KŌDAI MATH. SEM. REP.
20 (1968), 146—154

ON ANALYTIC MAPPINGS AMONG THREE-SHEETED SURFACES

BY MITSURU OZAWA

1. Introduction. Let R (resp. S) be a three-sheeted covering surface over the finite z-plane (resp. w-plane) defined by an irreducible equation

$$y^3 = A(z)y + B(z) \qquad (\text{resp. } Y^3 = a(w)Y + b(w)),$$

where A, B (resp. a, b) are meromorphic functions. Here we shall assume that R (resp. S) has an infinite number of branch points. Let $\mathfrak{p}_R$ (resp. $\mathfrak{p}_S$) be the projection map $(z, y) \to z$ (resp. $(w, Y) \to w$). Let φ be a non-trivial analytic mapping of R into S. If φ preserves the projection maps, that is, $\mathfrak{p}_S\varphi(p) = \mathfrak{p}_S\varphi(q)$ whenever $\mathfrak{p}_R p = \mathfrak{p}_R q$, then φ is called a rigid analytic mapping of R into S. In the sequel we make use of the inverse mapping $\mathfrak{p}_R^{-1}$ (resp. $\mathfrak{p}_S^{-1}$), as a three-valued analytic branch, of the z-plane (resp. the w-plane) onto R (resp. S). So there are three possible choices of $\mathfrak{p}_R^{-1}$ (resp. $\mathfrak{p}_S^{-1}$). Hiromi-Mutō [4] proved that φ is rigid. In this case $h = \mathfrak{p}_S \circ \varphi \circ \mathfrak{p}_R^{-1}$ is a single-valued regular function of z in $|z| < \infty$.

In the present paper we shall prove a necessary condition for the existence of a non-trivial analytic mapping φ of R into S. Several non-existence criteria for analytic mappings are established by making use of the necessary condition.

2. A necessary condition. Let φ be a non-trivial analytic mapping of R into S and h the corresponding projection of φ, that is, $h = \mathfrak{p}_S \circ \varphi \circ \mathfrak{p}_R^{-1}$. In our case h is independent of the choice of $\mathfrak{p}_R^{-1}$, since φ is rigid. Let Y^* be the analytic mapping of S into the finite plane, which induces Y in such a manner that $Y^* = Y \circ \mathfrak{p}_S$. Then $Y^* \circ \varphi$ gives an analytic mapping of R into the finite plane. Hence $Y^* \circ \varphi \circ \mathfrak{p}_R^{-1}$ can be represented by

$$f_0 + f_1 y + f_2 y^2,$$

where f_0, f_1 and f_2 are meromorphic functions of z in $|z| < \infty$. Further

$$Y^* \circ \varphi \circ \mathfrak{p}_R^{-1} = Y \circ \mathfrak{p}_S \circ \varphi \circ \mathfrak{p}_R^{-1} = Y \circ h.$$

Hence

$$Y \circ h = f_0 + f_1 y + f_2 y^2.$$

Since $Y^3 = aY + b$, we have

$$(f_0 + f_1 y + f_2 y^2)^3 = a \circ h(f_0 + f_1 y + f_2 y^2) + b \circ h.$$

Received September 25, 1967.

By $y^3=Ay+B$, we have

1) $f_0^3+Bf_1^3+6Bf_0f_1f_2+3ABf_1f_2^2+B^2f_2^3=a\circ hf_0+b\circ h$,

2) $3f_0^2f_1+Af_1^3+6Af_0f_1f_2+3Bf_1^2f_2+3Bf_0f_2^2+3A^2f_1f_2^2+2ABf_2^3=a\circ hf_1$,

3) $3f_0f_1^2+3f_0^2f_2+3Af_1^2f_2+3Af_0f_2^2+3Bf_1f_2^2+A^2f_2^3=a\circ hf_2$.

