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PROLONGATIONS OF HYPERSURFACES TO TANGENT BUNDLES

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Introduction.

The prolongations of tensor fields and connections to tangent bundles have been recently discussed in [1], [2], [3] and [4]. Ishihara, Kobayashi and Yano defined and studied prolongations called complete, vertical and horizontal lifts of tensor fields and connections. In this paper we introduce the notion of prolongations of surfaces to tangent bundle, which seems to be a natural one, and develop the theory of surfaces prolonged to the tangent bundle with respect to the metric tensor which is the complete lift of the metric tensor of the original manifold. We shall define in §3 the vertical and the complete lifts of the vector fields defined along the surface, and choose two kinds of lifts of the normal vector field of the surface as vector fields normal to the prolonged surface.

We shall recall in §4 some formulas for surfaces for the later use and give, for prolonged surfaces, some of fundamental formulas containing the so-called second fundamental tensors in §5. In the last section the equations of Gauss, of Weingarten, and the so-called structure equations, those of Gauss, of Codazzi and of Ricci, for the prolonged surface are formulated in the form of lifts of the corresponding equations of the surface given in the base space.

§1. Notations.

For any differentiable manifold N, we denote by $T(N)$ its tangent bundle with the projection π_N: $T(N)\to N$, and by $T_p(N)$ its tangent space at a point p of N. $\mathcal{T}^r_s(N)$ is the space of tensor fields of class C^∞ and of type (r, s), i.e., of contravariant degree r and covariant degree s in N. An element of $\mathcal{T}^0_0(N)$ is a C^∞-function defined on N. We denote by $\mathcal{T}(N)$ the tensor algebra on N, i.e., $\mathcal{T}(N)=\sum_{r,s}\mathcal{T}^r_s(N)$.

Let M be an n-dimensional differentiable manifold and V a coordinate neighborhood in M and (x^i) certain local coordinates defined in V. We introduce a system of coordinates (x^i, y^i) in $\pi_M^{-1}(V)$ such that (y^i) are cartesian coordinates in each tangent space $T_p(M)$, p being an arbitrary point of V, with respect to the natural frame $(\partial/\partial x^i)$ of local coordinates (x^i). We call (x^i, y^i) the coordinates induced in $\pi_M^{-1}(V)$ from (x^i), or simply the induced coordinates in $\pi_M^{-1}(V)$. (cf. [3], [4]).

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Let S be a manifold of dimension $n-1$ and ι: $S \to M$ its imbedding. The differential mapping $d\iota$ is a mapping from $T(S)$ into $T(M)$, which is called the tangential map of ι. We denote sometimes by B the tangential map $d\iota$. Then the mapping B induces its tangential map dB: $T(T(S)) \to T(T(M))$, which is denoted sometimes by $\tilde{B}$. $T(S)$ is a submanifold of dimension $2(n-1)$ in $T(M)$ and $T(T(S))$ a submanifold of dimension $4(n-1)$ in $T(T(M))$. If we put

$$T(S, M) = \pi_M^{-1}(\iota(S)),$$

$$T(T(S), T(M)) = \pi_{T(M)}^{-1}(B(T(S))),$$

we have

$$\begin{aligned} \iota: &\qquad S \to M, \\ d\iota = B: &\qquad T(S) \to T(S, M) \subset T(M), \\ dB = \tilde{B}: &\ T(T(S)) \to T(T(S), T(M)) = T(T(S, M)) \subset T(T(M)). \end{aligned}$$

In terms of local coordinates (x^i), ι has local expressions

$$x^i = x^i(u^a),$$

where (u^a) are local coordinates of S. Then B has local expressions

$$\begin{cases} x^i = x^i(u^a), \\ y^i = B_a{}^i v^a, \qquad B_a{}^i = \partial x^i / \partial u^a \end{cases}$$

with respect to the local coordinates (x^i, y^i), and (u^a, v^a) induced from (x^i) and (u^a), respectively. In the sequel we sometimes identify S with the image $\iota(S)$ and $T(S)$ with the image $B(T(S))$, respectively.

As for the tensor algebra, if we denote by $\mathcal{T}(S, M)$ the tensor algebra associated with $T(S, M)$, B induces an isomorphism from $\mathcal{T}(S)$ into $\mathcal{T}(S, M)$. Similarly dB induces an isomorphism from $\mathcal{T}(T(S))$ to $\mathcal{T}(T(S, M))$. A mapping $\bar{X}$ which assigns to each $p \in S$ a tangent vector at p of M,

$$\bar{X}_p \in T_p(M)$$

is called a vector field defined along S. $\mathcal{T}_0^1(S, M)$ is nothing but the set of all vector fields along S. Similarly a mapping $\bar{T}$ which assigns to each $p \in S$ a tensor of type (r, s) at p

$$\bar{T}_p \in T_p(M) \otimes \cdots \otimes T_p(M) \otimes T_p^*(M) \otimes \cdots \otimes T_p^*(M),$$

is called a tensor field of type (r, s) along S, where $T_p^*(M)$ is the dual space of $T_p(M)$. We denote by $\mathcal{T}_s^r(S, M)$ the space of tensor fields of type (r, s) along S, then we see that $\mathcal{T}_0^0(S, M) = \mathcal{T}_0^0(S)$. In the following, elements of $\mathcal{T}(S)$ are denoted by f, X, ω and so on. On the other hand, elements of $\mathcal{T}(S, M)$ are denoted by $\bar{f}, \bar{X}, \bar{\omega}$ and so on.

