$$

then

$$c_{11}=a_3-a_2^2,$$

$$c_{12}=a_4-2a_2a_3+a_2^3,$$

$$c_{22}=a_5-2a_2a_4-\frac{3}{2}a_3^2+4a_2^2a_3-\frac{3}{2}a_2^4.$$

Garabedian [3] proved the following inequality:

$$\Re\{c_{22}+2\lambda c_{12}+(\lambda^2+\mu)c_{11}+q(a_2+\lambda,\mu)\}$$

$$\leqq \frac{1}{2}+|\lambda|^2+\frac{\mu}{2}\Re(\lambda^2)+\frac{3}{16}\mu^2+\frac{\mu^2}{4}\log 6,$$

$$q(a_2+\lambda,\mu)=\frac{(a_2+\lambda)^4}{54}+\frac{\mu(a_2+\lambda)^2}{3}+\left[\frac{(a_2+\lambda)^3}{54}-\frac{5\mu(a_2+\lambda)}{36}\right]\sqrt{(a_2+\lambda)^2-3\mu}$$

$$+\frac{\mu^2}{4}\log\left[a_2+\lambda+\sqrt{(a_2+\lambda)^2-3\mu}\right],$$

where $\lambda=\xi+i\eta$ is restricted to lie inside or on the involute

$$\lambda_\mu(\theta)=4(\cos^3\theta+i\sin^3\theta)-\left(3\cos 2\theta-\frac{1}{4}\mu\right)e^{-i\theta}, \qquad 0\leqq\theta<2\pi$$

of the hypocycloid

$$\xi^{2/3}+\eta^{2/3}=4^{2/3}$$

and $-12\leqq\mu\leqq12$. This is best possible.

Here we need the following fact: If $\lambda=1+\mu/4$, then the extremal function is the Koebe function $z/(1-z)^2$. Now we put $\lambda=1$, $\mu=0$. Then

$$\Re\left\{c_{22}+2c_{12}+c_{11}+\frac{(a_2+1)^4}{27}\right\}\leqq\frac{3}{2}.$$

This is just

$$\Re\left\{a_5-2a_2a_4-\frac{3}{2}a_3^2+4a_2^2a_3-\frac{3}{2}a_2^4\right.$$

$$\left.+2(a_4-2a_2a_3+a_2^3)+a_3-a_2^2+\frac{(a_2+1)^4}{27}\right\}\leqq\frac{3}{2}.$$

Let us introduce the following notations:

$$a_2=p+ix'=2-x+ix',$$

$$a_3-\frac{3}{4}a_2^2=y+iy',$$

$$a_4-\frac{3}{2}a_2a_3+\frac{5}{8}a_2^3=\eta+i\eta'.$$

Then we have

$$*\equiv\Re\left\{a_5-a_2a_4-\frac{3}{2}a_3^2+2a_2^2a_3-\frac{5}{16}a_2^4\right\}$$

$$\leqq\frac{3}{2}+\Re a_2\left(a_4-2a_2a_3+\frac{19}{16}a_2^3\right)-2\Re(a_4-2a_2a_3+a_2^3)$$

$$-\Re(a_3-a_2^2)-\frac{1}{27}\Re(a_2+1)^4.$$

by Garabedian's inequality. We now rewrite this inequality

$$*\leqq\frac{5}{2}-3x+\frac{1}{2}x^2-\frac{19}{18}x^3+\left(\frac{3}{16}-\frac{1}{27}\right)x^4$$

$$-x\eta-\left(1-x+\frac{1}{2}x^2\right)y+\frac{1}{2}x'^2y$$

$$-x'^2\left(\frac{11}{4}-\frac{19}{6}x+\left(\frac{9}{8}-\frac{6}{27}\right)x^2-\left(\frac{3}{16}-\frac{1}{27}\right)x'^2\right)+(1-x)x'y'-x'\eta'.$$

By the area theorem

$$5\eta^2+5\eta'^2+3y^2+3y'^2+x'^2\leqq 4x-x^2,$$

we have

$$1-x+\frac{1}{2}x^2-\frac{1}{2}x'^2$$

$$\geqq 1-x+\frac{1}{2}x^2-2x+\frac{1}{2}x^2=1-3x+x^2>0$$

for $0\leqq x\leqq 0.3$. We now divide into two cases: i) $y\geqq 0$, ii) $y\leqq 0$.

