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A RENEWAL THEOREM ON (J, X)-PROCESSES

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1. Let $\boldsymbol{I}_r=\{1, 2, \cdots, r\}$ and $R=(-\infty, \infty)$, and let $\{(J_n, X_n);\ n=0, 1, 2, \cdots\}$ be a (J, X)-process with the state space $\boldsymbol{I}_r\times R$, or a two-dimensional stochastic process that satisfies $X_0\equiv 0$, and

$$P\{J_n=k,\ X_n\leqq x|(J_0, X_0), \cdots, (J_{n-1}, X_{n-1})\}=Q_{J_{n-1},\,k}(x) \qquad \text{(a.s.)},$$

for all $(k, x)\in\boldsymbol{I}_r\times R$, where $\{Q_{jk}(\cdot);\ j, k=1, 2, \cdots, r\}$ is a family of non-decreasing functions defined on R such that $Q_{jk}(-\infty)=0$ for $j, k=1, 2, \cdots, r$, and $\sum_{k=1}^{r}Q_{jk}(+\infty)=1$ for $j=1, 2, \cdots, r$. In the following we shall prove under some conditions that

$$\text{(1)} \qquad \lim_{x\to\infty}\sum_{n=1}^{\infty}P\{x\leqq X_1+\cdots+X_n\leqq x+h\}=\begin{cases} h/m & \text{if } m>0, \\ 0 & \text{if } m<0, \end{cases}$$

where m is a constant such that

$$p\text{-}\lim_{n\to\infty}\frac{X_1+\cdots+X_n}{n}=m.$$

2. Throughout this paper we set the following assumptions:

(i) There exists a positive integer M for which every element of the matrix P^M is positive, where $P=(p_{jk})$ is the $r\times r$ matrix with elements $p_{jk}\overset{\text{def}}{=}Q_{jk}(+\infty)$,

(ii) the conditional distribution of every X_n given $J_{n-1}=j$ and $J_n=k$ is a non-lattice distribution with the finite moment of 2nd order,
and

$$\text{(iii)} \qquad \overline{\lim_{|t|\to\infty}}\,|\phi_{jk}(t)|<1 \qquad (j\in\boldsymbol{I}_r,\ k\in\boldsymbol{I}_r),$$

where

$$\phi_{jk}(t)\overset{\text{def}}{=}E\{e^{itX_n}|J_{n-1}=j,\ J_n=k\}$$
$$=\frac{1}{p_{jk}}\int_{-\infty}^{\infty}e^{itx}dQ_{jk}(x) \qquad (i=\sqrt{-1}).$$

When $p_{jk}=0$, ϕ_{jk} may be chosen arbitrarily. There is some notational advantage, however, in choosing the characteristic function of a non-lattice distribution with the finite moment of 2nd order. Now we have the following

THEOREM. *Under the assumptions* (i), (ii) *and* (iii), *we have*

$$\text{(2)} \qquad \lim_{x\to\infty}\sum_{n=1}^{\infty}P\{x\leqq S_n\leqq x+h\}=\begin{cases} h/m & \textit{if } m>0, \\ 0 & \textit{if } m<0, \end{cases}$$

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where $S_n=X_1+\cdots+X_n$ *and* m *is a constant concerned with* $\{(J_n, X_n);\ n=0,1,2,\cdots\}$.

Proof. To prove (2) it is sufficient to prove that

$$\lim_{x\to\infty}\lim_{r\to 1-0}\int_{|t|\geqq\delta}\frac{1-\cos ht}{t^2}\cdot e^{-itx}\cdot\sum_{n=0}^{\infty} r^n\varphi_n(t)dt=0 \qquad \text{for any } \delta>0 \tag{3}$$

and

$$\lim_{\delta\to+0}\lim_{x\to\infty}\lim_{\varepsilon\to+0}\frac{1}{\pi}\int_{|t|<\delta}\frac{1-\cos ht}{t^2}\cdot\mathrm{Re}\left\{e^{-itx}\cdot\sum_{n=0}^{\infty}\frac{\varphi_n(t)}{(1+\varepsilon)^n}\right\}dt = \frac{h^2}{2}\left(\frac{1}{m}+\frac{1}{|m|}\right), \tag{4}$$

where $\varphi_n(t)\overset{\text{def}}{=}E\{e^{itS_n}\}$. This fact has been shown by Chung and Pollard [1]. From the assumptions (ii) and (iii) we have $\varepsilon(\delta)\overset{\text{def}}{=}\mathrm{Max}_{j,k=1,2,\cdots,r}\sup_{|t|\geqq\delta}|\psi_{jk}(t)|<1$ for all $\delta>0$. Since