In the next section, two kinds of isomorphisms of $\mathcal{T}(M)$ into $\mathcal{T}(T(M))$ or $\mathcal{T}(S)$ into $\mathcal{T}(T(S))$, which are called vertical and complete lifts, will be given. We shall give in §3 two kinds of isomorphisms of $\mathcal{T}(S, M)$ into $\mathcal{T}(T(S, M))$, so-called vertical and complete lifts from $\mathcal{T}(S, M)$ to $T(M)$.

§2. Vertical and complete lifts.

Let f, X, ω and F be a function, a vector field, a 1-form and a tensor field of type (1, 1) in M, respectively. We denote respectively by f^V, X^V, ω^V and F^V, their vertical lifts and by f^C, X^C, ω^C and F^C their complete lifts. For a function f in M, we have by definition

$$f^V=f\circ\pi_M \tag{2.1}$$

and

$$f^C=y^i\partial_i f \qquad (\partial_i=\partial/\partial x^i) \tag{2.2}$$

with respect to the induced coordinates. Moreover these lifts have the properties:

$$\begin{gathered}(fX)^V=f^VX^V, \qquad (fX)^C=f^CX^V+f^VX^C,\\ X^Vf^V=0, \qquad X^Vf^C=X^Cf^V=(Xf)^V, \qquad X^Cf^C=(Xf)^C,\\ \omega^V(X^V)=0, \qquad \omega^V(X^C)=\omega^C(X^V)=\omega(X)^V, \qquad \omega^CX^C=\omega(X)^C,\\ F^VX^C=(FX)^V, \qquad F^CX^C=(FX)^C,\\ [X, Y]^C=[X^C, Y^C], \qquad [X, Y]^V=[X^C, Y^V]=[X^V, Y^C]\end{gathered} \tag{2.3}$$

(cf. [3], [4]).

For a tensor field of the form $T=P\otimes Q$ where P and Q are arbitrary tensor fields, its vertical and complete lifts are given respectively by

$$\begin{gathered}T^V=(P\otimes Q)^V=P^V\otimes Q^V,\\ T^C=(P\otimes Q)^C=P^C\otimes Q^V+P^V\otimes Q^C\end{gathered} \tag{2.4}$$

(cf. [3], [4]). For the later use we note here the following formulas (cf. [3], [4]):

$$\begin{gathered}T^C(X^V, Y^C)=T(X, Y)^C,\\ T^C(X^C, Y^V)=T^V(X^C, Y^C)=T(X, Y)^V, \qquad T\in\mathcal{T}^0_2(M),\\ T^C(X^C, Y^C)=T^V(X^V, Y^C)=T^V(X^V, Y^V)=0,\end{gathered} \tag{2.5}$$

X and Y being arbitrary vector fields in M.

REMARK. Let $\overset{*}{X}$ and $\overset{*}{Y}$ be vector fields on $T(M)$ such that $\overset{*}{X}\hat{f}^C=\overset{*}{Y}\hat{f}^C$ for all

$\hat{f}\in\mathcal{T}_0^0(M)$, then $\overset{*}{X}=\overset{*}{Y}$. Let $\overset{*}{\omega}$ and $\overset{*}{\eta}$ be 1-forms on $T(M)$ such that $\overset{*}{\omega}(\hat{X}^C)=\overset{*}{\eta}(\hat{X}^C)$ for all $\hat{X}\in\mathcal{T}_0^1(M)$, then $\overset{*}{\omega}=\overset{*}{\eta}$. Consequently any tensor field on $T(M)$ are completely determined by its action on the set of $\hat{f}^C$ and $\hat{X}^C$, $\hat{f}$ and $\hat{X}$ being arbitrary elements of $\mathcal{T}_0^0(M)$ and $\mathcal{T}_0^1(M)$, respectively (cf. [5]).

§ 3. Vertical and complete lifts of $\mathcal{T}_s^r(S, M)$ to $T(M)$.

We define now the lifts of elements of $\mathcal{T}_s^r(S, M)$ to $T(M)$. Let $\bar{f}$ be a function defined on S. The vertical lift $\bar{f}^{\bar{V}}$ of $\bar{f}$ to $T(M)$ is defined by

$$\bar{f}^{\bar{V}}=\bar{f}\circ\pi_S.$$

In order to define the complete lift, for an arbitrary point p of S, we consider a sufficiently small neighborhood U of p in M. In U we can construct a function $\hat{f}$ such that $\hat{f}$ coincides with $\bar{f}$ on the connected component $(U\cap S)^0$ in $U\cap S$ containing p. We remark that a local extension $\hat{f}$ satisfies $\partial_a\hat{f}=\partial_a\bar{f}$ along $(U\cap S)^0$.

