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A RENEWAL TYPE THEOREM ON CONTINUOUS-TIME (J, X)-PROCESSES

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1. Let $\{X(t);\ t\geqq 0\}$ be a continuous-time, real-valued stochastic process. If the process $X(t)$ is measurable, then the expected time during which $X(t)$ stays in an interval $[x, x+h]$ is given by

$$E\left\{\int_0^\infty I_{[x,x+h]}(X(t))dt\right\}=\int_0^\infty P\{x\leqq X(t)\leqq x+h\}dt, \tag{1}$$

where $I_B(\cdot)$ is the indicator function of the set B. Thus theorems which state that the expression on the right in (1) converges to a limit as $x\to\infty$ may be regarded as continuous-time analogues of ordinary renewal theorems. In this paper we shall prove one of such theorems for a special class of stochastic processes, which are continuous-time versions of (J, X)-processes introduced by Pyke [4]. The method of the proof is essentially that of Chung, Pollard [1] and Maruyama [3].

2. Let $\{(J(t),\ X(t));\ t\geqq 0\}$ be a Markov process with the state space $\{1, 2,\cdots, N\}\times R^1$, having the following properties:

(a) $X(0)\equiv 0$.

(b) Its transition probability function is written as

$$P_t\{(j,\ x),\ \{k\}\times(-\infty,\ y]\}=Q_{tjk}(y-x) \tag{2}$$

for any $t>0$ and $j, k\in\{1, 2, \cdots, N\}$. It follows from this assumption that $\{J(t);\ t\geqq 0\}$ is a Markov process with the transition probability function $Q_{tjk}(+\infty)$.

(c) For every $j\neq k$

$$a_{jk}=\lim_{t\to+0}\frac{Q_{tjk}(+\infty)}{t}<\infty.$$

Put $a_{jj}=-\sum_{k\neq j}a_{jk}$, and denote the matrix (a_{jk}) by A.

(d) For each $j, k\in\{1, 2, \cdots, N\}$

$$\Pi_{tjk}(x)=Q_{tjk}(x)/Q_{tjk}(+\infty)$$

satisfies that

$$\lim_{t\to+0}\Pi_{tjk}(x)=\Pi_{jk}(x) \tag{3}$$

in distribution if $j\neq k$ and $a_{jk}>0$, and

$$\lim_{t\to+0}\Pi_{tjj}^{(n)}(x)=\Pi_{jj}(x) \tag{4}$$

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in distribution, where $H_{jk}, j, k\in\{1, 2, \cdots, N\}$, is some distribution function, $n=[1/t]$, and $H^{(n)}$ represents the n-fold convolution of H.

Denote by η_{jk} the characteristic function of H_{jk}. It follows from (4) that for every j H_{jj} is an infinitely divisible distribution function, and therefore its characteristic function has the expression $\eta_{jj}(\theta)=e^{\xi_j(\theta)}$ with

$$\xi_j(\theta)=im_j\theta-\frac{v_j}{2}\theta^2+\int_{-\infty}^{\infty}\left\{e^{i\theta u}-1-\frac{i\theta u}{1+u^2}\right\}d\nu_j(u),$$

where m_j is real, $v_j\geqq 0$ and ν_j is a measure defined on the class of Borel sets of the real line such that

$$\int_{|u|>1}d\nu_j(u)<\infty \quad \text{and} \quad \int_{|u|\leqq 1}u^2 d\nu_j(u)<\infty.$$

