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PSEUDO-UMBILICAL SUBMANIFOLDS WITH M-INDEX $\leqq 1$ IN EUCLIDEAN SPACES

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1. Pseudo-umbilical submanifolds with M-index 0.

In this note, the author will use the notations in Ōtsuki [5]. Let M^n be an n-dimensional manifold immersed in the $(n+N)$-dimensional Euclidean space E^{n+N} by a mapping ψ: $M^n \to E^{n+N}$.[1] We denote this simply by $M^n \Subset E^{n+N}$. Let ω_i, $\omega_{ij} = -\omega_{ji}$, $\omega_{i\alpha} = -\omega_{\alpha i}$, $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$, $i, j = 1, 2, \cdots, n$; $\alpha, \beta = n+1, \cdots, n+N$, are the differential 1-forms associated with the immersion ψ: $M^n \to E^{n-N}$ which are defined on B of all orthonormal frames $(p, e_1, \cdots, e_{n+N})$ such that $p \in M^n$, $e_1, \cdots, e_n \in T_pM^n$. As is well known, $\omega_{i\alpha}$ can be written as

$$(1.1)\qquad \omega_{i\alpha} = \sum_j A_{\alpha ij}\omega_j, \qquad A_{\alpha ij} = A_{\alpha ji}.$$

Let N_p be the normal tangent space to M^n at $p \in M^n$. For any normal unit vector $e = \sum \xi_\alpha e_\alpha \in N_p$, let

$$(1.2)\qquad \Phi_e(\omega, \omega) = \sum_{\alpha, i, j} A_{\alpha ij}\omega_i\omega_j$$

be the second fundamental form corresponding to e. Let $\bar{m}$: $N_p \to \boldsymbol{R}$ be the mapping as follows: For any $X = \sum_\alpha \xi_\alpha e_\alpha \in N_p$,

$$(1.3)\qquad \bar{m}(X) = \frac{1}{n} \sum_{\alpha, i} \xi_\alpha A_{\alpha ii}.$$

Let ${}^{M-}N_p$ be the kernel of $\bar{m}$ at p which is called the *minimal normal space* at p. Let k_1: $M^n \to \boldsymbol{R}$ be the first curvature of M^n as an immersed submanifold in E^{n+N}. At p such that $k_1(p) \neq 0$, let $\bar{e}(p)$ be the mean curvature normal unit vector, that is

$$(1.4)\qquad \sum_{\alpha, i} A_{\alpha ii} e_\alpha = k_1(p)\bar{e}(p) \qquad k_1(p) > 0.$$

At the point p, we make use of only the frames $b = (p, e_1, \cdots, e_{n+N})$ such that $e_{n+1} = \bar{e}(p)$. Then, we have

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1) We consider M^n as a Riemannian manifold with the metric induced from E^{n+N} by ψ.

(1. 5) $$\bar{m}(e_{n+1})=k_1(p), \qquad \bar{m}(e_{n+2})=\cdots=\bar{m}(e_{n+N})=0.$$

If we have

(1. 6) $$\Phi_{\bar{e}(p)}(\omega, \omega)=k_1(p)\sum_i \omega_i\omega_i,$$

that is, in matrix form,

(1. 6′) $$A_{\underline{n+1}ij}=k_1(p)\delta_{ij},$$ [2]

we call M^n is *pseudo-umbilical* at p. If M^n is pseudo-umbulical at each point of M, the immersion ψ: $M^n \to E^{n+N}$ is called *pseudo-umbilical*

If M^n is umbilical at p, then we have by definition

$$\Phi_e(\omega, \omega)=\lambda(e)\sum_i \omega_i\omega_i$$

for any normal unit vector $e \in N_p$, where $\lambda(e)$ is a real number depending on e. The above condition can be written as, in matrix form,

$$A_{\alpha ij}=\lambda(e)\delta_{ij}, \qquad \alpha=n+1, \cdots, n.$$

Hence we have $\lambda(e)=\bar{m}(e)$. For any $e \in {}^{M^-}N_p$, we get $\Phi_e(\omega, \omega)=0$, and so M-index at p is equal to 0. Accordingly, if M^n is not totally geodesic at p, then $\lambda(\bar{e}(p))\neq 0$ and so M^n is pseudo-umbilical. The converse is true. We have