Then the complete lift of $\hat{f}$ to $\pi_M^{-1}(U)$ is defined as

$$\hat{f}^C=y^i\partial_i\hat{f}$$

in the local coordinates in $\pi_M^{-1}(U)$. We see that the restriction of $\hat{f}^C$ to $\pi_M^{-1}((U\cap S)^0)$ is independent of the choice of $\hat{f}$. In fact if we denote by $\sharp$ the operation of taking restrictions to $\pi_M^{-1}((U\cap S)^0)$, we have

$$\sharp\hat{f}^C=\sharp(y^i\partial_i\hat{f})=v^aB_a{}^i\partial_i\hat{f}=v^a\partial_a\hat{f}=v^a\partial_a\bar{f} \tag{3. 2}$$

in $\pi_M^{-1}((U\cap S)^0)$. Then we can define a function which coincides with $\sharp\hat{f}^C$ in each coordinates neighborhood. We denote it by $\bar{f}^{\bar{C}}$ and call the complete lift of $\bar{f}$ to $T(M)$.

Let $\bar{X}$ be an element of $\mathcal{T}_0^1(S, M)$. Then $\bar{X}_p$ being a tangent vector at $p\in S$ we shall define the vertical lift $\bar{X}^{\bar{V}}$ to $T(M)$ by

$$\bar{X}^{\bar{V}}\hat{f}^C=(\bar{X}\hat{f})^{\bar{V}} \tag{3. 3}$$

and the complete lift to $T(M)$ by

$$\bar{X}^C\hat{f}^C=(\bar{X}\hat{f})^{\bar{C}} \tag{3. 4}$$

along S, where $\hat{f}$ is an arbitrary element of $\mathcal{T}_0^0(M)$. We see easily that this definition is equivalent to the one that $\bar{X}^{\bar{C}}$ is defined to be the restriction on $\pi_M^{-1}((U\cap S)^0)$ of $\hat{X}^C$, where $\hat{X}$ is a local extension of $\bar{X}$ in U. To see this, it is sufficient to prove

$$\bar{X}^{\bar{C}}\hat{f}^C=(\sharp\hat{X}^C)\hat{f}^C \qquad \text{for} \quad \hat{f}\in\mathcal{T}_0^0(M)$$

because of Remark in § 2. For an arbitrary vector field $\bar{X}$, $\bar{X}\hat{f}$ is a function which takes the value

$$(\bar{X}\bar{f})_p = \bar{X}_p \hat{f}.$$

at p. Therefore we have

$$\bar{X}^{\bar{C}} \hat{f}^C = (\bar{X}\hat{f})^{\bar{C}} = \#(\hat{X}\hat{f})^C = \#(\hat{X}^C \hat{f}^C) = (\#\hat{X}^C)\hat{f}^C.$$

Let $\bar{\omega}$ be an element of $\mathcal{T}_1^0(S, M)$. We define the vertical lift $\bar{\omega}^{\bar{V}}$ to $T(M)$ by

$$\bar{\omega}^{\bar{V}}(\bar{X}^{\bar{C}}) = \bar{\omega}(\bar{X})^{\bar{V}} \tag{3.5}$$

and the complete lift $\bar{\omega}^{\bar{C}}$ to $T(M)$ by

$$\bar{\omega}^{\bar{C}}(\bar{X}^{\bar{C}}) = \bar{\omega}(\bar{X})^{\bar{C}} \tag{3.6}$$

$\bar{X}$ being an arbitrary element of $\mathcal{T}_0^1(S, M)$. If we consider a 1-form $\hat{\omega}$ in a sufficiently small neighborhood U of p such that $\hat{\omega}(\hat{X})$ is a local extension of $\bar{\omega}(\bar{X})$, i.e.,

$$\hat{\omega}(\hat{X})|_{(U\cap S)^0} = \bar{\omega}(\bar{X})$$

where $\hat{X}$ is a local extension of $\bar{X}$. We call such $\hat{\omega}$ a local extension of $\bar{\omega}$ to U. Then we have the complete lift $\bar{\omega}^{\bar{C}}$

$$\bar{\omega}^{\bar{C}}(\bar{X}^{\bar{C}}) = \bar{\omega}(\bar{X})^{\bar{C}} = \#(\hat{\omega}(\hat{X})^C) = \#(\hat{\omega}^C(\hat{X}^C))$$

in $\pi_M^{-1}((U\cap S)^0)$, since $\bar{\omega}(\bar{X})^{\bar{C}}$ is defined by the restriction of the complete lift of a local extension. That is, $\hat{\omega}^C$ is a local extension of $\bar{\omega}^{\bar{C}}$ in the above sense.