Let φ_t denote the characteristic function of $X(t)$, and let

$$\varphi_{jt}(\theta)=E\{e^{i\theta X(t)}|\,J(0)=j\}.$$

Then we have from (2)

$$\begin{aligned}\varphi_{jt+\Delta t}(\theta)&=\sum_{k=1}^{N}\varphi_{kt}(\theta)\int_{-\infty}^{\infty}e^{i\theta x}dQ_{\Delta tjk}(x)\\&=(1+a_{jj}\Delta t)e^{\xi_j(\theta)\Delta t}\varphi_{jt}(\theta)+\sum_{k\neq j}a_{jk}\Delta t\eta_{jk}(\theta)\varphi_{kt}(\theta)+o(\Delta t)\end{aligned}$$

as $\Delta t\to 0$, thus obtaining

$$\frac{\partial\varphi_{jt}(\theta)}{\partial t}=(a_{jj}+\xi_j(\theta))\varphi_{jt}(\theta)+\sum_{k\neq j}a_{jk}\eta_{jk}(\theta)\varphi_{kt}(\theta),$$

i. e.,

$$\frac{\partial\boldsymbol{\varphi}_t(\theta)}{\partial t}=H(\theta)\boldsymbol{\varphi}_t(\theta), \tag{5}$$

where $\boldsymbol{\varphi}_t(\theta)$ represents the N-dimensional column vector whose j-th components are $\varphi_{jt}(\theta)$, and

$$H(\theta)=\begin{pmatrix} a_{11}+\xi_1(\theta) & a_{12}\eta_{12}(\theta) & \cdots & a_{1N}\eta_{1N}(\theta)\\ a_{21}\eta_{21}(\theta) & a_{22}+\xi_2(\theta) & \cdots & a_{2N}\eta_{2N}(\theta)\\ \vdots & \vdots & \cdots & \vdots\\ \vdots & \vdots & \cdots & \vdots\\ a_{N1}\eta_{N1}(\theta) & a_{N2}\eta_{N2}(\theta) & \cdots & a_{NN}+\xi_N(\theta)\end{pmatrix}.$$

Introducing the Laplace integral

$$\boldsymbol{\Phi}(\theta, s)=\int_0^{\infty}\boldsymbol{\varphi}_t(\theta)e^{-st}dt,$$

we obtain from (5) that

$$\boldsymbol{\Phi}(\theta, s)=(sI-H(\theta))^{-1}\boldsymbol{e}, \tag{6}$$

in distribution, where I is the N by N identity matrix, and $\boldsymbol{e}$ is the N-dimensional column vector whose components are all equal to 1.

Throughout the remainder of this paper we assume that the matrix $A=H(0)$ is indecomposable. This assumption implies that $s=0$ is a proper value of A with multiplicity 1, and that every proper value except $s=0$ has negative real part.

Let $\zeta(\theta)$ denote the proper value of $H(\theta)$ such that $\lim_{\theta\to0}\zeta(\theta)=0$. Then by the assumption on A, we can choose positive θ_0 and ε_0 so small that for $|\theta|<\theta_0$ $H(\theta)$ has no proper values except $\zeta(\theta)$ in the half plane $\Re(s)>-\varepsilon_0$. Moreover $\Re(\zeta(\theta))\leqq0$ holds for every θ. In fact, let $\boldsymbol{z}$ be a proper vector corresponding the proper value $\zeta(\theta)$: $H(\theta)\boldsymbol{z}=\zeta(\theta)\boldsymbol{z}$. We can assume that every component z_j of $\boldsymbol{z}$ does not exceed 1 in absolute value and some z_{j_0} is equal to 1. Then

$$\Re(\zeta(\theta))=\sum_{j\neq j_0}a_{j_0j}\Re(\eta_{j_0j}(\theta)z_j)+\Re(a_{j_0j_0}+\xi_{j_0}(\theta))z_{j_0}\leqq\sum_{j\neq j_0}a_{j_0j}+a_{j_0j_0}=0.$$

Now employing the same method as in [2], we can prove from (6) that

$$\Phi(\theta,s)=\frac{\sigma(\theta)}{s-\zeta(\theta)}+\Psi(\theta,s), \tag{7}$$

where $\Phi(\theta,s)$ is the Laplace transform of $\varphi_t(\theta)$, $\sigma(0)=1$, $\Psi(\theta,s)$ is uniformly bounded for $s>0$ in a neighborhood of $\theta=0$, and finite $\lim_{s\to+0}\Psi(\theta,s)$ exists. Moreover we can prove that there exist positive constants K and ε such that

$$|\varphi_t(\theta)-\sigma(\theta)e^{\zeta(\theta)t}|<Ke^{-\varepsilon t} \tag{8}$$

for $|\theta|<\theta_0$ and for $t>0$.

