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A CHARACTERIZATION OF THE ALMOST *O-MANIFOLD

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Dedicated to Professor Kentaro Yano on his sixtieth birthday

The theory of linear connections in an almost Hermitian manifold has been studied by Obata [2], Walker [4], Yano [5] and others. One of remarkable results obtained by these studies is a characterization of the complex manifold by the existence of a symmetric connection with respect to which the covariant derivative of the structure tensor J vanishes. So it may be expected that a special almost Hermitian manifold can be characterized by the existence of a certain linear connection. From this stand-point, we shall try in the present paper, to give such a characterization for the almost *O-manifold.

1. Preliminaries.

Let M be an almost complex manifold of real dimension $2n$, that is, a differentiable manifold which admits a tensor field J of type $(1, 1)$ satisfying

$$J^2 = -1, \tag{1.1}$$

where 1 denotes the identity mapping of the tangent bundle of M. The tensor field J is called an almost complex structure of M. It is well known that a necessary and sufficient condition for an almost complex manifold to be a complex manifold is that the Nijenhuis tensor N of J defined by

$$N(X, Y) = [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y] \tag{1.2}$$

vanishes identically.

First of all, for any tensor T of type $(0, 2)$, we define operators O and *O as follows:

$$\begin{aligned} 2O(T)(X, Y) &= T(X, Y) - T(JX, JY), \\ 2{*}O(T)(X, Y) &= T(X, Y) + T(JX, JY). \end{aligned} \tag{1.3}$$

Then it is easily verified that

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$$O+{}^*O=1,$$

$$O\cdot O=O,\qquad O\cdot{}^*O={}^*O\cdot O=0,\qquad {}^*O\cdot{}^*O={}^*O.$$

Thus the two conditions

$$O(T)=0 \quad\text{and}\quad {}^*O(T)=T$$

are equivalent to each other. Moreover, the two conditions

$$^*O(T)=0 \quad\text{and}\quad O(T)=T$$

are also equivalent to each other. We say that a tensor T is hybrid or pure if it satisfies

$$O(T)=0 \quad\text{or}\quad {}^*O(T)=0$$

respectively.

Now we assume that the almost complex manifold M admits a Riemannian metric g satisfying

$$O(g)=0. \tag{1.5}$$

A Riemannian metric g satisfying (1.5) is called a Hermitian metric. An almost complex manifold with a Hermitian metric is called an almost Hermitian manifold. In an almost Hermitian manifold the 2-form ω defined by

$$\omega(X, Y)=g(JX, Y) \tag{1.6}$$

is of rank $2n$. We now remark that, given an arbitrary positive definite Riemannian metric $\mathring{g}$, we can construct a Hermitian metric g in the following way:

$$g(X, Y)={}^*O(\mathring{g})(X, Y)=\frac{1}{2}(\mathring{g}(X, Y)+\mathring{g}(JX, JY)).$$

A connection ∇ satisfying

$$\nabla g=0 \tag{1.7}$$

is called a metric connection. Let ∇ be a metric connection and $\mathring{\nabla}$ the Levi-Civita connection constructed from the given Riemannian metric g. Then we can put

$$\nabla_X Y=\mathring{\nabla}_X Y+T(X, Y) \tag{1.8}$$

where T denotes a tensor field of type (1, 2).

Equations (1.7) and (1.8) show that for a metric connection ∇ we have

$$(\nabla_X g)(Y, Z)=-g(T(X, Y), Z)-g(Y, T(X, Z)). \tag{1.9}$$

The connection ∇, in general, has a torsion, so we put

$$2S(X, Y)=\nabla_X Y-\nabla_Y X-[X, Y]. \tag{1.10}$$

Using $S(X, Y)$, we find that the metric connection ∇ satisfies

$$(1.11)\quad \begin{aligned} g(\nabla_Y Z, X) = g(\mathring{\nabla}_Y Z, X) + g(S(X, Y), Z) \\ + g(S(Y, Z), X) + g(S(X, Z), Y) \end{aligned}$$

If an almost Hermitian manifold satisfies

$$(1.12)\quad \mathring{\nabla} J = 0,$$

$$(1.13)\quad d\omega = 0,$$

$$(1.14)\quad (\mathring{\nabla}_X J)(Y) + (\mathring{\nabla}_Y J)(X) = 0$$

or

$$(1.15)\quad {}^*O(\mathring{\nabla} J)(X, Y) = 0,$$

we call the almost Hermitian manifold a Kaehlerian manifold, an almost Kaehlerian manifold, an almost Tachibana manifold or an almost $^*O$-manifold, respetively.