We now extend these lifts to a linear mapping from $\mathcal{T}(S, M)$ to $\mathcal{T}(T(S, M))$ under the condition:

$$\begin{aligned} (\bar{P}\otimes\bar{Q})^{\bar{V}} &= \bar{P}^{\bar{V}}\otimes\bar{Q}^{\bar{V}}, \\ (\bar{P}\otimes\bar{Q})^{\bar{C}} &= \bar{P}^{\bar{V}}\otimes\bar{Q}^{\bar{C}} + \bar{P}^{\bar{C}}\otimes\bar{Q}^{\bar{V}}, \end{aligned} \tag{3.7}$$

where $\bar{P}$ and $\bar{Q}$ are arbitrary elements of $\mathcal{T}(S, M)$.

We shall now sum up some properties of lifts derived immediately from the definitions.

PROPOSITION 1. *The lifts of* $\mathcal{T}_0^0(S, M)$ *to* $T(M)$ *and the lifts of* $\mathcal{T}_0^0(S)$ to $T(S)$ *are related by*

$$\bar{f}^{\bar{V}} = \bar{f}^V, \qquad \bar{f}^{\bar{C}} = \bar{f}^C \qquad for \quad \bar{f}\in\mathcal{T}_0^0(S, M) = \mathcal{T}_0^0(S). \tag{3.8}$$

That is

$$\hat{f}^V\circ d\iota = (\hat{f}\circ\iota)^V, \qquad \hat{f}^C\circ d\iota = (\hat{f}\circ\iota)^C \qquad for \quad \hat{f}\in\mathcal{T}_0^0(M). \tag{3.9}$$

The lifts of vector fields tangent to S are tangent to $T(S)$, *i.e.*,

$$(BX)^{\bar{V}} = dB(X^V), \qquad (BX)^{\bar{C}} = dB(X^C) \qquad for \quad X\in\mathcal{T}_0^1(S). \tag{3.10}$$

Proof. (3. 8) and (3. 9) are easily seen from (2. 1), (2. 2), (3. 1) and (3. 2). As for (3. 10), by virtue of (3. 3), (3. 4), (3. 8) and (3. 9), we have

$$dB(X^V)\hat{f}^C = X^V(\hat{f}^C \circ B) = X^V(\hat{f}\circ\iota)^C = (X(\hat{f}\circ\iota))^V$$
$$= ((BX)\hat{f})^{\bar{V}} = (BX)^{\bar{V}}\hat{f}^C$$

and

$$dB(X^C)\hat{f}^C = X^C(\hat{f}^C \circ B) = X^C(\hat{f}\circ\iota)^C = (X(\hat{f}\circ\iota))^C$$
$$= ((BX)\hat{f})^{\bar{C}} = (BX)^{\bar{C}}\hat{f}^C$$

for an arbitrary element f of $\mathcal{T}^0_0(M)$. Consequently from Remark mentioned in § 2, we have (3. 10).

From definition of the lifts of elements of $\mathcal{T}^r_s(S, M)$, we have the formulas similar to (2. 3) and (2. 4). Summing up, we have the following formulas.

Let $\bar{f}$ and $\bar{X}$ be arbitrary elements of $\mathcal{T}^0_0(S)$ and $\mathcal{T}^1_0(S, M)$ respectively, then we have

$$\hat{T}^C(\bar{X}^{\bar{V}}, \bar{Y}^{\bar{C}}) = \hat{T}^V(\bar{X}^{\bar{C}}, \bar{Y}^{\bar{C}}) = \hat{T}(\bar{X}, \bar{Y})^V,$$
$$\hat{T}^C(\bar{X}^{\bar{C}}, \bar{Y}^{\bar{C}}) = \hat{T}(\bar{X}, \bar{Y})^C, \tag{3. 13}$$
$$\hat{T}^C(\bar{X}^{\bar{V}}, \bar{Y}^{\bar{V}}) = \hat{T}^V(\bar{X}^{\bar{C}}, \bar{Y}^{\bar{V}}) = \hat{T}^V(\bar{X}^{\bar{V}}, \bar{Y}^{\bar{V}}) = 0$$

along S.

§ 4. Formulas for surfaces.

Let there be given a Riemannian metric G in M. If we denote by g the induced metric on S from G, then by definition we have

$$g(X, Y) = G(BX, BY) \qquad \text{for} \quad X, Y \in \mathcal{T}^1_0(S).$$

We consider the Riemannian covariant differentiation $\hat{\nabla}$ determined by G in M. Then we have along S

$$\hat{\nabla}_{BX} BY = T_X Y + N_X Y \qquad \text{for} \quad X, Y \ni \mathcal{T}^1_0(S), \tag{4. 1}$$

where $T_X Y$ and $N_X Y$ are tangential and normal parts of $\hat{\nabla}_{BX} BY$, respectively. Then the correspondence T which assigns $T_X Y$ to a pair of two vector fields X and Y defines a covariant differentiation along S. Thus we introduce a connection ∇ on S by the condition

$$B\nabla_X Y = T_X Y \tag{4. 2}$$

X and Y being arbitrary elements of $\mathcal{T}^1_0(S)$. We can easily verify that ∇ thus defined is a Riemannian connection with respect to the induced metric g and we call ∇ the connection induced on S from $\hat{\nabla}$, or simply the induced connection on S. $N_X Y$ being normal to S, we can put

