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REMARKS ON EXCEPTIONAL VALUES OF MEROMORPHIC FUNCTIONS

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§1. In the present paper we are concerned with exceptional values of meromorphic functions. Throughout this paper we use the well-known symbols in Nevanlinna's theory.

Let $f(z)$ be a meromorphic function of order ρ (finite positive or infinite). A number A (finite or infinite) is said to be a Borel exceptional value of $f(z)$ if either the exponent of convergence of the A-points, $\rho(A)$, is less than ρ for $\rho<+\infty$ or $\rho(A)<+\infty$ for $\rho=+\infty$.

Valiron [8] had proved the following

THEOREM A. *Let $f(z)$ be a meromorphic function of finite order ρ. If two numbers A and B are Borel exceptional values of $f(z)$, then $\delta(A,f)=\delta(B,f)=1$ and $f(z)$ is completely regular growth and ρ is a positive integer. Further A and B are asymptotic values of $f(z)$.*

Here we note that it follows from Edrei and Fuchs [2] that A and B are two asymptotic values in the last part of Theorem A. Also Cartwright [1] has shown that for entire functions the similar theorem as above holds.

On the other hand for an arbitrary ρ, $1<\rho\leqq+\infty$, Goldberg [3] has constructed a meromorphic function $f(z)$ of order ρ, for which $\delta(\infty,f)=1$ and ∞ is not an asympotic value. Morover the ratio $T(r,f)/r^{\rho}$ for any $r>r_0$ is bounded from above and from below by positive constants, if $1<\rho<+\infty$ and $\log r=o\{\log T(r,f)\}$ if $\rho=\infty$; while $N(r,\infty,f)\sim Cr^{\beta}$, $\rho/(2\rho-1)<\beta<1$, $0<C<+\infty$. Thus ∞ is also a non-asymptotic Borel exceptional value. From this example we see that A is not always an asymptotic value, when a meromorphic function $g(z)$ has only one Borel exceptional value A.

We shall say in the sequel that a set $\{\Gamma_n\}$ is a sequence of arcs if it satisfies the following conditions:

(1) $\{\Gamma_n\}$ is a countable set of arcs.

(2) $\Gamma_i\cap\Gamma_j=\phi$ for $i\neq j$ if $n\geqq 2$.

(3) For an arbitrary $r>0$ there exist one arc Γ_n or two arcs Γ_m and Γ_{m+1} such that, for some θ, $0\leqq\theta\leqq 2\pi$,

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$$\Gamma_n \ni z = re^{i\theta}$$

or for some θ_1 and θ_2, $0 \leqq \theta_1, \theta_2 \leqq 2\pi$,

$$\Gamma_m \ni z = re^{i\theta_1} \quad \text{and} \quad \Gamma_{m+1} \ni z = re^{i\theta_2}, \text{ respectively.}$$

Then we shall prove the followings.

THEOREM 1. *Let $f(z)$ be a meromorphic function of lower order μ. If a number A is a Borel exceptional value of $f(z)$ such that $\rho(A) < \mu$, then there exists a sequence of arcs $\{\Gamma_n\}$ such that*

$$\lim_{\substack{z\to\infty \\ z\in\cup\Gamma_n}} f(z) = A \qquad (\textit{uniformly}).$$

THEOREM 2. *Let $f(z)$ be a meromorphic function of non-integral finite order and of very regular growth, i.e.,*

$$0 < \liminf_{r\to\infty} \frac{T(r,f)}{r^\rho} \leqq \limsup_{r\to\infty} \frac{T(r,f)}{r^\rho} < +\infty.$$

If $\delta(A,f) = 1$, then there exists a sequence of arcs $\{\Gamma_n\}$ such that

$$\lim_{\substack{z\to\infty \\ z\in\cup\Gamma_n}} f(z) = A \qquad (\textit{uniformly}).$$

Goldberg's example shows that the sequence of arcs $\{\Gamma_n\}$ in Theorem 1 or Theorem 2 cannot be replaced by a suitable curve.

We note that if a number A is an asymptotic value of $f(z)$, then there exists a sequence of arcs $\{\Gamma_n\}$ such that

$$\lim_{\substack{z\to\infty \\ z\in\cup\Gamma_n}} f(z) = A.$$

Thus the number of such values is at least that of asymptotic values. However for meromorphic functions of lower order less than 1/2 we have the following corollary.

COROLLARY. *Under the assumption of Theorem* 1 (*or Theorem* 2) *if $f(z)$ is of lower order μ, $\mu < 1/2$, then the value A is a unique value for which there exists a sequence of arcs $\{\Gamma_n\}$ in Theorem* 1 (*or Theorem* 2, *respectively*).

We do not know whether for entire functions there exists a non-asymptotic value for which a sequence of arcs exists.