LEMMA 1. *M^n is umbilical and not totally geodesic at p, if and only if M^n is pseudo-umbilical and of M-index* 0 *at p.*

Connecting with lemma and Theorem in [5], we get easily the following

THEOREM 1. *If M^n is an immersed submanifold in E^{n+N} which is pseudo-umbilical and of M-index* 0 *at every point, then M^n is an n-dimensional sphere or its subdomain in a linear subspace E^{n+1}.*

Proof. By the assumption, the index of relative nullity is identically 0. By Theorem 3 in [5], there exists an $(n+1)$-dimensional linear subspace E^{n+1} such that $M^n \Subset E^{n+1}$. Accordingly, M^n is hypersurface in E^{n+1} which is umbilical at every point. Hence M^n is a hypersphere or its subdomain in E^{n+1}.

2. Pseudo-umbilical submanifolds with M-index 1.

In this section, we suppose that M^n is an n-dimensional manifold immersed in E^{n+N} which is pseudo-umbilical and of M-index 1 at every point. Then, the first curvature k_1: $M^n \to \boldsymbol{R}$ is not zero everywhere. Since M-index is constant 1, we take only such frame $b=(p, e_1, \cdots, e_{n+N}) \in B$ that

2) In the following we use the notation $\underline{n+1}$ in place of $(n+1)$ for sufixes.

(2. 1) $$e_{n+1}=\bar{e}(p),$$

(2. 2) $$\begin{cases} A_{n+1}=(A_{\underline{n+1}ij})=k_1(p)(\delta_{ij}), \\ A_{n+2}=(A_{\underline{n+2}ij})\neq 0, \\ A_\beta=(A_{\beta ij})=0, \qquad \beta=n+3, \cdots, n+N, \end{cases}$$

and denote the submanifold of B composed of these frames by B_2. On B_2, from $\omega_{i\beta}=0$ $(\beta>n+2)$ and the structure equations of the immersion ψ: $M^n \to E^{n+N}$, we get

$$0=d\omega_{i\beta}=\sum_j \omega_{ij}\wedge\omega_{j\beta}+\omega_{i\underline{n+1}}\wedge\omega_{\underline{n+1}\beta}+\omega_{i\underline{n+2}}\wedge\omega_{\underline{n+2}\beta}+\sum_{\gamma>n+2}\omega_{i\gamma}\wedge\omega_{\gamma\beta},$$

that is

(2. 3) $$k_1\omega_i\wedge\omega_{\underline{n+1}\beta}+\omega_{i\underline{n+2}}\wedge\omega_{\underline{n+2}\beta}=0.$$

Now, we take a frame $(p, e_1, \cdots, e_n)$ of M^n such that

(2. 4) $$\omega_{i\underline{n+2}}=h_i\omega_i, \qquad i=1, \cdots, n.$$

$\{h_i, \cdots, h_n\}$ are the eigen values of the second fundamental form $\Phi_{e_{n+2}}(\omega, \omega)$. Since $\bar{m}(e_{n+2})=0$ and $A_{n+2}\neq 0$, we have

(2. 5) $$h_1+h_2+\cdots+h_n=0, \qquad (h_1, \cdots, h_n)\neq(0, \cdots, 0).$$

It is clear that the number of distinct eigen values of $\Phi_e(\omega, \omega)$, $e\in N_p$, $e\cdot e_{n+2}\neq 0$, is the same. We call such e a generic normal unit vector. Making use of this frame (2. 3) becomes

$$\omega_i\wedge(k_1\omega_{\underline{n+1}\beta}+h_i\omega_{\underline{n+2}\beta})=0,$$

hence we can write the second factor as

(2. 4) $$k_1\omega_{\underline{n+1}\beta}+h_i\omega_{\underline{n+2}\beta}=\rho_i\omega_i$$

for fixed β $(>n+2)$. Accordingly, we have

$$(h_i-h_j)\omega_{\underline{n+2}\beta}=\rho_i\omega_i-\rho_j\omega_j.$$

If the number of distinct eigen values out of $\{h_1, \cdots, h_n\}$ is not less than 3, then we have easily