From 2) and 3) we have

$$(3f_0+2Af_2)(Bf_2^3+Af_1f_2^2-f_1^3)=0.$$

Since $y^3=Ay+B$ is irreducible, there is no single-valued solution of the equation. Hence

$$f_1^3-Af_1f_2^2-Bf_2^3\not\equiv 0,$$

whence follows $3f_0+2Af_2=0$. Substituting this into 3), we have

$$f_2\left(Af_1^2+\frac{1}{3}A^2f_2^2+3Bf_1f_2\right)=a\circ hf_2.$$

If $f_2=0$, then $f_0=0$. Thus $f_1\not\equiv 0$ and hence

4) $$\begin{cases} a\circ h=Af_1^2, \\ b\circ h=Bf_1^3. \end{cases}$$

If $f_2\not\equiv 0$, then we have

5) $$\begin{aligned} a\circ h&=Af_1^2+A^2f_2^2/3+3Bf_1f_2, \\ b\circ h&=\left(-\frac{2}{27}A^3+B^2\right)f_2^3+ABf_1f_2^2+\frac{2}{3}A^2f_1^2f_2+Bf_1^3. \end{aligned}$$

Now we compute the discriminants D_R and $D_{S\circ\varphi}$. Then we have

6) $$\begin{aligned} D_{S\circ\varphi}\equiv D_S\circ h&=27(b\circ h)^2-4(a\circ h)^3 \\ &=D_R[f_1^3-Af_1f_2^2-Bf_2^3]^2, \qquad D_R=27B^2-4A^3. \end{aligned}$$

The case $f_2=0$ gives

$$D_{S\circ\varphi}\equiv D_S\circ h=D_Rf_1^6,$$

which is included in 6). Our necessary condition for the existence of analytic mappings is 5) and 6).

3. A regularly branched three-sheeted surfaces. If $a\equiv 0$, then S is called a regularly branched three-sheeted surface. In this case all the branch points of S are three-sheeted locally like $z^{1/3}$. Now we shall prove the following Proposition.

PROPOSITION 1. *If all the zeros and poles of D_R are of even order and R has* *an infinite number of branch points, then R is conformally equivalent to a regularly branched three-sheeted surface S.*

Proof. Consider a new surface R_1 defined by $y_1^3=A_1y_1+B_1$ with $B_1=Bf_1^3$, $A_1=Af_1^2$. Then by $y_1=f_1y$ we have $y^3=Ay+B$, which defines the R. Hence the correspondence $(z, y_1)\leftrightarrow(z, y)$ gives the conformal equivalence of R_1 and R. Further the evenness of orders of all the zeros and poles of D_R implies that of D_{R_1}. By the above process we may assume that A and B are entire functions. We shall use a notation $\operatorname{ord}(z_j, T)$ as α defined by

$$T(z)=(z-z_j)^{\alpha}S(z), \qquad \lim_{z\to z_j} S(z)\neq 0.$$

It is easy to prove that R has no branch point of order 2. Hence all the branch points of R are of order 3 [2]. Then every branch point of R, which lies over z_j, satisfies

$$\frac{1}{3}\operatorname{ord}(z_j, B)<\frac{1}{2}\operatorname{ord}(z_j, A), \qquad \operatorname{ord}(z_j, B)=3n\pm 1.$$

Further by considering a new surface, if necessary, we may assume that

$$\begin{cases}\operatorname{ord}(z_j, B)=1\\ \operatorname{ord}(z_j, A)\geqq 1\end{cases} \quad \text{or} \quad \begin{cases}=2\\ \geqq 2.\end{cases}$$