(4. 3) $$N_X Y = h(X, Y)N,$$

N being the nomal vector field and h being a certain tensor field of type (0, 2) on S. We call h the second fundamental tensor field and we define the tensor field H of type (1, 1) by

$$g(HX, Y) = h(X, Y).$$

If we denote by K the curvature tensor field for the induced connection ∇, the equations of Weingarten, Gauss and Codazzi for S in M are written respectively as

(4. 4) $$\hat{\nabla}_{BX} N = -B(HX),$$

(4. 5) $$g(K(X, Y)Z, W) = G(\hat{K}(BX, BY)BZ, BW) + g((HX)h(Y, Z) - (HY)h(X, Z), W),$$

(4. 6) $$G(\hat{K}(BX, BY)N, BW) = g(\nabla_X HY - \nabla_Y HX, W)$$

X, Y and Z being arbitrary elements of $\mathcal{T}_0^1(S)$.

S is said to be totally umbilical if there exists a scalar field m such that

$$h(X, Y) = mg(X, Y)$$

for arbitrary elements X, Y of $\mathcal{T}_0^1(S)$. We call m the mean curvature of S, and have

$$m = \frac{1}{n-1} \text{Trace } H.$$

If a totally umbilical hypersurface has the vanishing mean curvature, it is said to be totally geodesic.

§ 5. The induced metric and connectioc on $T(S)$.

Let G be the Riemannian metric given in M. Then the complete lift G^C of G is the pseudo-Riemannian metric in $T(M)$. We say that two vector field $\overset{*}{X}$ and $\overset{*}{Y}$ are orthogonal on $T(S)$ with respect to G^C, when we have

$$G^C(\overset{*}{X}, \overset{*}{Y}) = 0$$

on $T(S)$ and we say that $\overset{*}{N}$ is a normal vector field to $T(S)$ when we have

$$G^C(\overset{*}{N}, \tilde{B}\tilde{X}) = 0 \qquad \text{for} \quad \tilde{X} \in \mathcal{T}_0^1(T(S)).$$

If we denote a mapping which assigns to each $p \in S$ a normal vector N_p to S by N, N is a vector field along S. We can define its vertical lift $N^{\bar{V}}$ and complete lift $N^{\bar{C}}$ to $T(M)$ according to § 3. Then we find that for each point $x \in T(S)$, $(N^{\bar{V}})_x$

and $(N^{\bar{C}})_x$ are normal vectors to $T(S)$ with respect to G^C and they are self-orthogonal but not mutually orthogonal, i.e.,

$$
\begin{aligned}
&G^C(N^{\bar{V}}, \tilde{B}X^C)=G^C(N^{\bar{C}}, \tilde{B}X^C)=0, \\
(5.1) \qquad &G^C(N^{\bar{C}}, N^{\bar{C}})=G^C(N^{\bar{V}}, N^{\bar{V}})=0, \\
&G^C(N^{\bar{V}}, N^{\bar{C}})=1
\end{aligned}
$$

for $X\in\mathcal{T}_0^1(S)$. These are direct consequences of (3.13). Moreover we can chose $\{N^{\bar{V}}, N^{\bar{C}}\}$ as the basis of normal space to $T(S)$.

If we denote by $\tilde{g}$ the induced metric on $T(S)$ from G^C then we have

$$
(5.2) \qquad \tilde{g}(X^C, Y^C)=G^C(\tilde{B}X^C, \tilde{B}Y^C) \qquad \text{for} \quad X, Y\in\mathcal{T}_0^1(S).
$$

The complete lift $\hat{\nabla}^C$ of $\hat{\nabla}$ to $T(M)$ is by definition an affine connection in $T(M)$ characterized by the property

$$
(5.3) \qquad \hat{\nabla}^C_{\hat{X}^C} \hat{Y}^C=(\hat{\nabla}_{\hat{X}}\hat{Y})^C \qquad \text{for} \quad \hat{X}, \hat{Y}\in\mathcal{T}_0^1(M),
$$

from which we also have

$$
(5.4) \qquad \hat{\nabla}^C_{\hat{X}^C}\hat{Y}^V=(\hat{\nabla}_{\hat{X}}\hat{Y})^V \qquad \text{for} \quad \hat{X}, \hat{Y}\in\mathcal{T}_0^1(M).
$$

It is known that, $\hat{\nabla}$ being the Riemannian connection with respect to G, $\hat{\nabla}^C$ is the Riemannian connection of $T(M)$ with respect to the pseudo-Riemannian metric G^C. (cf. [4] Prop. 7.5). Similarly the complete lift ∇^C of the induced connection ∇ on S is the Riemannian connection with respect to g^C.