(2. 5) $$\omega_{\underline{n+2}\beta}=0 \quad \text{and} \quad \omega_{\underline{n+1}\beta}=0.$$

If the number of distinct eigen values out of $\{h_1, \cdots, h_n\}$ is equal to 2, we may put

$$h_1=h_2=\cdots=h_{\nu_0}, \qquad h_{\nu_0+1}=\cdots=h_n, \qquad 1\leqq\nu_0\leqq n-1.$$

If $1<\nu_0<n-1$, then we get also (2. 5) from (2. 4). If $\nu_0=n-1$ and $n\geqq 3$, then (2. 4) becomes

$$k_1\omega_{\underline{n+1}\beta}+h_1\omega_{\underline{n+2}\beta}=0,$$

$$k_1\omega_{\underline{n+1}\beta}+h_n\omega_{\underline{n+2}\beta}=\rho_n\omega_n,$$

hence we may put

$$\omega_{\underline{n+1}\beta}=\lambda_\beta\omega_n \qquad \text{and} \qquad \omega_{\underline{n+2}\beta}=-\frac{k_1}{h_1}\lambda_\beta\omega_n.$$

Furthermore, we can choose $e_{n+3}, \cdots, e_{n+N}$ so that

$$\lambda_{n+4}=\cdots=\lambda_{n+N}=0,$$

that is

$$(2.6) \qquad \begin{cases} \omega_{\underline{n+1}\,\underline{n+3}}=\lambda\omega_n, & \omega_{\underline{n+2}\,\underline{n+3}}=-\dfrac{k_1}{h_1}\lambda\omega_n, \\ \omega_{\underline{n+1}\beta}=\omega_{\underline{n+2}\beta}=0 & (\beta=n+4, \cdots, n+4),\ \lambda\neq 0. \end{cases}$$

The case $\nu_0=1$ is analogous to the case $\nu_0=n-1$. Making use of these facts, we get the following

THEOREM 2. *Let $M^n(n\geqq 3)$ be an n-dimensional submanifold immersed in E^{n+N} which is pseudo-umbilical and of M-index* 1 *at every point. Then the number of eigen values of the second fundamental form $\Phi_e(\omega, \omega)$ for a generic normal unit vector e is not less than* 2. *Furthermore,*

i) *if this number is not less than* 3 *or if it is equal to* 2 *and the dimensions of the eigen spaces corresponding to the two eigen values are greater than* 1, *then there exists an $(n+2)$-dimensional linear subspace E^{n+2} of E^{n+N} such that $E^{n+2}\Supset M^n$;*

ii) *if this number is equal to* 2 *and the dimensisns of the eigen spaces are $n-1$ and* 1 *at every point, then we can choose frames $b=(p, e_1, \cdots, e_{n+N})\in B$ such that*

$$\omega_{i\underline{n+1}}=k_1\omega_i,$$

$$\omega_{a\underline{n+2}}=h_1\omega_a\ (a=1, \cdots, n-1), \qquad \omega_{n\underline{n+2}}=-(n-1)h_1\omega_n\ (h_1\neq 0),$$

$$\omega_{i\alpha}=0 \qquad (\alpha=n+3, \cdots, n+N),$$

$$\omega_{\underline{n+1}\,n+3}=\lambda\omega_n, \qquad \omega_{\underline{n+1}\beta}=0 \qquad (\beta=n+4, \cdots, n+N),$$