Then

$$\operatorname{ord}(z_j, D_R)=2 \qquad \text{or} \quad 4$$

respectively. Assume that the first case occurs. In this case we have

$$\frac{1}{2}\left(B+\frac{\sqrt{D_R}}{3\sqrt{3}}\right)=(z-z_j)U(z), \qquad U(z_j)\neq 0, \infty$$

and

$$\frac{1}{2}\left(B-\frac{\sqrt{D_R}}{3\sqrt{3}}\right)=\frac{1}{27}\frac{A^3}{\frac{1}{2}\left(B+\frac{\sqrt{D_R}}{3\sqrt{3}}\right)}=\frac{1}{27}(z-z_j)^{3p-1}V(z), \qquad V(z_j)\neq 0, \infty,$$

where p is an integer $\geqq 1$. Let $b(z)$ be an entire function satisfying

$$b(z)=(z-z_j)L(z), \qquad L(z_j)\neq 0, \infty$$

around z_j. Then

$$\frac{3\sqrt{3}\,B+\sqrt{D_R}}{2\cdot 3\sqrt{3}\,b}$$

is regular and non-zero around z_j and further

$$\frac{3\sqrt{3}\,B-\sqrt{D_R}}{6\sqrt{3}\,b^2}=(z-z_j)^{3p-3}M(z), \qquad M(z_j)\neq 0, \infty$$

around z_j. Hence the last term may be zero of order $3(p-1)$.

If the second case occurs, then

$$\frac{1}{2}\left(B+\frac{\sqrt{D_R}}{3\sqrt{3}}\right)=(z-z_j)^2U(z), \qquad U(z_j)\neq 0, \infty,$$

$$\frac{1}{2}\left(B-\frac{\sqrt{D_R}}{3\sqrt{3}}\right)=\frac{1}{27}\frac{2A^3}{\left(B+\frac{\sqrt{D_R}}{3\sqrt{3}}\right)}=\frac{1}{27}(z-z_j)^{3p-2}V(z), \qquad V(z_j)\neq 0, \infty,$$

where p is an integer $\geqq 2$. In this case we put $b(z)=(z-z_j)^2L(z)$, $L(z_j)\neq 0$, around z_j. Then

$$\frac{3\sqrt{3}\,B-\sqrt{D_R}}{6\sqrt{3}\,b}=W(z), \qquad W(z_j)\neq 0, \infty$$

and

$$\frac{3\sqrt{3}\,B-\sqrt{D_R}}{6\sqrt{3}\,b^2}=(z-z_j)^{3p-6}N(z), \qquad N(z_j)\neq 0, \infty.$$

Let z_0 be a point satisfying

$$\frac{1}{3}\operatorname{ord}(z_0, B)<\frac{1}{2}\operatorname{ord}(z_0, A), \qquad \operatorname{ord}(z_0, B)=3n.$$

Then we may assume that $\operatorname{ord}(z_0, B)=0$, since we may consider a new surface R_1 if necessary. Then

$$\frac{1}{2}\left(B+\frac{\sqrt{D_R}}{3\sqrt{3}}\right)=T(z), \qquad T(z_0)\neq 0, \infty,$$

$$\frac{1}{2}\left(B-\frac{\sqrt{D_R}}{3\sqrt{3}}\right)=(z-z_0)^{3p}X(z), \qquad X(z_0)\neq 0, \infty,$$

where $p=\operatorname{ord}(z_0, A)\geqq 1$. In this case we put $b(z_0)\neq 0, \infty$. Then

$$\frac{3\sqrt{3}\,B+\sqrt{D_R}}{6\sqrt{3}\,b}=P(z), \qquad P(z_0)\neq 0, \infty,$$

$$\frac{3\sqrt{3}\,B-\sqrt{D_R}}{6\sqrt{3}\,b^2}=(z-z_0)^{3p}Q(z), \qquad Q(z_0)\neq 0, \infty$$

around z_0.

