KŌDAI MATH. SEM. REP.
19 (1967), 103–128

SUBMANIFOLDS IN FUBINIAN MANIFOLDS

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In the theory of surfaces in the 3-dimensional Euclidean space E^3, the following well known theorem of Bonnet plays an important role. Theorem of Bonnet: Let S be a 2-dimensional Riemannian space with the fundamental tensor g and there be given on S a symmetric tensor h of type $(0, 2)$ which satisfies equations of Gauss and Codazzi, then S is realized in E^3 as a surface having g and h as the first and the second fundamental tensors respectively; Moreover, this realization is unique up to motions in E^3.

In the case of complex space, we can easily foresee a theorem similar to that of Bonnet, but we do not find it in literature. One of the purposes of the present paper is to establish theorem of Bonnet for complex analytic hypersurfaces in a *Fubinian manifold* (i.e. the complex projective space with the natural Kählerian metric) and we shall study some topics in the theory of surfaces in spaces, Fubinian or locally Fubinian. A Kählerian manifold of constant holomorphic curvature is called a *locally Fubinian manifold* (Tashiro and Tachibana [5], Yano [7]).

We introduce, in § 0, terminologies, notations and the structure equations for surfaces in a Riemannian or Kählerian manifold. In § 1, we prepare formulae for a complex hypersurface in a locally Fubinian manifold which are useful in the next three sections. As a direct result of those formulae we obtain a proposition (Smyth [4]) for an invariant Einstein hypersurface in a locally Fubinian manifold. Recently we had an opportunity to see [4], where Smyth has accomplished the classification of simply connected complex hypersurfaces of Fubinian manifold which are complete and Einstein. § 2 is devoted to prove the theorem of Bonnet in a Fubinian manifold. In §§ 3 and 4, we study some properties of submanifolds not necessarily invariant in a locally Fubinian manifolds and characterize totally geodesic or totally umbilical submanifolds in a locally Fubinian manifold.

§ 0. Formulae for surfaces in manifolds, Riemannian or Kählerian.

Let M be a complex manifold of complex dimension n with complex structure F and S a complex manifold of complex dimension $n-1$ with complex structure f. We assume that there is given a complex analytic immersion ξ: $S \rightarrow M$, whose differential B: $T_p(S) \rightarrow T_{\xi(p)}(M)$ is injective at each point p of S, where $T_p(S)$ and $T_{\xi(p)}(M)$ denote the tangent space of S at p and the tangent space of M at $\xi(p)$, respectively. Thus, ξ being complex analytic, we have by definition

Received July 7, 1966.

$$(0.1)\qquad B\circ f = F\circ B,$$

which implies that the subspace $BT_p(S)$ of $T_{\xi(p)}(M)$ is invariant under the action of F at each point $\xi(p)$ belonging to $\xi(S)$. The pair (S, ξ) of an $(n-1)$-dimensional complex manifold S and a complex analytic immersion ξ: $S \to M$ will be called a *complex analytic hypersurface* in M. However, we denote simply by S a complex analytic hypersurface (S, ξ) and identify the complex analytic manifold S with its image $\xi(S)$ by the immersion ξ. A complex analytic hypersurface S is sometimes called briefly a *complex hypersurface* or an *invariant hypersurface*.

Before going further we introduce here the structure equations for a submanifold of a Riemannian manifold.

Let M be a Riemannian manifold of dimension $2n$ and S a differentiable[1] manifold of dimension $2(n-1)$ and ξ a differentiable immersion from S into M whose differential B is injective. The pair (S, ξ) of such S and ξ is called a *surface* of M and simply denoted by S. It is well known that a Riemannian metric g can be induced on S from the Riemannian metric G in M by the immersion ξ, that is

$$(0.2)\qquad g = B^*G$$

B^* being the dual map of the differential B of the immersion ξ. Let us denote by (ξ^h)[2] local coordinates in each coordinate neighborhood of M and by $G=(G_{ji})$ the Riemannian metric tensor in M with respect to (ξ^h). (The quantities in parentheses denote the components in the local coordinate system (ξ^h).) We denote by $\langle X, Y\rangle$ the inner product of two vectors X and Y on M with respect to G, i.e.

$$\langle X, Y\rangle = G_{ji}X^jY^i,$$ [3]

X^j and Y^i being respective components of X and Y with respect to (ξ^h).

Hereafter, X, Y, Z and W will denote vector fields on S, that is, fields of vectors tangent to S. Then BX is a vector field in M tangent to S which is ~~denoted~~ by $\bar{X}$, and so on. Denoting by $\bar{\nabla}$ the covariant differentiation along M with respect to G, we can put

$$(0.3)\qquad \bar{\nabla}_{\bar{X}}\bar{Y} = \nabla_X Y + N_X Y, \qquad \bar{X} = BX \quad\text{and}\quad \bar{Y} = BY,$$

where $\nabla_X Y$ is the tangential part of $\bar{\nabla}_{\bar{X}}\bar{Y}$ and $N_X Y$ is the normal part of $\bar{\nabla}_{\bar{X}}\bar{Y}$. As an immediate consequence of the definition (0.3) for ∇, we easily verify that the correspondence ∇, which assigns a vector field $\nabla_X Y$ to a pair of two vector fields X and Y tangent to S, is a metric covariant differentiation along S (i.e. $\nabla_X g = 0$) and torsionless. That is to say, ∇ is the Riemannian connection determined by the

1) The differentiability of manifolds, mappings and geometric objects are assumed to be of class C^∞ throughout this paper.

