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ON LOCAL MAXIMALITY FOR THE COEFFICIENTS a_6 and a_8

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1. In our previous papers [1], [2], [3] we proved the local maximality of $\Re a_6$ and $\Re a_8$ at the Koebe function $z/(1-z)^2$. In this note we shall prove the local maximality of $|a_6|$ and $|a_8|$ at the Koebe function $z/(1-e^{i\theta}z)^2$, that is, the following theorems.

THEOREM 1. *Let $f(z)$ be a normalized regular function univalent in the unit circle*

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n.$$

Then there is a positive constant ε such that $|a_6|\leqq 6$ holds for $0\leqq 2-|a_2|\leqq\varepsilon$. Equality occurs only for the Koebe function $z/(1-e^{i\theta}z)^2$.

THEOREM 2. *$|a_8|\leqq 8$ holds for $0\leqq 2-|a_2|\leqq\varepsilon$. Equality occurs only for the Koebe function $z/(1-e^{i\theta}z)^2$.*

In the sequel we shall use the same notations as in [1], [2], [3]. Further we put $p=2-x$, $x'=kp$.

2. Proof of theorem 1. By the well-known rotation it is sufficient to prove that

$$\text{(A)} \qquad \Re a_6 < 6$$

for $0\leqq 2-|a_2|\leqq\varepsilon$, $|\arg a_2|\leqq\pi/4$, unless $a_2=2$. Then we can use our earlier result in [1], [3]:

$$\Re a_6\leqq 6-A(2-\Re a_2), \qquad A>0$$

holds for $0\leqq 2-\Re a_2\leqq\varepsilon_1$. Here equality occurs only for the Koebe function $z/(1-z)^2$.

Hence there are positive constants ε_2 and δ' such that $\Re a_6<6$ for $0\leqq 2-|a_2|\leqq\varepsilon_2$, $|\arg a_2|\leqq\delta'$, unless $a_2=2$. Hence we may assume that $0<\tan\delta'=\delta\leqq|k|\leqq 1$.

By Grunsky's inequality $|b_{55}|\leqq 1$ we have

$$\left| a_6-2a_2a_5-3a_3a_4+4a_4a_2^2+\frac{21}{4}a_2a_3^2-\frac{59}{8}a_3a_2^3+\frac{689}{320}a_2^5\right|\leqq\frac{2}{5}.$$

By taking the real part we have

$$\Re a_6\leqq\frac{2}{5}+\Re\Big\{2(p+ix')(\xi+i\xi')+3(y+iy')(\eta+i\eta')+\frac{5}{4}(p+ix')^2(\eta+i\eta')$$

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$$+\frac{3}{4}(p+ix')(y+iy')^2+\frac{11}{8}(p+ix')^3(y+iy')\Big\}$$

$$+\frac{7}{40}(p^5-10p^3x'^2+5px'^4).$$

We put the right hand side

$$\frac{2}{5}+\frac{7}{40}p^5-\frac{7}{8}k^2p^5(2-k^2)+L.$$

Now by the area theorem for $f(z^2)^{-1/2}$ we have

$$7(\xi^2+\xi'^2)+5(\eta^2+\eta'^2)+3(y^2+y'^2)+p^2(1+k^2)\leqq 4.$$

Let ε_3 be $2/\sqrt{1+k^2}-p$. Then all of $\xi, \xi', \eta, \eta', y, y'$ are $O(\varepsilon_3^{1/2})$. Since L is a polynomial of each variable, L tends to 0 uniformly for k in $\delta\leqq|k|\leqq 1$ as $\varepsilon_3\to 0$ decreasingly. For $|k|\leqq 1$

$$k^2p^5(2-k^2)\geqq 0.$$

Thus we have

$$\Re a_6<6$$

for $0\leqq 2/\sqrt{1+k^2}-p\leqq\varepsilon_4$, $\delta\leqq|k|\leqq 1$. Take ε as min $(\varepsilon_2, \varepsilon_4\sqrt{1+\delta^2})$. Then we have the desired result (A).

3. Proof of theorem 2. The same idea as in theorem 1 does work in this case. Again it is sufficient to prove that $\Re a_8<8$ for $0\leqq 2-|a_2|\leqq\varepsilon$, $|\arg a_2|\leqq\pi/6$, unless $a_2=2$.

In the first place we shall use our earlier result in [2] and determine positive constants ε_2 and δ. Then starting from Grunsky's inequality

$$|b_{77}|\leqq 1$$

and using the area theorem

$$11|b_{11}|^2+9|b_9|^2+7|b_7|^2+5|b_5|^2+3|b_3|^2+|b_1|^2\leqq 1.$$

we can prove the desired result.

References

[1] JENKINS, J. A., AND M. OZAWA, On local maximality for the coefficient a_6. Nagoya Math. Journ. **30** (1967), 71–78.

[2] JENKINS, J. A., AND M. OZAWA, On local maximality for the coefficient a_8. Illinois Journ. Math. **11** (1967), 596–602.

[3] OZAWA, M., An elementary proof of local maximality for a_6. Kōdai Math. Sem. Rep. **20** (1968), 437–439.

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