If we assume that every $\eta_{jk}(\theta)$ has continuous second derivatives in a neighborhood of $\theta=0$, then it is easy to show that $\zeta(\theta)$ and $\sigma(\theta)$ have the same property. Applying the method of [2], we see that $m=-i\zeta'(0)$ is real and $\zeta''(0)<0$. From $\sigma(-\theta)=\overline{\sigma(\theta)}$ it follows that $\sigma'(0)$ is pure imaginary.

Remark 1. The inequality (8) with the assumption above enables us to prove a central limit theorem for the process $X(t)$. The method of the proof is similar to that of [2].

Lastly we add the following assumption: for every $\theta\neq0$, either at least one $\xi_j(\theta)\neq0$ or there exist $j,k\in\{1,2,\cdots,N\}$ $(j\neq k)$ such that $a_{jk}>0$ and $|\eta_{jk}(\theta)|<1$. This assumption implies that for every $\theta\neq0$ the matrix $H(\theta)$ is regular. In fact, if $\det H(\theta)=0$, $\theta\neq0$, then there exists a column vector $\boldsymbol{z}\neq\boldsymbol{0}$ such that $H(\theta)\boldsymbol{z}=\boldsymbol{0}$. We may assume that every component z_j of $\boldsymbol{z}$ does not exceed 1 in absolute value and some z_{j_0} is equal to 1. Then

$$-a_{j_0j_0}-\xi_{j_0}(\theta)=\sum_{j\neq j_0}z_ja_{j_0j}\eta_{j_0j}(\theta). \tag{9}$$

From $\Re(\xi_{j_0}(\theta))\leqq0$, $|\eta_{j_0j}(\theta)|\leqq1$ and $|z_j|\leqq1$, (9) holds only if $\xi_{j_0}(\theta)=0$ and $z_j\eta_{j_0j}(\theta)=1$ for every j such that $a_{j_0j}>0$. Therefore $|z_j|=1$ if $a_{j_0j}>0$. Since the equality (9) replaced j_0 by j_1 must hold if $|z_{j_1}|=1$, and since A is indecomposable, it follows that $\xi_j(\theta)=0$ for every j and $|\eta_{jk}(\theta)|=1$ for every j,k such that $a_{jk}>0$. This proves our assertion.

3. We shall now prove the following

THEOREM. *Under the assumptions in the preceding section*

$$\lim_{x\to\infty}\int_0^\infty P\{x\leqq X(t)\leqq x+h\}dt=\begin{cases}\dfrac{h}{m} & \text{if } m>0,\\ 0 & \text{if } m<0.\end{cases} \tag{10}$$

Proof. The theorem follows from the following:

$$\lim_{x\to\infty}\int_0^\infty E\{F(X(t)-x)\}dt=\begin{cases}\dfrac{1}{m}\displaystyle\int_{-\infty}^\infty F(x)dx & \text{if } m>0,\\ 0 & \text{if } m<0,\end{cases} \tag{11}$$

where $F(x)\in L^1(R^1)$ is an arbitrary non-negative continuous even function such that its Fourier transform

$$f(\theta)=\int_{-\infty}^\infty F(x)e^{-i\theta x}dx$$

belongs to $L^1(R^1)$ and vanishes outside a finite interval $(-c, c)$. In order to prove that (11) implies (10), we may apply the same method as that employed by Maruyama [3], and therefore we do not reproduce it. The remaining part of the proof, i. e., the proof of (11) is also quite similar to [1] and [3].