2) The indices $h, i, j, \cdots$ run over the range $1, 2, \cdots, 2n$ and indices a, b, c, d, e, f the range $1, 2, \cdots, 2n-2$, where $2n-2$ is the real dimension of S.

3) We use the summation convention.

induced metric g on S. The metric connection thus introduced is called the *induced connection* on S.

Let U be a sufficiently small coordinate neighborhood of S, in which there exist two fields C and D of unit normal vectors to S which are mutually orthogonal at each point of U. The second term $N_X Y$ in the right hand side of (0. 3) is then written, by using C and D, as

$$N_X Y=\tilde{h}(X, Y)C+\tilde{k}(X, Y)D, \tag{0. 4}$$

where $\tilde{h}$ and $\tilde{k}$ are both symmetric bilinear forms on tangent space at each point of S.

The equations of Weingarten (cf. for example, Yano [7]) are reduced to

$$\begin{aligned} \bar{\nabla}_{\bar{X}}\bar{Y}&=\nabla_X Y+\tilde{h}(X, Y)C+\tilde{k}(X, Y)D \\ \bar{\nabla}_{\bar{X}}C&=-T_X \qquad\qquad +\tilde{l}(X)D, \\ \bar{\nabla}_{\bar{X}}D&=-T_X'-\tilde{l}(X)C, \end{aligned} \tag{0. 5}$$

for any two vector fields X and Y on S, where T_X and T_X' are respectively the tangential parts of $\bar{\nabla}_{\bar{X}}C$ and $\bar{\nabla}_{\bar{X}}D$ along S and $\tilde{l}$ is a 1-form on S. The two vector fields T_X and T_X' thus defined satisfy

$$(T_X, Y)=\tilde{h}(X, Y)$$

and

$$(T_X', Y)=\tilde{k}(X, Y),$$

where (,) denotes the inner product with respect to the induced metric g. In fact, we have $\langle\bar{\nabla}_{\bar{X}}C, C\rangle=0$ since $\langle C, C\rangle=1$. This implies that the normal part of $\bar{\nabla}_{\bar{X}}C$ is expressed by D alone and thus we get the second equation of (0. 5). Similarly we can show that the normal part of $\bar{\nabla}_{\bar{X}}D$ is expressed by C alone. On the other hand, we have

$$\langle\bar{\nabla}_{\bar{X}}C, D\rangle+\langle C, \bar{\nabla}_{\bar{X}}D\rangle=0$$

since C and D are mutually orthogonal. From this we see that if we denote by $\tilde{l}(X)D$ the normal part of $\bar{\nabla}_{\bar{X}}C$, then the normal part of $\bar{\nabla}_{\bar{X}}D$ is $-\tilde{l}(X)C$. If we take as Y a tangent vector field on S, then we have $\langle C, \bar{Y}\rangle=0$. Differentiating covariantly the both sides of this equation, we have

$$\langle\bar{\nabla}_{\bar{X}}C, \bar{Y}\rangle+\langle C, \bar{\nabla}_{\bar{X}}\bar{Y}\rangle=0.$$

Substituting the first and the second equations of (0. 5) into the equation above we get $(T_X, Y)=\tilde{h}(X, Y)$. By the same method we can verify $(T_X', Y)=\tilde{k}(X, Y)$.

We shall now write down equations of Gauss, Codazzi and Ricci for a $(2n-2)$-dimensional surface in a $2n$-dimensional manifold, for the later use. By virtue of

Ricci identity, we have

$$\langle \bar{\nabla}_{\bar{W}}\bar{\nabla}_{\bar{V}}\bar{Y}-\bar{\nabla}_{\bar{V}}\bar{\nabla}_{\bar{W}}\bar{Y}-\bar{\nabla}_{[\bar{W},\bar{V}]}\bar{Y}, \bar{X}\rangle=\bar{K}(\bar{W}, \bar{V}, \bar{Y}, \bar{X}), \tag{0. 6}$$

where $\bar{K}$ is the curvature tensor of M and $\bar{W}=BW$, $\bar{V}=BV$, $\bar{Y}=BY$ and $\bar{X}=BX$, W, V, Y and X being vector fields on S. Differentiating covariantly the both sides of (0. 5)

$$\bar{\nabla}_{\bar{V}}\bar{Y}=\nabla_V Y+\tilde{h}(V, Y)C+\tilde{k}(V, Y)D$$

along S, we have

$$\begin{aligned}\bar{\nabla}_{\bar{W}}\bar{\nabla}_{\bar{V}}\bar{Y}=&\nabla_W\nabla_V Y-\tilde{h}(V, Y)T_W-\tilde{k}(V, Y)T_W'\\ &+\{\tilde{h}(W, \nabla_V Y)+W(\tilde{h}(V, Y))-\tilde{l}(W)\tilde{k}(V, Y)\}C\\ &+\{\tilde{k}(W, \nabla_V Y)+W(\tilde{k}(V, Y))+\tilde{l}(W)\tilde{h}(V, X)\}D.\end{aligned}$$