Let z_0 be a point satisfying

$$\frac{1}{2}\operatorname{ord}(z_0, A)<\frac{1}{3}\operatorname{ord}(z_0, B), \qquad \operatorname{ord}(z_0, A)=2n.$$

Then we may assume that $\operatorname{ord}(z_0, A)=0$. Hence $D_R(z_0)\neq 0, \infty$. Then we have

$$\frac{1}{2}\left(B+\frac{\sqrt{D_R}}{3\sqrt{3}}\right)\neq 0, \qquad \text{and} \qquad \frac{1}{2}\left(B-\frac{\sqrt{D_R}}{3\sqrt{3}}\right)\neq 0,$$

around z_0. In this case we put $b(z_0)\neq 0, \infty$. Then

$$\frac{3\sqrt{3}B+\sqrt{D_R}}{6\sqrt{3}b}\neq 0, \infty \qquad \text{and} \qquad \frac{3\sqrt{3}B-\sqrt{D_R}}{6\sqrt{3}b^2}\neq 0, \infty$$

around z_0.

Let z_0 be a point satisfying

$$\frac{1}{2}\operatorname{ord}(z_0, A)=\frac{1}{3}\operatorname{ord}(z_0, B), \qquad \operatorname{ord}(z_0, D_R)=2p.$$

Then $\operatorname{ord}(z_0, A)=2s$ and $\operatorname{ord}(z_0, B)=3t$ with two integers satisfying $s=t$, $6t\leqq 2p$. In this case by considering a new surface we may assume that $A(z_0)\neq 0$, $B(z_0)\neq 0$, $\operatorname{ord}(z_0, D_R)=2q$, $q\geqq 0$. In this case we put $b(z_0)\neq 0, \infty$. Then

$$\frac{3\sqrt{3}B+\sqrt{D_R}}{6\sqrt{3}b}\neq 0, \infty \qquad \text{and} \qquad \frac{3\sqrt{3}B-\sqrt{D_R}}{6\sqrt{3}b^2}\neq 0, \infty$$

around z_0.

By our assumption that D_R has only zeros of even order, the following two cases do not occur:

(i) $$\frac{1}{2}\operatorname{ord}(z, A)<\frac{1}{3}\operatorname{ord}(z, B), \qquad \operatorname{ord}(z, A)=2n-1,$$

(ii) $$\frac{1}{2}\operatorname{ord}(z, A)=\frac{1}{3}\operatorname{ord}(z, B), \qquad \operatorname{ord}(z, D_R)=2n-1.$$

Because each case implies an absurdity relation $\operatorname{ord}(z, D_R)=2q-1$.

We have enumerated all the possible cases. There is at least one entire function $b(z)$ satisfying the required zero-point conditions. Then there are two suitable entire functions L_1 and L_2 such that

$$\frac{3\sqrt{3}B+\sqrt{D_R}}{6\sqrt{3}b}=L_1^3, \qquad \frac{3\sqrt{3}B-\sqrt{D_R}}{6\sqrt{3}b^2}=L_2^3.$$

Let $Y=L_1y+L_2y^2$. Then by $Y^3=AY+B$

$$(y^3-b)\{(L_2y+L_1)^3+bL_2^3\}=0,$$

which implies either $y^3=b$ or $(L_2y+L_1)^3=-bL_2^3$. The latter case is equivalent to $y^{*3}=b$.

If $y^3=b$ and $Y=L_1y+L_2y^2$, then $Y^3=AY+B$. Therefore the correspondence $(z, Y)\leftrightarrow(z, y)$ gives the conformal equivalence between R and a regularly branched three-sheeted surface.

4. Non-existence criteria for analytic mappings. We may assume that all the coefficients of the defining equations are entire.

THEOREM 1. *Let R be a three-sheeted surface defined by $y^3=Ay+B$ whose discriminant D_R has at least one zero of odd order. Let S be a regularly branched three-sheeted surface defined by $Y^3=b$. Then there is no non-trivial analytic mapping of R into S.*

Proof. By 6) we have

$$D_S\circ h=D_R[f_1^3-Af_1f_2^2-Bf_2^3]^2,$$

if φ exists. By the assumption $D_S=27b^2$. This is a contradiction. By the way we give another proof. By 5) we have

$$Af_1^2+\frac{A^2}{3}f_2^2+3Bf_1f_2=0,$$

since $a\equiv 0$. Since f_1 and f_2 are single-valued, the discriminant of this quadratic equation must be of perfectly square form. Hence

$$9B^2-\frac{4}{3}A^3=\frac{1}{3}D_R$$

has no zero of odd order, which contradicts our assumption.