It is easily verified that a Kaehlerian manifold is an almost Kaehlerian manifold and is also an almost Tachibana manifold and that an almost Kaehlerian manifold and an almost Tachibana manifold are both almost $^*O$-manifolds. We also see that an almost $^*O$-manifold with vanishing Nijenhuis tensor is a Kaehlerian manifold. Examples of an almost $^*O$-manifold which is not almost Kaehlerian and not almost Tachibana are $E^4 \times S^2$ and $E^2 \times S^4$. These examples are given in [6].

2. A characterization of an almost *O-manifold.

Let M^{2n} be an almost Hermitian manifold and (J, g) the Hermitian structure. We call an affine connection ∇ satisfying $\nabla_X J = 0$, X being an arbitrary vector fiield, a J-connection. We need the following

LEMMA 2.1 [2]. *In an almost complex manifold, if the torsion tensor of a J-connection is proportional to the Nijenhuis tensor, then the proportional factor should be equal to* 1/8, *that is,*

$$(2.1)\quad S(X, Y) = \frac{1}{8} N(X, Y).$$

Now we suppose that there exists a metric J-connection whose torsion tensor $S(X, Y)$ is proportional to the Nijenhuis tensor $N(X, Y)$. Then, by the lemma above we have $S(X, Y) = (1/8)N(X, Y)$ and consequently

$$(2.2)\quad g(\nabla_Y Z, X) = g(\mathring{\nabla}_Y Z, X) + \frac{1}{8}\{g(N(Y, Z), X) + g(N(X, Y), Z) + g(N(X, Z), Y)\}.$$

Using this connection, we have

$$
\begin{aligned}
g((\nabla_X J)Y, Z) = g((\mathring{\nabla}_X J)Y, Z) \\
+\frac{1}{8}\{&g(N(X,JY),Z)+g(N(Z,X),JY)+g(N(Z,JY),X) \\
&+g(N(X,Y),JZ)+g(N(JZ,X),Y)+g(N(JZ,Y),X)\}=0.
\end{aligned}
\tag{2.3}
$$

On the other hand, using $\mathring{\nabla}_X Y-\mathring{\nabla}_Y X=[X,Y]$, we can write the Nijenhuis tensor as follows:

$$
N(X,Y)=J\mathring{\nabla}_Y J(X)-J\mathring{\nabla}_X J(Y)+(\mathring{\nabla}_{JX}J)Y-(\mathring{\nabla}_{JY}J)X. \tag{2.4}
$$