The integral on the left in (11) is written as

$$\begin{aligned}&\lim_{\alpha\to+0}\int_0^\infty e^{-\alpha t}E\{F(X(t)-x)\}dt\\ &=\lim_{\alpha\to+0}\frac{1}{2\pi}\int_0^\infty e^{-\alpha t}dt\int_{-c}^{c} e^{-i\theta x}f(\theta)\varphi_t(\theta)d\theta\\ &=\lim_{\alpha\to+0}\frac{1}{2\pi}\int_{-c}^{c} e^{-i\theta x}f(\theta)\Phi(\theta,\alpha)d\theta.\end{aligned} \tag{12}$$

For any $\delta>0$, it follows from the regularity of matrix $H(\theta)$, $\theta\neq 0$, that $\lim_{\alpha\to+0}\Phi(\theta,\alpha)=\Phi(\theta,0)$ is bounded on every compact interval excluding the origin, and therefore by Riemann-Lebesgue lemma

$$\lim_{x\to\infty}\lim_{\alpha\to+0}\frac{1}{2\pi}\int_{c>|\theta|>\delta} e^{-i\theta x}f(\theta)\Phi(\theta,\alpha)d\theta=0. \tag{13}$$

If δ is sufficiently small, then by (7) and again by Riemann-Lebesgue lemma

$$\lim_{x\to\infty}\lim_{\alpha\to+0}\frac{1}{2\pi}\int_{-\delta}^{\delta} e^{-i\theta x}f(\theta)\left\{\Phi(\theta,\alpha)-\frac{\sigma(\theta)}{\alpha-\zeta(\theta)}\right\}d\theta=0. \tag{14}$$

Now we shall evaluate

$$\lim_{x\to\infty}\lim_{\alpha\to+0}\frac{1}{2\pi}\int_{-\delta}^{\delta} e^{-i\theta x}\frac{f(\theta)\sigma(\theta)}{\alpha-\zeta(\theta)}d\theta=\lim_{x\to\infty}\lim_{\alpha\to+0}\frac{1}{2\pi}\int_{-\delta}^{\delta} f(\theta)\Re\left\{\frac{e^{-i\theta x}\sigma(\theta)}{\alpha-\zeta(\theta)}\right\}d\theta. \tag{15}$$

Denote by $R(\theta)$, $I(\theta)$ and $R_1(\theta)$, $I_1(\theta)$ the real part and imaginary part of $-\zeta(\theta)$ and $\sigma(\theta)$ respectively. Then $R(\theta)=O(\theta^2)$ is non-negative for small θ, $I(\theta)=-m\theta+O(\theta^2)$, $R_1(\theta)=1+O(\theta^2)$ and $I_1(\theta)=-i\sigma'(0)\theta+O(\theta^2)$. Divide the integrand on the right in (15) as follows:

$$\frac{\alpha f\cdot R_1(\cos\theta x-1)}{(\alpha+R)^2+I^2}+\frac{f\cdot(RR_1+II_1)}{(\alpha+R)^2+I^2}\cos\theta x + \frac{\alpha f\cdot R_1}{(\alpha+R)^2+I^2}+\frac{f\cdot\{(\alpha+R)I_1-IR_1\}}{(\alpha+R)^2+I^2}\sin\theta x. \tag{16}$$

We have easily

$$\lim_{\alpha\to+0}\frac{1}{2\pi}\int_{-\delta}^{\delta}\frac{\alpha f\cdot R_1(\cos\theta x-1)}{(\alpha+R)^2+I^2}d\theta=0, \tag{17}$$

and by Riemann-Lebesgue lemma

$$\lim_{x\to\infty}\lim_{\alpha\to+0}\frac{1}{2\pi}\int_{-\delta}^{\delta}\frac{f\cdot(RR_1+II_1)}{(\alpha+R)^2+I^2}\cos\theta x d\theta=\lim_{x\to\infty}\frac{1}{2\pi}\int_{-\delta}^{\delta}\frac{f\cdot(RR_1+II_1)}{R^2+I^2}\cos\theta x d\theta=0. \tag{18}$$