$$|\varphi_n(t)|=\left|\sum_{j,j_1,\cdots,j_n} P\{J_0=j\}p_{jj_1}p_{j_1j_2}\cdots p_{j_{n-1}j_n}\psi_{jj_1}(t)\psi_{j_1j_2}(t)\cdots\psi_{j_{n-1}j_n}(t)\right| \leqq\varepsilon(\delta)^n \qquad \text{for } |t|\geqq\delta,$$

we have

$$\left|\sum_{n=0}^{\infty} r^n\varphi_n(t)\right|\leqq\sum_{n=0}^{\infty}\varepsilon(\delta)^n=\frac{1}{1-\varepsilon(\delta)} \qquad \text{for } 0<r<1 \text{ and } |t|\geqq\delta,$$

which implies with Riemann-Lebesgue lemma that

$$\lim_{x\to\infty}\lim_{r\to 1-0}\int_{|t|\geqq\delta}\frac{1-\cos ht}{t^2}\cdot e^{itx}\cdot\sum_{n=0}^{\infty} r^n\varphi_n(t)dt =\lim_{x\to\infty}\int_{|t|\geqq\delta}\frac{1-\cos ht}{t^2}\cdot e^{itx}\cdot\sum_{n=0}^{\infty}\varphi_n(t)dt =0 \qquad \text{for every } \delta>0.$$

Therefore we have (3). The equation $\det(\delta_{jk}-zp_{jk}\psi_{jk}(t))=0$ on z has a root $z=\zeta_0(t)$ for small t such that $\zeta_0(t)\to 1$ as $t\to 0$. We have under the assumption (i) that

$$\left|\varphi_n(t)-\frac{\tau_0(t)}{\zeta_0(t)^n}\right|<K\cdot\frac{(r-1)(n+1)^{r-2}}{(1+\varepsilon_1)^{n-r+2}} \tag{5}$$

for $n=1,2,\cdots$, and $|t|<t_0$, where K, ε_1 and t_0 are positive constants. The functions $\zeta_0(t)$ and $\tau_0(t)$ have continuous derivatives of 2nd order for $|t|<t_0$, respectively, and $\lim_{t\to 0}\tau_0(t)=\tau_0(0)=1$. These facts have been proved in [2]. Putting $\varphi_n(t)=\tau_0(t)/\zeta_0(t)^n+\rho_n(t)$ for $|t|<t_0$, we have from (5)

$$|\rho_n(t)|\leqq\frac{K_1}{(1+\varepsilon_1/2)^n} \qquad \text{for } |t|<t_0, \tag{6}$$

where K_1 is a positive constant independent of n and t. We shall now prove (4). Since

$$\left|\mathrm{Re}\left\{\sum_{n=0}^{\infty}\frac{\rho_n(t)}{(1+\varepsilon)^n}\right\}\right|\leqq\frac{K_1}{1-(1+\varepsilon_1/2)^{-1}}$$

for $\varepsilon>0$ and $|t|<t_0$, we have for every $\delta<t_0$ that

$$\lim_{x\to\infty}\lim_{\varepsilon\to+0}\int_{|t|<\delta}\frac{1-\cos ht}{t^2}\cdot e^{-itx}\cdot\sum_{n=0}^{\infty}\frac{\rho_n(t)}{(1+\varepsilon)^n}dt=0. \tag{7}$$

Thus in order to prove (4) it remains to show that

$$\begin{aligned}&\lim_{\delta\to+0}\lim_{x\to\infty}\lim_{\varepsilon\to+0}\frac{1}{\pi}\int_{|t|<\delta}\frac{1-\cos ht}{t^2}\cdot\mathrm{Re}\left\{e^{-itx}\cdot\sum_{n=0}^{\infty}\frac{\tau_0(t)}{[(1+\varepsilon)\zeta_0(t)]^n}\right\}dt\\ &=\lim_{\delta\to+0}\lim_{x\to\infty}\lim_{\varepsilon\to+0}\frac{1}{\pi}\int_{|t|<\delta}\frac{1-\cos ht}{t^2}\cdot\mathrm{Re}\left\{e^{-itx}\cdot\frac{\tau_0(t)}{1+\varepsilon-\zeta_0(t)^{-1}}\right\}dt\\ &=\frac{h^2}{2}\left(\frac{1}{m}+\frac{1}{|m|}\right).\end{aligned} \tag{8}$$