Thus we have

$$
\begin{aligned}
g((\nabla_X J)Y, Z) &= g((\mathring{\nabla}_X J)Y, Z) \\
&-\frac{1}{8}\{g(J\mathring{\nabla}_X J(JY)-J\mathring{\nabla}_{JY}J(X)-\mathring{\nabla}_Y J(X)-\mathring{\nabla}_{JX}J(JY),Z) \\
&+g(J\mathring{\nabla}_Z J(X)-J\mathring{\nabla}_X J(Z)+\mathring{\nabla}_{JX}J(Z)-\mathring{\nabla}_{JZ}J(X),JY) \\
&+g(J\mathring{\nabla}_Z J(JY)-J\mathring{\nabla}_{JY}J(Z)-\mathring{\nabla}_Y J(Z)-\mathring{\nabla}_{JZ}J(JY),X) \\
&+g(J\mathring{\nabla}_X J(Y)-J\mathring{\nabla}_Y J(X)+\mathring{\nabla}_{JY}J(X)-\mathring{\nabla}_{JX}J(Y),JZ) \\
&+g(J\mathring{\nabla}_{JZ}J(X)-J\mathring{\nabla}_X J(JZ)+\mathring{\nabla}_{JX}J(JZ)+\mathring{\nabla}_Z J(X),Y) \\
&+g(J\mathring{\nabla}_{JZ}J(Y)-J\mathring{\nabla}_Y J(JZ)+\mathring{\nabla}_{JY}J(JZ)+\mathring{\nabla}_Z J(Y),X)\} \\
&= g((\mathring{\nabla}_X J)Y, Z) \\
&-\frac{1}{8}\{g(J\mathring{\nabla}_X J(JY)-J\mathring{\nabla}_{JY}J(X)-\mathring{\nabla}_Y J(X)-\mathring{\nabla}_{JX}J(JY),Z) \\
&-g(-\mathring{\nabla}_Z J(X)+\mathring{\nabla}_X J(Z)+J\mathring{\nabla}_{JX}J(Z)-J\mathring{\nabla}_{JZ}J(X),Y) \\
&+g(J\mathring{\nabla}_Z J(JY)-J\mathring{\nabla}_{JY}J(Z)-\mathring{\nabla}_Y J(Z)-\mathring{\nabla}_{JZ}J(JY),X) \\
&-g(-\mathring{\nabla}_X J(Y)+\mathring{\nabla}_Y J(X)+J\mathring{\nabla}_{JY}J(X)-J\mathring{\nabla}_{JX}J(Y),Z) \\
&+g(J\mathring{\nabla}_{JZ}J(X)-J\mathring{\nabla}_X J(JZ)+\mathring{\nabla}_{JX}J(JZ)+\mathring{\nabla}_Z J(X),Y) \\
&+g(J\mathring{\nabla}_{JZ}J(Y)-J\mathring{\nabla}_Y J(JZ)+\mathring{\nabla}_{JY}J(JZ)+\mathring{\nabla}_Z J(Y),X)\}.
\end{aligned}
$$

Since $\mathring{\nabla}_X 1=-\mathring{\nabla}_X J\cdot J-J\cdot\mathring{\nabla}_X J=0$, the equation above reduces to

$$
\begin{aligned}
g((\nabla_X J)Y, Z) = g((\mathring{\nabla}_X J)Y, Z) \\
-\frac{1}{4}\{&g(\mathring{\nabla}_X J(Y),Z)+g(\mathring{\nabla}_{JY}J(X),JZ)-g(\mathring{\nabla}_Y J(X),Z) \\
&-g(\mathring{\nabla}_{JX}J(Y),JZ)-g(\mathring{\nabla}_X J(Z),Y)-g(\mathring{\nabla}_{JZ}J(X),JY) \\
&+g(\mathring{\nabla}_{JX}J(Z),JY)+g(\mathring{\nabla}_Z J(X),Y)+g(\mathring{\nabla}_Z J(Y),X) \\
&+g(\mathring{\nabla}_{JY}J(Z),JX)-g(\mathring{\nabla}_Y J(Z),X)-g(\mathring{\nabla}_{JZ}J(Y),JX)\}.
\end{aligned}
\tag{2.5}
$$

We need to get further results

LEMMA 2.2. *For any* $X, Y, Z \in T(M)$, *we have*

$$g(\mathring{\nabla}_Z J(Y), X) = -g(\mathring{\nabla}_Z J(X), Y), \tag{2.6}$$

$$g((\mathring{\nabla}_{JZ} J)X, JY) = -g((\mathring{\nabla}_{JZ} J)Y, JX) \tag{2.7}$$

Proof. Differentiating

$$g(J(X), Y) = -g(X, J(Y)) \tag{2.8}$$

covariantly, we have

$$\begin{aligned} &g(\mathring{\nabla}_Z J(X), Y) + g(J(\mathring{\nabla}_Z X), Y) + g(J(X), \mathring{\nabla}_Z Y) \\ &= -g(\mathring{\nabla}_Z X, J(Y)) - g(X, \mathring{\nabla}_Z J(Y)) - g(X, J(\mathring{\nabla}_Z Y)). \end{aligned}$$

Thus, using (2.8) in the above, we have (2.6).

On the other hand, by (2.6)

$$\begin{aligned} g((\mathring{\nabla}_{JZ} J)X, JY) &= -g(X, (\mathring{\nabla}_{JZ} J)JY) \\ &= g(X, J(\mathring{\nabla}_{JZ} J)Y) = -g((\mathring{\nabla}_{JZ} J)Y, JX), \end{aligned}$$

which proves (2.7).