Case i). In this case we have for $0\leqq x\leqq 0.3$

$$*\leqq\frac{5}{2}-3x+\frac{1}{2}x^2-\frac{19}{18}x^3+\left(\frac{3}{16}-\frac{1}{27}\right)x^4-x\eta$$

$$-x'^2\left\{\frac{11}{4}-\frac{19}{6}x+\left(\frac{9}{8}-\frac{6}{27}\right)x^2-\left(\frac{3}{16}-\frac{1}{27}\right)x'^2\right\}$$
$$+(1-x)x'y'-x'\eta'.$$

By the trivial inequality

$$-x\eta\leqq\frac{\alpha}{2}x^2+\frac{1}{2\alpha}\eta^2,\qquad \alpha>0.$$

and by the area theorem we have

$$*\leqq\frac{5}{2}-\frac{3-2\alpha-2}{4}x^2-\frac{19}{18}x^3+\left(\frac{3}{16}-\frac{1}{27}\right)x^4+\left(\frac{1}{2\alpha}-\frac{15}{4}\right)\eta^2$$
$$-\left[x'^2\left\{\frac{14}{4}-\frac{19}{6}x+\frac{65}{72}x^2-\frac{65}{432}x'^2\right\}-(1-x)x'y'+\frac{9}{4}y'^2+x'\eta'+\frac{15}{4}\eta'^2\right].$$

Now we put $\alpha=2/15$. Then

$$*\leqq\frac{5}{2}-x^2P(x)-Q,$$

$$P(x)=\frac{13}{60}+\frac{19}{18}x-\frac{65}{432}x^2,$$

$$Q=\left\{\frac{14}{4}-\frac{19}{6}x+\frac{65}{72}x^2-\frac{65}{432}x'^2\right\}x'^2-(1-x)x'y'+\frac{9}{4}y'^2+x'\eta'+\frac{15}{4}\eta'^2.$$

It is very easy to prove

$$P(x)>0,\qquad Q\geqq0$$

for $0\leqq x\leqq1$.

Case ii). In this case

$$y\left(x-\frac{1}{2}x^2\right)+\frac{1}{2}x'^2y\leqq0.$$

Further by $|a_2^2-a_3|\leqq1$

$$-y\leqq x-\frac{1}{4}x^2+\frac{1}{4}x'^2.$$

Hence

$$*\leqq\frac{5}{2}-2x+\frac{1}{4}x^2-\frac{19}{18}x^3+\frac{65}{432}x^4-x\eta$$

$$-x'^2\left(\frac{10}{4}-\frac{19}{6}x+\frac{65}{72}x^2-\frac{65}{432}x'^2\right)+x'y'(1-x)-x'\eta'.$$

By the trivial inequality and the area theorem

$$*\leqq\frac{5}{2}-\frac{1-2\alpha}{4}x^2-\frac{19}{18}x^3+\frac{65}{432}x^4-\left(\frac{5}{2}-\frac{1}{2\alpha}\right)\eta^2-Q,$$

$$Q=x'^2\left(\frac{10}{4}-\frac{19}{6}x+\frac{65}{72}x^2-\frac{65}{432}x'^2\right)-(1-x)x'y'+\frac{3}{2}y'^2+x'\eta'+\frac{5}{2}\eta'^2.$$

Now we put $\alpha=1/5$. Then

$$*\leqq\frac{5}{2}-x^2P(x)-Q,$$

$$P(x)=\frac{3}{20}+\frac{19}{18}x-\frac{65}{432}x^2.$$

Again it is very easy to prove $P(x)>0$ and $Q\geqq 0$ for $0\leqq x\leqq 0.23$.

Summing up the results, we have the desired result.

10. A proof of $|a_4-2a_2a_3+a_2^3|\leqq 2/3$. By Schiffer's variational method any extremal functions $w(z)$ for the problem $\max_S \Re(a_4-2a_2a_3+a_2^3)$ satisfy

$$z^2\frac{w'^2}{w^5}(a_2w+1)$$

$$=\frac{1}{z^3}\{1+(a_3-a_2^2)z^2+3(a_4-2a_2a_3+a_2^3)z^3+(\bar{a}_3-\bar{a}_2^2)z^4+z^6\}.$$