Writing $m=i\zeta_0'(0)$, $m'=\zeta_0''(0)$, m and m' are real constants. In fact, we have that $(X_1+\cdots+X_n)/n$ converges in probabilty to m and $(X_1+\cdots+X_n-nm)/\sqrt{n}$ converges in distribution to $N(0, m^2+m')$, which are found in [2]. Since

$$\begin{aligned}\zeta_0(t)^{-1}=\frac{1}{\zeta_0(t)}&=\frac{1}{1-imt+(m'/2)t^2+o(t^2)}\\ &=1+imt-(m^2+m'/2)t^2+o(t^2),\end{aligned}$$

we have that

$$\begin{aligned}&R=R(t)\overset{\text{def}}{=}\mathrm{Re}\,(1-\zeta_0(t)^{-1})=O(t^2),\\ &I=I(t)\overset{\text{def}}{=}\mathrm{Im}\,(imt-\zeta_0(t)^{-1})=o(t^2),\\ &R_1=R_1(t)\overset{\text{def}}{=}\mathrm{Re}\,(\tau_0(t))=1+O(t)\end{aligned} \tag{9}$$

and

$$I_1=I_1(t)\overset{\text{def}}{=}\mathrm{Im}\,(\tau_0(t))=O(t).$$

Moreover, we have

$$R(t)\geqq 0. \tag{10}$$

We shall prove it. There exists a r-dimensional vector $\begin{bmatrix}x_1\\ \vdots\\ x_r\end{bmatrix}\neq\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}$ such that

$$(p_{jk}\phi_{jk}(t))\begin{bmatrix}x_1\\ \vdots\\ x_r\end{bmatrix}=\frac{1}{\zeta_0(t)}\begin{bmatrix}x_1\\ \vdots\\ x_r\end{bmatrix}.$$

Taking $j_0\in\boldsymbol{I}_r$ such that $\mathrm{Max}_{j=1,2,\cdots,r}|x_j|=|x_{j_0}|>0$, we have

$$\begin{aligned}\left|\frac{x_{j_0}}{\zeta_0(t)}\right|&=\left|\sum_{k=1}^{r}p_{j_0k}\phi_{j_0k}(t)\cdot x_k\right|\\ &\leqq\sum_{k=1}^{r}p_{j_0k}|x_k|\leqq|x_{j_0}|,\end{aligned}$$

which implies that

$$\left|\frac{1}{\zeta_0(t)}\right|\leqq 1$$

and so

$$\mathrm{Re}\,(\zeta_0(t)^{-1}) \leqq 1.$$

Hence we have (10). Now, we have that

$$\mathrm{Re}\left\{e^{-itx}\cdot\frac{\tau_0(t)}{1+\varepsilon-\zeta_0(t)^{-1}}\right\}=\frac{P\cos tx+Q\sin tx}{(\varepsilon+R)^2+(mt-I)^2},$$

where

$$P=(\varepsilon+R)R_1+(mt-I)I_1$$

and

$$Q=(\varepsilon+R)I_1-(mt-I)R_1.$$

Since $Q=O(t)$ as $t\to 0$, it follows by using (10) that

$$\frac{|t\cdot Q|}{(\varepsilon+R)^2+(mt-I)^2}\leqq\frac{|t\cdot Q|}{R^2+(mt-I)^2}\leqq K_2 \qquad \text{for } 0<\varepsilon<\varepsilon_0 \text{ and } |t|<t_0,$$

where ε_0 is a constant and K_2 is a constant independent of t and ε, which implies, with the fact that

$$\frac{\{RI_1-(mt-I)R_1\}\cdot t}{R^2+(mt-I)^2}$$

is of bounded variation,

$$\begin{aligned}(11)\qquad &\lim_{x\to\infty}\lim_{\varepsilon\to+0}\frac{1}{\pi}\int_{|t|<\delta}\frac{1-\cos ht}{t^2}\cdot\frac{t\cdot Q}{(\varepsilon+R)^2+(mt-I)^2}\cdot\frac{\sin tx}{t}dt\\ &=\lim_{t\to 0}\frac{1-\cos ht}{t^2}\cdot\frac{\{RI_1-(mt-I)R_1\}\cdot t}{R^2+(mt-I)^2}=\frac{h^2}{2m}.\end{aligned}$$

To prove (8) we must now show that

$$(12)\qquad \lim_{\delta\to+0}\lim_{x\to\infty}\lim_{\varepsilon\to+0}\frac{1}{\pi}\int_{|t|<\delta}H\cdot\frac{P\cos tx}{(\varepsilon+R)^2+A^2}dt=\frac{h^2}{2|m|},$$

where

$$H=H(t)\overset{\mathrm{def}}{=}\frac{1-\cos ht}{t^2} \qquad\text{and}\qquad A=A(t)\overset{\mathrm{def}}{=}mt-I.$$