2. Lemmas. From a theorem on the maximum modulus of an entire function in Varilon [7] we have the following

LEMMA 1. *If $g(z)$ is an entire function, then there exists a sequence of arcs $\{\Gamma_n\}$ such that*

$$|g(z)| = M(r, g) \qquad \textit{for any} \qquad z = re^{i\theta} \in \cup\Gamma_n,$$

where $M(r, g)=\max_{|z|=r} |g(z)|$.

Hardy [4] had constructed examples showing that the curve of the maximum modulus can actually show discontinuities.

The followings are well known.

LEMMA 2. ([5]). *Let $f(z)$ be an entire function. Then*

$$T(r, f)\leqq \log M(r, f)\leqq 3T(2r, f).$$

LEMMA 3. *Let $f(z)$ be a meromorphic function of order ρ and of lower order μ. If $\rho<+\infty$, then* $\lim T(r, f)/r^{\lambda}=0$ *for any $\lambda>\rho$. If $\mu>0$, then* $\lim T(r, f)/r^{\lambda}=+\infty$ *for any $\lambda<\mu$.*

LEMMA 4. ([5]). *Let a_{ν} be a sequence of non-zero complex number and let q the least integer such that $\sum_{\nu=1}^{\infty}|a_{\nu}|^{-q}$ converges. Then the product $\Pi_{\nu=1}^{\infty}E(z/a_{\nu}, q-1)$ converges absolutely and uniformly in any bounded part of the plane to an entire function $\pi(z)$ having the same order ρ as the sequence a_{ν} and the same type-class if ρ is not an integer.*

By Ostrovskii [6] we have the following

LEMMA 5. *Let $f(z)$ be a meromorphic function of lower order μ ($\mu<1/2$). If $\delta(\infty, f)>1-\cos \pi\mu$, then there exists a sequence of circles $|z|=r_n$ ($r_n\to\infty$), on which the function $f(z)$ uniformly converges to infinity.*

3. Proof of Theorem 1. From Lemma 4 we can construct an entire function $E(z)$ of order $\rho(A)$ such that $\{f(z)-A\}/E(z)$ has no zeros. We put

$$f(z)-A=\frac{E(z)}{R(z)}, \tag{3.1}$$

where $R(z)$ is entire and of lower order μ because of our assumption.

We apply Lemma 1 to $R(z)$. Then there exists a sequence of arcs $\{\Gamma_n\}$ such that

$$M(r, R)=|R(z)| \qquad \text{for any} \quad z\in \cup \Gamma_n.$$

Hence it follows from (3.1) and Lemma 2 that, for large $r=|z|$ and $z\in \cup\Gamma_n$,

$$|f(z)-A|\leqq \frac{M(r, E)}{M(r, R)}\leqq e^{-T(r,R)+3T(2r,E)}. \tag{3.2}$$

Further we have by Lemma 3

$$\begin{aligned} -T(r, R)+3T(2r, E)&=-T(r, R)\left\{1-3\cdot\frac{T(2r, E)}{T(r, R)}\right\} \\ &=-T(r, R)\left\{1-3\cdot 2^{\lambda}\cdot\frac{T(2r, E)}{(2r)^{\lambda}}\cdot\frac{r^{\lambda}}{T(r, R)}\right\} \qquad (3.3) \\ &\to -\infty \end{aligned}$$

as $r\to+\infty$, for $\rho(A)<\lambda<\mu$ since $\rho(A)<\mu\leqq\infty$. Thus by (3. 2) and (3. 3) we have

$$|f(z)-A|\to 0 \quad \text{as} \quad z\to\infty, \quad z\in\bigcup\Gamma_n.$$

Hence the proof of Theorem 1 is completed.

4. Proof of Theorem 2. Since $\delta(A,f)=1$ and $f(z)$ is of very regular growth we have

$$\lim_{r\to\infty}\frac{N(r, A, f)}{r^{\rho}}=0. \tag{4. 1}$$

If $\rho(A)<\rho$, by Theorem 1 there is nothing to prove. Thus we may assume that $\rho(A)$ is not an integer. Hence by Lemma 4 we can construct an entire function $E(z)$ such that $\{f(z)-A\}/E(z)$ has no zeros and by (4. 1)

$$\lim_{r\to\infty}\frac{T(r, E)}{r^{\rho}}=0 \qquad (\rho=\rho(A)). \tag{4. 2}$$

Thus by the same discussion as in the proof of theorem 1 with (4. 1), (4. 2) and our assumption $\liminf_{r\to\infty} T(r,f)/r^{\rho}>0$ we have Theorem 2.

5. Proof of Corollary. By our assumption we have $\delta(A,f)=1$. If $|A|=+\infty$, by Lemma 5 Corollary is valid. If $|A|<+\infty$, then we consider $f(z)=1/\{f(z)-A\}$ instead of $f(z)$. We also have $\delta(\infty, F)=1$, so that by Lemma 5 Corollary is valid.

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