$$\omega_{\underline{n+2}\,\underline{n+3}}=\mu\omega_n, \qquad \omega_{\underline{n+2}\beta}=0 \qquad (\beta=n+4, \cdots, n+N),$$

where

$$k_1\lambda+h_1\mu=0.$$

COROLLARY. *In order that there exists an E^{n+2} such that $M^n\Subset E^{n+2}$ under the same assumptions of Theorem* 2, *it is necessary and sufficient that the linear mapping*

$$\varphi_2\colon\ T_p(M^n)\to e_{n+2}{}^{\perp}\cap {}^{M-}N_p$$

is trivial, where φ_2 is defined by

$$\varphi_2(X)=\sum_{\beta=n+3}^{n+N}\omega_{\underline{n+2}\beta}(X)e_\beta.$$

3. Pseudo-umbilical submanifolds in E^{n+2} with M-index 1.

Let M^n be an n-dimensional submanifold imbedded in E^{n+2} which is pseudo-umbilical and of M-index 1 at every point. Then we have a linear mapping φ_1: $T_p(M^n)\rightarrow {}^{M-}N_p=\boldsymbol{R}e_{n+2}$ defined by

$$\varphi_1(X)=\omega_{\underline{n+1}\,\underline{n+2}}(X)e_{n+2},\quad X\in T_p(M^n).$$

Then the second curvature of M^n at p is defined by

$$k_2(p)=\max\{|\omega_{\underline{n+1}\,\underline{n+2}}(X)|\,;\ X\in T_p(M^n),\ ||X||=1\}. \tag{3. 1}$$

Now, making use of the fact that k_1 does not vanish everywhere, we consider the following mapping ψ: $M^n\rightarrow E^{n+2}$ by

$$q=\psi(p)=p+\frac{1}{k_1(p)}\bar{e}(p) \tag{3. 2}$$

where p and q denote the position vectors in E^{n+2}.

Case $k_2(p)\neq 0$ *at every point* $p\in M^n$.

In this case, we can choose frames $b=(p, e_1, \cdots, e_n)$ such that

$$\omega_{\underline{n+1}\,\underline{n+2}}=k_2\omega_n. \tag{3. 3}$$

From this we get

$$\begin{aligned} d\omega_{\underline{n+1}\,\underline{n+2}}&=dk_2\wedge\omega_n+k_2d\omega_n\\ &=\sum_i\omega_{\underline{n+1}i}\wedge\omega_{i\underline{n+2}}=-k_1\sum_i\omega_i\wedge\omega_{i\underline{n+2}}=0, \end{aligned}$$

hence

$$d\omega_n=-d\log k_2\wedge\omega_n, \tag{3. 4}$$

which shows that the Pfaff equation

$$\omega_n=0 \tag{3. 5}$$

is completely integrable. Let the family of integral hypersurfaces of (3. 5) be $Q(v)$ and we may suppose that v is the arclength of an orthogonal trajectory of this family. By means of the Gauss' lemma, we have

$$\omega_n=dv. \tag{3. 6}$$

By (3. 4) and (3. 6), k_2 is a positive function of v. Differentiating (3. 2) and making use of $\omega_{in+1}=k_1\omega_i$, (3. 3) and (3. 6), we have

$$dq=-\frac{dk_1}{k_1^2}\bar{e}+\frac{k_2dv}{k_1}e_{n+2}. \tag{3. 7}$$

This shows that $\psi(M)$ is generally two dimensional. If $dk_1 \neq 0$ along $Q(v)$, then $\psi(Q(v))$ is a curve whose tangent direction is that of $\bar{e}$. But $\bar{e}$ varies $(n-1)$-dimensionally on $Q(v)$. This is impossible since $n-1 \geqq 2$. Hence, we have $dk_1=0$ along $Q(v)$, in other words k_1 is also a function of v. Hence the image of $Q(v)$ by ψ is a point denoted by $q=q(v)$ and $Q(v)$ is contained in a hyper sphere $S^{n+1}(v)$ with centor $q(v)$ and radius $1/k_1(v)$. Then, (3. 7) can be written as