Making use of (2.6) and (2.7), we can rewrite (2.5) as follows:

$$\begin{aligned} g((\nabla_X J)Y, Z) &= g((\mathring{\nabla}_X J)Y, Z) - \frac{1}{2}\{g((\mathring{\nabla}_X J)Y, Z) - g(\mathring{\nabla}_{JX} J(Y), JZ)\} \\ &= \frac{1}{2}\{g((\mathring{\nabla}_X J)Y, Z) + g(\mathring{\nabla}_{JX} J(JY), Z)\} \\ &= \frac{1}{2}\{g(\mathring{\nabla} J(X, Y), Z) + g(\mathring{\nabla} J(JX, JY), Z)\} \\ &= \frac{1}{2} g({}^*O(\mathring{\nabla} J)(X, Y), Z). \end{aligned}$$

Thus, if the connection ∇ is a J-connection, we have

$$^*O(\mathring{\nabla} J) = 0.$$

This shows that, if there exists a metric J-connection whose torsion tensor is proportional to the Nijenhuis tensor, then the almost Hermitian manifold must be an almost $^*O$-manifold.

Conversely, in an almost $^*O$-manifold, we consider the connection defined by (2.2). Then this is a metric J-connection whose torsion tensor is proportional to the Nijenhuis tensor. Thus we get

THEOREM 2.3. *In order that an almost Hermitian manifold M is an almost *O-manifold it is necessary and sufficient that there exists in M a metric J-connection whose torsion tensor is proportional to the Nijenhuis tensor.*

Since an almost *O-manifold with vanishing Nijenhuis tensor is a Kaehlerian manifold, as a special case of Theorem 2.3, we have the following well known result.

COROLLARY 2.4 [5]. *In order that an almost Hermitian manifold M is a Kaehlerian manifold it is necessary and sufficient that there exists in M a symmetric metric J-connection.*

3. Metric J-connection in S^6 as an almost Tachibana manifold.

We take a seven dimensional Euclidean space E^7 and consider it as the space of pure imaginary parts of Cayley numbers. In such E^7 we consider a hypersphere S^6. Then, it is well known that the S^6 is an almost Tachibana manifold, which is not Kaehlerian. The almost Tachibana structure on S^6 has been studied by Fukami and Ishihara [1]. They introduced on S^6 a metric J-connection defined by

$$\nabla_X Y = \mathring{\nabla}_X Y + \frac{1}{2}(\mathring{\nabla}_{JY} J)X. \tag{3.1}$$

In the following, we shall show that this connection is identical with the connection introduced by (2.2).

The torsion tensor $S(X, Y)$ of the connection defined by (3.1) is given by

$$\begin{aligned} 2S(X, Y) &= \nabla_X Y - \nabla_Y X - [X, Y] \\ &= \frac{1}{2}((\mathring{\nabla}_{JY} J)X - (\mathring{\nabla}_{JX} J)Y). \end{aligned} \tag{3.2}$$

On the other hand, using (2.6), we get

$$\begin{aligned} N(X, Y) &= J\mathring{\nabla}_Y J(X) - J\mathring{\nabla}_X J(Y) + (\mathring{\nabla}_{JX} J)Y - (\mathring{\nabla}_{JY} J)X \\ &= \mathring{\nabla}_X J(JY) - \mathring{\nabla}_Y J(JX) + (\mathring{\nabla}_{JX} J)Y - (\mathring{\nabla}_{JY} J)X. \end{aligned} \tag{3.3}$$

Since S^6 is an almost Tachibana manifold, substituting JY in (1.4) for Y, we have

$$\mathring{\nabla}_X J(JY) = -(\mathring{\nabla}_{JY} J)X$$

and

$$\mathring{\nabla}_Y J(JX) = -(\mathring{\nabla}_{JX} J)Y.$$

Thus, in an almost Tachibana manifold, we get

$$N(X, Y) = 2((\mathring{\nabla}_{JX} J)Y - (\mathring{\nabla}_{JY} J)X). \tag{3.4}$$

Comparing (3.2) and (3.4), we find

$$S(X, Y) = \frac{1}{8} N(X, Y). \tag{3.5}$$

The connection ∇ being metric J-connection, this relation, together with theorem 2.3, shows that ∇ is identical with the connection introduced by (2.2).

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