The boundedness of $(d/d\theta)(\zeta(\theta)/\theta)$ implies that $(f\cdot(RI_1-IR_1)\theta)/(R^2+I^2)$ is of bounded variation at the origin, and therefore

$$\begin{aligned}&\lim_{x\to\infty}\lim_{\alpha\to+0}\frac{1}{2\pi}\int_{-\delta}^{\delta}\frac{f\cdot\{(\alpha+R)I_1-IR_1\}\theta}{(\alpha+R)^2+I^2}\cdot\frac{\sin\theta x}{\theta}d\theta\\ &=\lim_{x\to\infty}\frac{1}{2\pi}\int_{-\delta}^{\delta}\frac{f\cdot(RI_1-IR_1)\theta}{R^2+I^2}\cdot\frac{\sin\theta x}{\theta}d\theta\\ &=\lim_{\theta\to0}\frac{f\cdot(RI_1-IR_1)\theta}{2(R^2+I^2)}\\ &=\frac{f(0)}{2m}.\end{aligned} \tag{19}$$

To evaluate the integral corresponding the third term of (16), we note that

$$\lim_{\delta\to+0}\lim_{\alpha\to+0}\frac{1}{2\pi}\int_{-\delta}^{\delta}\frac{\alpha f\cdot R_1}{\alpha^2+m^2\theta^2}d\theta=\frac{f(0)}{2|m|}, \tag{20}$$

and

$$\lim_{\delta\to+0}\lim_{\alpha\to+0}\frac{1}{2\pi}\int_{-\delta}^{\delta}f\cdot R_1\left\{\frac{\alpha}{\alpha^2+m^2\theta^2}-\frac{\alpha}{(\alpha+R)^2+I^2}\right\}d\theta=0. \tag{21}$$

To prove (21) let us write the integral in (21) as follows:

$$\int_{-\delta}^{\delta}f\cdot R_1\frac{\alpha(R^2+I^2-m^2\theta^2)}{(\alpha^2+m^2\theta^2)((\alpha+R)^2+I^2)}d\theta+\int_{-\delta}^{\delta}f\cdot R_1\frac{2\alpha^2R}{(\alpha^2+m^2\theta^2)((\alpha+R)^2+I^2)}d\theta.$$

The first integral does not exceed

$$\int_{-\delta}^{\delta}f\cdot R_1\frac{\alpha}{\alpha^2+m^2\theta^2}\cdot\frac{|R^2+I^2-m^2\theta^2|}{R^2+I^2}d\theta$$

in absolute value, which converges as $\alpha\to+0$ to

$$\lim_{\theta\to 0} f\cdot R_1 \frac{|R^2+I^2-m^2\theta^2|}{R^2+I^2}=0.$$

The integrand of the second integral is uniformly bounded and converges to 0 as $\alpha\to+0$, hence the integral converges to 0. From (20) and (21) it follows that

$$\lim_{\delta\to+0}\lim_{\alpha\to+0}\frac{1}{2\pi}\int_{-\delta}^{\delta}\frac{\alpha f\cdot R_1}{(\alpha+R)^2+I^2}d\theta=\frac{f(0)}{2|m|}. \tag{22}$$

(17), (18), (19) and (22) together prove

$$\lim_{\delta\to+0}\lim_{x\to\infty}\lim_{\alpha\to+0}\frac{1}{2\pi}\int_{-\delta}^{\delta}e^{-i\theta x}\frac{f(\theta)\sigma(\theta)}{\alpha-\zeta(\theta)}d\theta=\frac{f(0)}{2}\left(\frac{1}{m}+\frac{1}{|m|}\right), \tag{23}$$

and the theorem follows from (13), (14) and (23).

Remark 2. This proof is also applicable to show that under the assumptions of the theorem

$$\lim_{x\to\infty}\int_0^\infty P\{J(t)=k,\ x\leqq X(t)\leqq x+h\}dt=\begin{cases}\dfrac{\pi_k h}{m} & \text{if } m>0,\\ 0 & \text{if } m<0\end{cases}$$

holds, where $\pi_k=\lim_{t\to\infty}P\{J(t)=k\}$.

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