THEOREM 2. *Let R and S be the same as in theorem 1. Further assume that D_R has at least three zeros of odd order. Then there is no non-trivial analytic mapping of S into R.*

Proof. In this case we have

$$D_R\circ h=D_S[f_1^3-bf_2^3]^2.$$

Assume that h is a transcendental entire function. Then by the Nevanlinna ramification relation h has at most two perfectly branched values. Let w_j be a zero of D_R of odd order, then $h(z)=w_j$ allows only solutions of even multiplicity or only a finite number of solutions. Here j runs from 1 to 3, which contradicts the ramification relation. Next assume that h is a polynomial of degree d. Then the equation $h(z)=w_j$, $j=1, 2, 3$, has no simple zero. Hence the number of simple zero must be equal to zero. But it is not less than $3d-d$. This is a contradiction.

THEOREM 3. *Let R and S be two general three sheeted surfaces. Suppose that D_R has m zeros of odd order $(1\leqq m<\infty)$ and D_S has n zeros of odd order. Assume that $n>\max(2, m)$. Then there is no non-trivial analytic mapping of R into S.*

Proof. In this case we have

$$D_S\circ h=D_R[f_1^3-Af_1f_2^2-Bf_2^3]^2.$$

Since $n>3$ and $m<\infty$, h is not transcendental by the Nevanlinna ramification relation. Assume that h is a polynomial of degree d. Consider $h(z)=w_j$, $j=1,2,3,\cdots,n$, where w_j is a zero of D_S of odd order. These equations have simple zeros, which are not less than $nd-d$ in number. Further these simple zeros correspond to some zeros of D_R of odd order. Hence

$$m\geqq nd-d=(n-1)d\geqq md.$$

This implies $d=1$. If $d=1$, then evidently $m=n$, which contradicts our assumption.

THEOREM 4. *Let R and S be the same as in theorem* 3. *If $n=m\geqq 3$, then there is no non-trivial analytic mapping of R into S unless R and S are conformally equivalent. If there is a positive integer t satisfying $(t+1)(n-1)>m>nt$ and if $n\geqq 3$, then there is no non-trivial analytic mapping of R into S.*

Proof. If $n=m\geqq 3$ and if there is a non-trivial analytic mapping of R into S, then the corresponding h must be a polynomial. Let d be the degree of h. Then we have $m\geqq(n-1)d=(m-1)d$, which implies $d=1$. Hence h must be a linear function $\alpha z+\beta$. Then R and S must be conformally equivalent.

Assume that the latter assumption holds, then the degree d of h satisfies

$$nd\geqq m\geqq(n-1)d,\qquad (t+1)(n-1)>m>nt.$$

Hence

$$t+1>d>t,$$

which is a contradiction. Thus we have the desired result.

THEOREM 5. *Let ρ_R be the order of the Nevanlinna N-function of branch points of R of order* 2 *and ρ_S the corresponding quantity for S. Assume $\rho_R<\infty$ and $0<\rho_S<\infty$ and there is a non-trivial analytic mapping φ of R into S. Then $\rho_R=\nu\rho_S$ with a positive integer ν, which is just the degree of h.*

Proof. We should remark the following fact. Every branch point of R of order 2 corresponds to a zero of D_R of odd order and vice versa. Hence the Nevanlinna N-function of branch points of R of order 2 is equal to the N-function N of zeros of D_R of odd order, counting only once. Now we can make use of 6)

$$D_S\circ h=D_R[f_1^3-Af_1f_2^2-Bf_2^3]^2.$$

Now we can use the same procedure to the above equation as in [3].