$$\frac{dq}{dv}=\frac{k_2}{k_1}e_{n+2}-\frac{k_1'}{k_1^2}e_{n+1}$$

and so the right hand side depends only on v. Making use of $\omega_{i\underline{n+1}}=k_1\omega_i$, (3. 3) and (3. 6), we have

$$d\omega_{i\underline{n+1}}=\sum_j \omega_{ij}\wedge\omega_{j\underline{n+1}}+\omega_{i\underline{n+2}}\wedge\omega_{\underline{n+2}\,\underline{n+1}}$$

$$=k_1\omega_j\wedge\omega_{ji}+k_2 dv\wedge\omega_{i\underline{n+2}},$$

$$d(k_1\omega_i)=k_1' dv\wedge\omega_i+k_1\sum_j \omega_j\wedge\omega_{ji}$$

and so

$$dv\wedge\omega_{i\underline{n+2}}=\frac{k_1'}{k_2}dv\wedge\omega_i.$$

Substituting $\omega_{i\underline{n+2}}=\sum_j A_{\underline{n+2}ij}\omega_j$ in the above equation and making use of $\bar{m}(e_{n+2})=0$, we get

$$\text{(3. 8)}\qquad \begin{cases} A_{\underline{n+2}ab}=\dfrac{k_1'}{k_2}\delta_{ab}, \qquad A_{\underline{n+2}nb}=0 \qquad (a, b=1, 2, \cdots, n-1), \\ A_{\underline{n+2}nn}=-\dfrac{(n-1)k_1'}{k_2}. \end{cases}$$

Since M-index is 1 at every point of M^n, $A_{n+2}\neq 0$, hence

$$\text{(3. 9)}\qquad k_1'(v)\neq 0.$$

Let us use the vector field of E^{n+2} defined over M^n by

$$\text{(3. 10)}\qquad X=k_1^2\frac{dq}{dv}=k_1k_2e_{n+2}-k_1'e_{n+1},$$

which depends only on v and is normal to M^n. by means of (3. 3) and (3. 8), we get

$$dX=\{(k_1k_2)'e_{n+2}-k_1''e_{n+1}\}dv$$

$$+k_1k_2\Big(\sum_i \omega_{\underline{n+2}i}e_i+\omega_{\underline{n+2}\,\underline{n+1}}e_{n+1}\Big)$$

$$-k_1'(\sum_i \omega_{\underline{n+1}i}e_i+\omega_{\underline{n+1}\,\underline{n+2}}e_{n+2})$$

$$=\{(k_1k_2)'e_{n+2}-k_1''e_{n+1}\}dv$$

$$+k_1k_2\left\{-\frac{k_1'}{k_2}\sum_a\omega_a e_a+\frac{(n-1)k_1'dv}{k_2}e_n-k_2dve_{n+1}\right\}$$

$$-k_1'\left\{-k_1(\sum_a\omega_a e_a+dve_n)+k_2dve_{n+2}\right\},$$

that is

$$\frac{dX}{dv}=k_1k_2'e_{n+2}-(k_1''+k_1k_2^2)e_{n+1}+nk_1k_1'e_n. \tag{3.11}$$

This shows that dX/dv is linearly independent of X and normal to each $Q(v)$. Since $X(v)$ and $X'(v)$ are constant vectors along $Q(v)$, there exist linear subspaces $E_1^{n+1}(v)$ and $E_2^{n+1}(v)$ such that

$$Q(v)\Subset E_1^{n+1}(v), \qquad E_1^{n+1}(v)\perp X(v)$$

and

$$Q(v)\Subset E_2^{n+1}(v), \qquad E_2^{n+1}(v)\perp X'(v).$$

Since e_{n+1}, $X(v)$ and $X'(v)$ are linearly independent, we can put

$$S^{n-1}(v)=S^{n+1}(v)\cap E_1^{n+1}(v)\cap E_2^{n+1}(v).$$

Hence $Q(v)$ is imbedded in $S^{n-1}(v)$. M^n can be considered as a locus of moving $(n-1)$-sphere $S^{n-1}(v)$ depending on one parameter v.

