KŌDAI MATH. SEM. REP.
21 (1969), 233—235

A NOTE ON PSEUDO-UMBILICAL SUBMANIFOLDS WITH *M*-INDEX 1 AND CODIMENSION 2 IN EUCLIDEAN SPACES

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In the proof of the case $k_2 \neq 0$ of Theorem 3 in [2], the author made a mistake by using the Gauss' lemma. In this note he will show that the same results holds. We rewrite the related part of the theorem.

THEOREM. *Let M^n $(n \geqq 3)$ be an n-dimensional submanifold in $(n+2)$-dimensional Euclidean space E^{n+2} which is pseudo-umbilical and of M-index 1 and whose second curvature is not zero everywhere. Then M^n is a locus of a moving $(n-1)$-sphere $S^{n-1}(v)$ depending on a parameter v such that the radius is not constant, the locus of the centor has the tangent direction orthogonal to the tangeht space to M^n at the corresponding point and intersects obliquely the n-dimensional linear subspace containing $S^{n-1}(v)$, and $S^{n-1}(v)$ is umbilical in M^n.*

Proof. Using the notations §§ 1, 2 in [2], let $k_1(p)$ and $k_2(p)$ be the first and second curvatures at p of M^n in E^{n+2}. Let ψ: $M^n \to E^{n+2}$ be the mapping defined by

$$q = \psi(p) = p + \frac{1}{k_1(p)} \bar{e}(p), \tag{1}$$

where $\bar{e}(p)$ is the mean curvature unit vector at p. Making use of the frame $(p, e_1, \cdots, e_{n+2})$ such that

$$\omega_{i n+1} = k_1 \omega_i, \qquad \omega_{n+1, n+2} = k_2 \omega_n, \tag{2}$$

where $k_1 \neq 0$, $k_2 \neq 0$ by the assumption. Differentiating the second of (2) and using them and the structure equation of M^n, we get

$$d\omega_n = -d \log k_2 \wedge \omega_n, \tag{3}$$

which shows that the Pfaff equation

$$\omega_n = 0 \tag{4}$$

is completely integrable. Let $Q(v)$ be the integral hypersurface of (4) depending on a parameter v. Then, we put

$$\omega_n = f dv, \tag{5}$$

Received November 4, 1968.

where f is a function defined on some neighborhood in M^n. From (3) and (5), we see that

$$\bar{k}_2 = k_2 f \tag{6}$$

depends on v only. Differentiating (1), we get

$$dq = -\frac{dk_1}{k_1^2}\bar{e} + \frac{\bar{k}_2 dv}{k_1} e_{n+2}. \tag{7}$$

This shows that $\psi(M^n)$ is generally two-dimensional and $dk_1=0$ along $Q(v)$ by (2) and $n \geqq 3$. Hence k_1 depends on v only. Therefore, the image of $Q(v)$ by the mapping ψ is a point which is denoted by $q=q(v)$. $Q(v)$ is contained in a hypersphere $S^{n+1}(v)$ in E^{n+2} with center $q(v)$ and radius $1/k_1(v)$. Therefore, (7) can be written as

$$\frac{dq}{dv} = \frac{\bar{k}_2}{k_1} e_{n+2} - \frac{k_1'}{k_1^2}\bar{e}.$$

Differentiating the first of (2) and using them, we get

$$\omega_n \wedge \omega_{in+2} = \frac{k_1'}{\bar{k}_2}\omega_n \wedge \omega_i,$$

in which substituting $\omega_{in+2} = \sum_j A_{n+2ij}\omega_j$, we get

$$A_{n+2ab} = \frac{k_1'}{\bar{k}_2}\delta_{ab}, \qquad A_{n+2nb}=0, \qquad A_{n+2nn} = -\frac{(n-1)k_1'}{\bar{k}_2} \qquad (a, b=1, 2, \cdots, n-1) \tag{8}$$

by $\bar{m}(A_{n+2})=0$. Since M-index$=1$, $A_{n+2} \neq 0$ and

$$k_1' = \frac{dk_1}{dv} \neq 0. \tag{9}$$

Now consider a vector field with the domain of v defined by

$$X = k_1^2 \frac{dq}{dv} = k_1\bar{k}_2 e_{n+2} - k_1' e_{n+1}, \tag{10}$$

which has the tangent direction of the locus of $q(v)$, depends on v only, and is orthogonal to the tangent space M_p^n, $p \in Q(v)$. From (2), (8) and (10), we get

$$X' = \frac{dX}{dv} = k_1\bar{k}_2' e_{n+2} - (k_1'' + k_1\bar{k}_2^2)e_{n+1} + nk_1k_1' f e_n. \tag{11}$$

This shows that X' is linearly independent of X and normal to the tangent space to $Q(v)$. Since X and X' are constant along $Q(v)$, there exist two linear subspaces $E_1^{n+1}(v)$ and $E_2^{n+1}(v)$ which are orthogonal to $X(v)$ and $X'(v)$ respectively and contain $Q(v)$. Since e_{n+1}, $X(v)$ and $X'(v)$ are linearly independent, we can put

$$S^{n-1}(v)=S^{n+1}(v)\cap E_1^{n+1}(v)\cap E_2^{n+1}(v)$$

and $Q(v)$ is contained in $S^{n-1}(v)$. $k_1'\neq 0$ and (10) show that the curve $q(v)$ intersects obliquely the n-dimensional linear subspaces containing $S^{n-1}(v)$.

Lastly, we consider the second fundamental form of $Q(v)$ as a hypersurface of M^n. Differentiating (11) along $Q(v)$ and using (2) and (8), we get

$$0=k_1\bar{k}_2'\sum \omega_{n+2a}e_a-(k_1''+k_1\bar{k}_2^2)\sum \omega_{n+1a}e_a+nk_1k_1'f\sum \omega_{na}e_a+nk_1k_1'dfe_n$$

$$=\{-k_1k_1'(\log \bar{k}_2)'+k_1(k_1''+k_1\bar{k}_2^2)\}\sum \omega_a e_a+nk_1k_1'f\sum \omega_{na}e_a+nk_1k_1'dfe_n.$$

Hence we have $df=0$ along $Q(v)$, which shows that the function f may be consider as a function of v only, therefore we may put $f=1$. Then we have

$$\omega_{an}=\frac{1}{n}\left\{-\frac{k_2'}{k_2}+\frac{k_1''+k_1k_2^2}{k_1'}\right\}\omega_a$$

from the above equality, which shows that $Q(v)$ is umbilical in M^n. q.e.d.

References

[1] Ōtsuki, T., A theory of Riemannian submanifolds. Kōdai Math. Sem. Rep. **20** (1968), 282–295.

[2] Ōtsuki, T., Pseudo-umbilical submanifolds with M-index 1 in Euclidean spaces. Kōdai Math. Sem. Rep. **20** (1968), 296–304.

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