Denoting by $\tilde{\nabla}$ the connection induced on $T(S)$ from $\hat{\nabla}^C$, along $T(S)$ we have

$$
(5.5) \qquad \hat{\nabla}^C_{\tilde{B}X^C}\tilde{B}Y^C=\tilde{B}(\tilde{\nabla}_{X^C}Y^C)+N_{X^C}Y^C \qquad \text{for} \quad X, Y\in\mathcal{T}_0^1(S),
$$

where $N_{X^C}Y^C$ is the normal part of $\hat{\nabla}^C_{\tilde{B}X^C}\tilde{B}Y^C$. Then we can put

$$
(5.6) \qquad N_{X^C}Y^C=\tilde{h}(X^C, Y^C)N^{\bar{V}}+\tilde{k}(X^C, Y^C)N^{\bar{C}},
$$

where $\tilde{h}$ and $\tilde{k}$ are certain tensor fields of type (0, 2) which are called the second fundamental tensor fields with respect to $N^{\bar{V}}$ and $N^{\bar{C}}$, respectively.

PROPOSITION 2. *The connection $\tilde{\nabla}$ induced on $T(S)$ from $\hat{\nabla}^C$ is the complete lift of the connection ∇ induced on S from $\hat{\nabla}$. That is to say, $\tilde{\nabla}$ is the Riemannian connection of $T(S)$ with respect to g^C satisfying the condition*

$$
\tilde{\nabla}_{X^C}Y^C=(\nabla_X Y)^C \qquad \textit{for} \quad X, Y\in\mathcal{T}_0^0(S).
$$

Proof. First we shall show

$$
(5.7) \qquad \hat{\nabla}^C_{\tilde{B}X^C}\tilde{B}Y^C=(\hat{\nabla}_{BX}BY)^{\bar{C}} \qquad \text{for} \quad X, Y\in\mathcal{T}_0^1(S).
$$

Recalling the definitions of lifts of BX in § 3, we introduce vector fields $\hat{X}$ and $\hat{Y}$

on a sufficiently small neighborhood U such that $\hat{X}$ and $\hat{Y}$ coincide respectively with BX and BY on $(U\cap S)^0$. Then we have

$$\hat{\nabla}^C_{\tilde{B}X^C}\tilde{B}Y^C=\hat{\nabla}^C_{(BX)^{\bar{C}}}(BY)^{\bar{C}}=\#\hat{\nabla}^C_{\hat{X}^C}\hat{Y}^C=\#(\hat{\nabla}_{\hat{X}}\hat{Y})^C=(\hat{\nabla}_{BX}BY)^{\bar{C}},$$

since $\hat{\nabla}_{\hat{X}}\hat{Y}$ is a vector field on U which coincides with $\hat{\nabla}_{BX}BY$ on $(U\cap S)^0$. By the the same reason we have the following formulas which will be used in the next section.

$$(5.8)\qquad \begin{aligned}&\hat{\nabla}^C_{\tilde{B}X^C}N^{\bar{C}}=(\hat{\nabla}_{BX}N)^{\bar{C}},\\ &\nabla^C_{\tilde{B}X^C}N^{\bar{V}}=(\nabla_{BX}N)^{\bar{V}}\end{aligned}\qquad \text{for}\quad X\in\mathcal{T}^1_0(S).$$

Now by virtue of (3. 11), (3. 13), (4. 1), (4. 3) and (5. 7), we get

$$(5.9)\qquad \nabla^C_{\tilde{B}X^C}\tilde{B}Y^C=\tilde{B}(\nabla_XY)^C+h^C(X^C,\ Y^C)N^{\bar{V}}+h^V(X^C,\ Y^C)N^{\bar{C}}.$$

On the other hand by (5. 5) and (5. 6) we have

$$(5.10)\qquad \nabla^C_{\tilde{B}X^C}\tilde{B}Y^C=\tilde{B}(\tilde{\nabla}^C_{X^C}Y^C)+\tilde{h}(X^C,\ Y^C)N^{\bar{V}}+\tilde{k}(X^C,\ Y^C)N^{\bar{C}}.$$

Therefore we obtain

$$(\nabla_XY)^C=\tilde{\nabla}_{X^C}Y^C. \qquad \text{(q.e.d.)}$$

REMARK. It is known that if $\hat{K}$ is the curvature tensor fiele of $\hat{\nabla}$, then $\hat{K}^C$ is the curvature tensor field of $\hat{\nabla}^C$. (cf. [4]). Similarly, the complete lift of the curvature tensor field K of the induced connection ∇ on S is the curvature tensor field of ∇^C. Therefore from this Proposition the curvature tensor $\tilde{K}$ of $\tilde{\nabla}$ $(=\nabla^C)$ is the complete lift of the curvature tensor field of ∇.