The integrand in (12) can be written as

$$\begin{aligned}&H\cdot\frac{P(\cos tx-1)}{(\varepsilon+R)^2+A^2}+H\cdot\frac{\varepsilon R_1}{(\varepsilon+R)^2+A^2}+H\cdot\frac{RR_1+(mt-I)I_1}{(\varepsilon+R)^2+A^2}\\ &=J_1+J_2+J_3, \qquad \text{(say)}.\end{aligned}$$

Since

$$\frac{RR_1+(mt-I)I_1}{R^2+A^2}=O(1) \qquad \text{as } t\to 0,$$

we have

$$\begin{aligned}
&\lim_{\delta\to+0}\lim_{x\to\infty}\lim_{\varepsilon\to+0}\int_{|t|<\delta} J_3 dt = \lim_{\delta\to+0}\lim_{\varepsilon\to+0}\int_{|t|<\delta} H\cdot\frac{RR_1+(mt-I)I_1}{(\varepsilon+R)^2+A^2}dt \\
(13)\quad &= \lim_{\delta\to+0}\int_{|t|<\delta} H\cdot\frac{R}{R^2+A^2}dt = 0.
\end{aligned}$$

Since

$$\frac{|P\cdot(\cos tx-1)|}{(\varepsilon+R)^2+A^2} \leqq \frac{K_3 t^2}{R^2+A^2} \leqq K_4$$

for fixed x, $|t|<\delta$ and $0<\varepsilon<\varepsilon_0$, where K_3 and K_4 are independent of t, and

$$\left[\frac{P}{(\varepsilon+R)^2+A^2}\right]_{\varepsilon=0} = \frac{RR_1+(mt-I)I_1}{R^2+A^2} = O(1) \qquad \text{as } t\to 0,$$

we have by Riemann-Lebesgue lemma that

$$\begin{aligned}
&\lim_{\delta\to+0}\lim_{x\to\infty}\lim_{\varepsilon\to+0}\int_{|t|<\delta} J_1 dt \\
(14)\quad &= \lim_{\delta\to+0}\lim_{x\to\infty}\int_{|t|<\delta} H\cdot\frac{(\cos tx-1)\cdot\{RR_1+(mt-I)I_1\}}{R^2+A^2}dt \\
&= -\lim_{\delta\to+0}\int_{|t|<\delta} H\cdot\frac{RR_1+(mt-I)I_1}{R^2+A^2}dt = 0.
\end{aligned}$$

Now, we shall estimate $\int_{|t|<\delta} J_2 dt$. We have

$$\begin{aligned}
&\frac{h^2}{2|m|} - \lim_{\delta\to+0}\lim_{\varepsilon\to+0}\frac{1}{\pi}\int_{|t|<\delta} J_2 dt \\
(15)\quad &= \lim_{\delta\to+0}\lim_{\varepsilon\to+0}\frac{1}{\pi}\int_{|t|<\delta} H\cdot\left\{\frac{\varepsilon}{\varepsilon^2+m^2t^2} - \frac{\varepsilon R_1}{(\varepsilon+R)^2+A^2}\right\}dt \\
&= \lim_{\delta\to+0}\lim_{\varepsilon\to+0}\frac{1}{\pi}\int_{|t|<\delta} H\cdot\frac{\varepsilon(R^2+A^2-m^2t^2R_1)+2\varepsilon^2\cdot R+\varepsilon^3(1-R_1)}{(\varepsilon^2+m^2t^2)[(\varepsilon+R)^2+A^2]}dt = 0,
\end{aligned}$$

because we have by using $A=mt+o(t)$ and (9) that

$$\frac{|R^2+A^2-m^2t^2R_1|}{(\varepsilon+R)^2+A^2} \leqq \frac{|R^2+A^2-m^2t^2R_1|}{R^2+A^2} = o(1) \qquad \text{as } t\to 0,$$

$$H\cdot\frac{2\varepsilon^2 R}{(\varepsilon^2+m^2t^2)[(\varepsilon+R)^2+A^2]} \leqq \frac{2RH}{R^2+A^2} = O(1) \qquad \text{as } t\to 0$$

and

$$H\cdot\frac{\varepsilon^3\cdot|1-R_1|}{(\varepsilon^2+m^2t^2)[(\varepsilon+R)^2+A^2]} \leqq H\cdot\frac{|1-R_1|}{\sqrt{R^2+A^2}} = O(1) \qquad \text{as } t\to 0.$$

(13), (14) and (15) imply (12), whence the desired result.

References

[1] Chung, K. L., and H. Pollard, An extension of renewal theory. Proc. Amer. Math. Soc. **3** (1952), 303–309.

[2] Hatori, H., and T. Mori, An improvement of a limit theorem on (J, X)-processes. Kōdai Math. Sem. Rep. **18** (1966), 347–352.

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