5. Analytic mappings of R into itself.

PROPOSITION 2. *Let R be a general three-sheeted surface. Assume that D_R has either $n(\geqq 3)$ zeros of odd order or no zero of odd order. Then any non-trivial analytic mapping φ of R into itself induces*

$$h(z)=e^{2\pi ip/q}z+\beta$$

with a suitable rational number p/q and a constant β.

Proof. If D_R has no zero of odd order, then R is a regularly branched three-sheeted surface by Proposition 1. Hence we have the desired result by [3]. Assume that D_R has n zeros of odd order. When n is finite, we can apply the first part of Theorem 4. Thus $h(z)=\alpha z+\beta$. When n is infinite, we can make use of the same method for 6) as in [3]. This implies $h(z)=\alpha z+\beta$. In this case we can use again the same method as in [3] and then we have the desired result. If n is finite, then we make the p-th iteration h_p of h. Then h_p makes a permutation of the underlying set of branch points of order 2. Hence there is an integer ν for which h_ν fixes all points of the set. Since h_ν is a linear function, h_ν must be the identity. Thus $h(z)$ has the desired form.

We prove the following theorem.

THEOREM 6. *Let R be a general three-sheeted surface. Then any non-trivial analytic mapping φ or R into itself is a conformal mapping onto itself and the corresponding $h(z)$ has the following form $e^{2\pi ip/q}z+\beta$, where p/q is a rational number.*

Proof. By Proposition 2 it is sufficient to prove our result in two cases: (A) D_R has only one zero of odd order and (B) D_R has only two zeros of odd order.

In order to prove this theorem we need a result due to Kubota [5], which is a generalization of Heins' theorem [1]: In any hyperbolic surface R whose fundamental group is not abelian, φ satisfies either i) for some integer n the n-th iteration φ_n of φ coincides with the identity mapping or ii) φ_n tends to some ideal boundary or iii) φ_n tends to a point p in R and $\varphi(p)=p$.

Case (B). By 6) $D_R\circ h=D_R(f_1^3-Af_1f_2^2-Bf_2^3)^2$. Let w_1, w_2 be two zeros of D_R of odd order. Then $h(w_j)$ must be a zero of D_R of odd order and hence $h(w_j)$ must be w_k. Assume $h(w_1)=w_1$. Then φ has two fixed points on R, which lie over the same w_1. Hence ii) and iii) do not occur. Assume $h(w_1)=w_2$, then $h(w_2)=w_1$ or w_2. The latter case can be omitted. Hence the second iteration h_2 of h has two fixed points and hence φ has four fixed points. Hence ii) and iii) do not occur.

Case (A). This case implies a contradiction unless i) occurs.

In both cases for some n $h_n=z$. Thus $h(z)$ must be of the desired form.

The following problem seems to be very important. Determine the class of Riemann surfaces on which any non-trivial analytic mapping into itself reduces to a conformal automorphism.

REFERENCES

[1] HEINS, M., On the iteration of functions which are analytic and single-valued in a given multiply-connected region. Amer. Journ. Math. **63** (1941), 461–480.

[2] HENSEL, K., AND G. LANDSBERG, Theorie der algebraischen Funktionen einer Variabeln und ihre Anwendung auf algebraische Kurven und Abelsche Integrale. (1902).

[3] HIROMI, G., AND H. MUTŌ, On the existence of analytic mappings, I. Kōdai Math. Sem. Rep. **19** (1967), 236–244.

[4] HIROMI, G., AND H. MUTŌ, On the existence of analytic mappings, II. Kōdai Math. Sem. Rep. **19** (1967), 439–450.

[5] KUBOTA, Y., On an analytic mapping of an open Riemann surface into itself. Kanazawa symposium in function-theory (July, 1967).

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.