Moreover from (5. 9) and (5. 10) we have

PROPOSITION 3. *The complete and vertical lifts of the second fundamental tensor field of S are the second fundamental tensor fields with respect to $N^{\bar{V}}$ and $N^{\bar{C}}$, respectively.*

$T(S)$ is said to be totally umbilic if and only if at each point of $T(S)$, there exists differentiable functions λ and μ such

$$h^V(\tilde{X},\tilde{Y})=\lambda\tilde{g}(\tilde{X},\tilde{Y})\qquad h^C(\tilde{X},\tilde{Y})=\mu\tilde{g}(\tilde{X},\tilde{Y})$$

for any $\tilde{X},\tilde{Y}\in\mathcal{T}^1_0(T(S))$. Then we find

$$(5.11)\qquad \begin{aligned}&\lambda=\frac{1}{2(n-1)}\text{Trace } H^V=0,\\ &\mu=\frac{1}{2(n-1)}\text{Trace } H^C,\end{aligned}$$

in terms of local coordinates. If both λ and μ vanish, $T(S)$ is said to be totally geodesic. The mean curvature vector field $\tilde{M}$ of $T(S)$ is defined by

$$\tilde{M}=\lambda N^{\bar{V}}+\mu N^{\bar{C}},$$

which is independent of the basis chosen in the space normal to $T(S)$. The mean curvature $\overset{*}{m}$ of $T(S)$ in $T(M)$ is defined to be the magnitude of the mean curvature vector field, (i.e. $\overset{*}{m}=G^{C}(\tilde{M}, \tilde{M})$).

PROPOSITION 4. *If $T(S)$ is totally umbilic, then S is totally geodesic. $T(S)$ is totally geodesic if and only if S is totally geodesic in M.*

Proof. If we assume that $T(S)$ is totally umbilic, then the second fundamental tensor of S always vanishes, because Trace h^{V} is zero. Conversely, if S is totally geodesic, $T(S)$ is also totally geodesic from Proposition 3.

PROPOSITION 5. *The mean curvature of $T(S)$ vanishes.*

Proof. If we denote the mean curvature by $\overset{*}{m}$, by virtue of (5. 1) we have

$$\overset{*}{m}=G^{C}(\tilde{M}, \tilde{M})=2\lambda\mu, \qquad 2(n-1)\lambda=\text{Trace } H^{V}, \qquad 2(n-1)\mu=\text{Trace } H^{C}.$$

Now from (5. 11), we have Proposition 5.

§6. The structure equation of $T(S)$.

In this section we shall investigate the equations of Gauss and Weingarten and the structure equations (i.e., the equations of Gauss, Codazzi and Ricci) on $T(S)$ in $T(M)$. These are written in the following form:

$$\left.\begin{aligned}\nabla^{C}_{\tilde{B}X^{C}}N^{\bar{V}}&=-\tilde{B}H^{V}X^{C},\\ \nabla^{C}_{\tilde{B}X^{C}}N^{\bar{C}}&=-\tilde{B}H^{C}X^{C}\end{aligned}\right\}\quad \text{for}\quad X\in\mathcal{T}^{1}_{0}(S), \tag{6. 1}$$

$$\begin{aligned}\tilde{g}(\tilde{K}(X^{C}, Y^{C})Z^{C}, W^{C})=&\,G^{C}(\hat{K}^{C}(\tilde{B}X^{C}, \tilde{B}Y^{C})\tilde{B}Z^{C}, \tilde{B}W^{C})\\ &+\tilde{g}((H^{C}X^{C})h^{V}(Y^{C}, Z^{C})+(H^{V}X^{C})h^{C}(Y^{C}, Z^{C})\\ &-(H^{C}Y^{C})h^{V}(X^{C}, Z^{C})-(H^{V}Y^{C})h^{C}(X^{C}, Z^{C}), W^{C})\end{aligned} \tag{6. 2}$$

where $\tilde{K}$ is the Riemannian curvature tensor field of ∇^{C} on $T(S)$.

$$\begin{aligned}\hat{K}^{C}(\tilde{B}X^{C}, \tilde{B}Y^{C})N^{\bar{V}}&=\tilde{B}(\nabla^{C}_{X^{C}}H^{V}Y^{C}-\nabla^{C}_{Y^{C}}H^{V}X^{C}),\\ \hat{K}^{C}(\tilde{B}X^{C}, \tilde{B}Y^{C})N^{\bar{C}}&=\tilde{B}(\nabla^{C}_{X^{C}}H^{C}Y^{C}-\nabla^{C}_{Y^{C}}H^{C}X^{C}),\\ \hat{K}^{C}(N^{\bar{C}}, N^{\bar{V}})\tilde{B}X^{C}&=0.\end{aligned} \tag{6. 3}$$

Proof. By virtue of (2. 3), (3. 10), (4. 4) and (5. 8), (6. 1) is reduced to

$$\hat{\nabla}^{C}_{\tilde{B}X^{C}}N^{\bar{V}}=(\hat{\nabla}_{BX}N)^{\bar{V}}=-(BHX)^{\bar{V}}=-\tilde{B}H^{V}X^{C},$$

and

$$\hat{\nabla}^C_{\tilde{B}X^C} N^{\bar{C}} = (\hat{\nabla}_{BX} N)^{\bar{C}} = -(BHX)^{\bar{C}} = -\tilde{B} H^C X^C.$$

Next as for (6. 2), we note first that $\tilde{K}$ is the complete lift of K, then we find

$$\tilde{K}(X^C, Y^C)Z^C = K^C(X^C, Y^C)Z^C, \qquad \text{for} \quad X, Y \in \mathcal{T}^1_0(S)$$

because of Remark mentioned in § 5.