Now, we consider the second fundamental form of $Q(v)$ as a submanifold of M^n. Since the right hand side of (3.11) depends only on v, using (3.3), (3.6) and (3.8), along $Q(v)$, we have

$$0=k_1k_2'de_{n+2}-(k_1''+k_1k_2^2)de_{n+1}+nk_1k_1'de_n$$

$$=k_1k_2'\sum_a\omega_{\underline{n+2}a}e_a-(k_1''+k_1k_2^2)\sum_a\omega_{\underline{n+1}a}e_a+nk_1k_1'\sum_a\omega_{na}e_a$$

$$=-k_1k_1'(\log k_2)'\sum_a\omega_a e_a+(k_1''+k_1k_2^2)k_1\sum_a\omega_a e_a+nk_1k_1'\sum_a\omega_{na}e_a,$$

hence we have

$$\omega_{an}=\frac{1}{n}\left\{-\frac{k_2'}{k_2}+\frac{k_1''+k_1k_2^2}{k_1'}\right\}\omega_a \qquad \text{on} \quad Q(v). \tag{3.12}$$

On the other hand, we get from (3.4) and (3.6)

$$0=d\omega_n=\sum_a\omega_a\wedge\omega_{an}=0 \qquad \text{on} \quad M^n.$$

This shows that (3.12) is true on M^n. Along $Q(v)$, e_n is its normal unit vector field and we have

$$de_n = -\sum_a \omega_{an} e_a.$$

According to the principle of Levi-Civita, the second fundamental form of $Q(v)$ as a hypersurface of M^n is

$$\sum_a \omega_{an}\omega_a = \frac{1}{n}\left\{-\frac{k_2'}{k_2}+\frac{k_1''+k_1k_2{}^2}{k_1'}\right\}\sum_a \omega_a\omega_a.$$

This shows that $Q(v)$ is umbilical in M^n.

Case $k_2=0$.

In this case, the mapping φ_1 is trivial at each point of M^n. Since $\omega_{\underline{n+1}\,\underline{n+2}}=0$, we get easily

$$dq = -\frac{dk_1}{k_1^2} e_{n+1}.$$

If $n\geqq 2$, by analogous argument as in Case $k_2\neq 0$, we see that k_1 is a constant and q is a fixed point. M^n must be contained in an $(n+1)$-dimensional sphere S^{n+1} with centor q and radius $1/k_1$. e_{n+2} is the normal unit vector field of M^n in S^{n+1}. In this case, we have

$$de_{n+2} = \sum_i \omega_{\underline{n+2}i} e_i$$

and so the second fundamental form of M^n in S^{n+1} is given by

$$\omega_i\omega_{i\underline{n+2}} = \sum_{i,j} A_{\underline{n+2}ij}\omega_i\omega_j.$$

Since $\sum_i A_{\underline{n+2}ii}=0$, M^n must be an minimal hypersurface in S^{n+1}. Conversely, if a minimal hypersurface in S^{n+1} can be considered as a submanifold M^n in E^{n+2} in this case. Thus we get the following theorem.

THEOREM 3. *Let M^n be an n-dimensional submanifold in imbedded in E^{n+2} which is pseudo-umbilical and of M-index* 1 *at every point. If the second curvature k_2 of M^n is not equal to zero at every point, then M^n is imbedded in a submanifold which is a locus of a moving $(n-1)$-dimensional sphere $S^{n-1}(v)$ such that the radius $r(v)$ is not constant, the curve of the centor $q(v)$ is orthogonal to this submanifold at the corresponding points and not to the n-dimensional linear subspace containing $S^{n-1}(v)$, and $S^{n-1}(v)$ is umbilical hypersurface in the locus. If $k_2=0$, then M^n is a minimal submanifold in a $(n+1)$-dimensional sphere in E^{n+2}.*

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