Let ζ be $1/w$ and consider the quadratic differential

$$-(a_2+\zeta)d\zeta^2.$$

Let Γ be the image of $|z|=1$ by ζ. Let $\zeta=\xi+i\eta$ and $a_2=x_2+iy_2$. Then on the trajectories

$$(x_2+\xi)(-d\xi^2+d\eta^2)+2d\xi d\eta(y_2+\eta)>0,$$

$$(y_2+\eta)(-d\xi^2+d\eta^2)-2d\xi d\eta(x_2+\xi)=0.$$

If $x_2+\xi\neq 0$. Then

$$(x_2+\xi)\left\{\left(\frac{d\eta}{d\xi}\right)^2-1\right\}>0.$$

On the other hand we have

$$(y_2+\eta)\frac{d\eta}{d\xi}>0.$$

Now we may assume that $|\arg a_2|\leqq\pi/3$. In what follows we make use of the term "the first quadrant, the second $\cdots$ around a point A" as if we set a coordinate system, being parallel to the original coordinate axis, at A.

1) $0<|\arg a_2|\leqq\pi/4$. We treat only $0<\arg a_2\leqq\pi/4$. If (ξ, η) belongs to the first quadrant I_1 around $-a_2$, then $d\eta/d\xi>1$. Hence Γ, which has tangent vector at the origin with the argument $\pi/2-(\arg a_2)/2$, does meet the open segment joining two points $-y_2-iy_2$, $-iy_2$. Γ does not meet the closed segment joining $-a_2$ and $-y_2-iy_2$. This fact can be deduced by $d\eta/d\xi>1$ in the first quadrant I_1 around $-a_2$ and by

$$-a_2=\frac{1}{2\pi}\int_0^{2\pi}\zeta(e^{i\theta})d\theta.$$

Thus Γ enter into the fourth quadrant I_4 around $-a_2$. Then we have $-\infty<d\eta/d\xi<-1$ in I_4, since $d\eta/d\xi\equiv\infty$ implies that Γ should be a segment parallel to the imaginary axis. Hence

$$\Re\frac{1}{2\pi}\int_0^{2\pi}\zeta(e^{i\theta})d\theta>-x_2,$$

which is a contradiction.

2) $\arg a_2=0$. Then Γ is tangent to the imaginary axis at the origin. Then Γ does not enter into the second and the third quadrants around the origin. This is again a contradiction.

3) $\pi/4<\arg a_2\leqq\pi/3$. The case $-\pi/4>\arg a_2\geqq-\pi/3$ is similar. If Γ intersects to the open segment joining $-a_2$ and $-iy_2$, then the situation is the same as in 1). Thus Γ should meet the segment joining $-a_2$ and $-x_2-x_2i$. Then we have two subcases: Γ meets $-a_2$ or Γ does not meet $-a_2$. Assume that Γ does not meet $-a_2$. Then Γ enter into the second quadrant I_2 around $-a_2$, in which $0<d\eta/d\xi<1$. If Γ does not meet the straight line $y=-y_2$, then Γ lies in the upper half plane of the straight line $z=-a_2+te^{i\pi/4}$, $-\infty<t<\infty$. This contradicts

$$-a_2=\frac{1}{2\pi}\int_0^{2\pi}\zeta(e^{i\theta})d\theta.$$

Hence Γ must meet $y=-y_2$. Consider $d\eta/d\xi$ around this intersection point. In the upper half-plane $0<d\eta/d\xi<1$ and in the lower half plane $d\eta/d\xi<-1$. This contradicts the continuity of $d\eta/d\xi$ or the analyticity of Γ. Hence we have a contradiction. Thus Γ should meet $-a_2$. Since $-a_2$ is a simple zero of $-(a_2+\zeta)d\zeta^2$, Γ, then, forks at $-a_2$. Returning to w, we have the second expression

$$z^2 \frac{w'^2}{w^5}(a_2 w+1)$$

$$=\frac{1}{z^3}(z^3+Pz^2+Qz+R)^2 \equiv g(z).$$

Further $g(1/\bar{z})=g(z)$. Thus we have

$$R^2=1, \qquad \bar{P}=\bar{R}Q,$$

$$2QR=0,$$

$$2PR+Q^2=a_3-a_2^2,$$

$$2R+2PQ=3(a_4-2a_2a_3+a_2^3).$$

This shows that $Q=P=0$. Hence $a_3-a_2^2=0$ and

$$a_4-2a_2a_3+a_2^3=\pm\frac{2}{3}.$$

This gives the desired result. The equality statement is easily obtained by integrating the equation.

In the above discussion we have assumed that $a_2 \neq 0$. If $a_2=0$, then the origin is a simple zero of $-\zeta d\zeta^2$. Hence the similar second expression remains true. Hence we have the desired result.

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