We shall derive the formulas for the later use

$$\begin{aligned} & K^C(X^C, Y^C)Z^C = (K(X, Y)Z)^C, \\ & \hat{K}^C(\tilde{B}X^C, \tilde{B}Y^C)\tilde{B}Z^C = (\hat{K}(BX, BY)BZ)^{\bar{C}}, \\ (6.\ 4) \qquad & \hat{K}^C(\tilde{B}X^C, \tilde{B}Y^C)N^{\bar{V}} = (\hat{K}(BX, BY)N)^{\bar{V}}, \\ & \hat{K}^C(\tilde{B}X^C, \tilde{B}Y^C)N^{\bar{C}} = (\hat{K}(BX, BY)N)^{\bar{C}}, \\ & \hat{K}^C(N^{\bar{V}}, N^{\bar{C}})\tilde{B}X^C = (\hat{K}(N, N)BX)^{\bar{C}} = 0. \end{aligned}$$

In fact, from (2. 3) and (5. 3) we have,

$$\begin{aligned} K^C(X^C, Y^C)Z^C &= \nabla^C_{X^C}\nabla^C_{Y^C}Z^C - \nabla^C_{Y^C}\nabla^C_{X^C}Z^C - \nabla^C_{[X^C, Y^C]}Z^C \\ &= \nabla^C_{X^C}(\nabla_Y Z)^C - \nabla^C_{Y^C}(\nabla_X Z)^C - \nabla^C_{[X,Y]^C}Z^C \\ &= (\nabla_X\nabla_Y Z)^C - (\nabla_Y\nabla_X Z)^C - (\nabla_{[X,Y]}Z)^C \\ &= (K(X, Y)Z)^C. \end{aligned}$$

By making use of (5. 6) and (5. 7), the others are also obtained.

Now we have

$$\begin{aligned} \tilde{g}(\tilde{K}(X^C, Y^C)Z^C, W^C) &= \tilde{g}(K^C(X^C, Y^C)Z^C, W^C) = \tilde{g}((K(X, Y)Z)^C, W^C) \\ &= G^C((\hat{K}(BX, BY)BZ)^{\bar{C}}, (BW)^{\bar{C}}) \\ &\quad + \tilde{g}((HX)h(Y, Z) - (HY)h(X, Z))^C, W^C) \\ &= G^C(\hat{K}^C(\tilde{B}X^C, \tilde{B}Y^C)\tilde{B}Z^C, \tilde{B}W^C) \\ &\quad + \tilde{g}((H^C X^C)h^V(Y^C Z^C) + (H^V X^C)h^C(Y^C, Z^C) \\ &\quad - (H^C Y^C)h^C(X^C, Z^C) - (H^V Y^C)h^C(X^C, Z^C), W^C) \end{aligned}$$

from (4. 4).

As for (6. 3), by making use of the equations of Codazzi and Ricci (4. 5), (4. 6) and (6. 4), we have

$$\begin{aligned}
\hat{K}^C(\tilde{B}X^C, \tilde{B}Y^C)N^{\bar{V}} &= (\hat{K}(BX, BY)N)^{\bar{V}} \\
&= (B(\nabla_X HY - \nabla_Y HX))^{\bar{V}} \\
&= \tilde{B}(\nabla_X HY - \nabla_Y HX)^V \\
&= \tilde{B}(\nabla^C_{X^C} H^V Y^C - \nabla^C_{Y^C} H^V X^C), \\
\hat{K}^C(\tilde{B}X^C, \tilde{B}Y^C)N^{\bar{C}} &= (\hat{K}(BX, BY)N)^{\bar{C}} \\
&= (B(\nabla_X HY - \nabla_Y HX))^{\bar{C}} \\
&= \tilde{B}(\nabla_X HY - \nabla_Y HX)^C \\
&= \tilde{B}(\nabla^C_{X^C} H^C Y^C - \nabla^C_{Y^C} H^C X^C),
\end{aligned}$$

and

$$\hat{K}^C(N^{\bar{V}}, N^{\bar{C}})\tilde{B}X^C = 0.$$

Thus we have (6. 1), (6. 2) and (6. 3), which are the equations of Weingarten, Gauss and Codazzi and Ricci on $T(S)$ in $T(M)$, respectively.

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