By the definition of the induced connection, we have

$$\nabla_{[\bar{W},\bar{V}]}\bar{Y}=\nabla_{[W,V]}Y+\tilde{h}([W, V], Y)C+\tilde{k}([W, V], Y)D$$

because of $[\bar{W}, \bar{V}]=[W, V]$. Thus the left hand side of (0. 6) are reduced to, for a vector field X on S,

$$\begin{aligned}&\langle \bar{\nabla}_{\bar{W}}\bar{\nabla}_{\bar{V}}\bar{Y}-\bar{\nabla}_{\bar{V}}\bar{\nabla}_{\bar{W}}\bar{Y}-\bar{\nabla}_{[\bar{W},\bar{V}]}\bar{Y}, \bar{X}\rangle\\ &=(\nabla_W\nabla_V Y-\nabla_V\nabla_W Y-\nabla_{[W,V]}Y, X)\\ &\quad-\tilde{h}(V, Y)\tilde{h}(W, X)+\tilde{h}(W, Y)\tilde{h}(V, X)-\tilde{k}(V, Y)\tilde{k}(W, X)+\tilde{k}(W, Y)\tilde{k}(V, X)\end{aligned}$$

because of $(T_W, X)=\tilde{h}(W, X)$ and $(T_W', X)=\tilde{k}(W, X)$. Therefore, if we take account of (0. 6), we have, denoting by K the curvature tensor of S,

$$\begin{aligned}\bar{K}(BW, BV, BY, BX)=K(W, V, Y, X)-\{&\tilde{h}(W, X)\tilde{h}(V, Y)-\tilde{h}(W, Y)\tilde{h}(V, X)\\ &+\tilde{k}(W, X)\tilde{k}(V, Y)-\tilde{k}(W, Y)\tilde{k}(V, X)\},\end{aligned} \tag{0. 7}$$

which is the *equation of Gauss*.

The *equation of Codazzi* follows by replacing X by C in (0. 6). It is written as

$$\begin{aligned}\bar{K}(BW, BV, BY, C)=&\tilde{h}(W, \nabla_V Y)+W(\tilde{h}(V, Y))-\tilde{l}(W)\tilde{k}(V, Y)\\ &-\tilde{h}(V, \nabla_W Y)-V(\tilde{h}(W, Y))+\tilde{l}(V)k(W, Y)-\tilde{h}([W, V], Y).\end{aligned} \tag{0. 8}$$

On the other hand, differentiating $\tilde{h}(V, Y)$ and $\tilde{h}(W, Y)$ covariantly, we obtain respectively

$$W(\tilde{h}(V, Y))-\tilde{h}(V, \nabla_W Y)=\tilde{h}(\nabla_W V, Y)+((\nabla_W h)V, Y)$$

and

$$-V(\tilde{h}(W, Y))+\tilde{h}(W, \nabla_V Y)=-\tilde{h}(\nabla_V W, Y)-((\nabla_V h)W, Y).$$

Thus, if we substitute these two equations above into (0. 8), the equation of Codazzi reduces to

$$\bar{K}(BW, BV, BY, C)=((\nabla_W h)V-(\nabla_V h)W-\tilde{l}(W)kV+\tilde{l}(V)kW, Y). \tag{0. 9}$$

Replacing C by D in (0. 9), we get

$$\bar{K}(BW, BV, BY, D)=((\nabla_W k)V-(\nabla_V k)W+\tilde{l}(W)hV-\tilde{l}(V)hW, Y). \tag{0. 10}$$

A direct computation shows that the normal part of $\bar{\nabla}_{\bar{W}}\bar{\nabla}_{\bar{V}}C$ has the form

$$\{-\tilde{l}(V)\tilde{l}(W)-\tilde{h}(hV, W)\}C+\{W(\tilde{l}(V))-\tilde{k}(hV, W)\}D$$

and that of $\bar{\nabla}_{[\bar{W},\bar{V}]}C$ has the form

$$\tilde{l}([W, V])D.$$

On the other hand, as is well known, we find

$$W(\tilde{l}(V))-V(\tilde{l}(W))-\tilde{l}([W, V])=2(d\tilde{l})(W, V),$$

where $d\tilde{l}$ denotes the exterior differential form of the 1-form $\tilde{l}$. Thus we have

$$\bar{K}(BW, BV, C, D)=2(d\tilde{l})(W, V)-\tilde{k}(hV, W)+\tilde{k}(hW, V), \tag{0. 11}$$

which is the *equation of Ricci*.

Summing up (0. 7), (0. 8), (0. 10) and (0. 11), we have

$$\bar{K}(BW, BV, BY, BX)=K(W, V, Y, X)-\{\tilde{h}(W, X)\tilde{h}(V, Y)-\tilde{h}(W, Y)\tilde{h}(V, X)$$
$$+\tilde{k}(W, X)\tilde{k}(V, Y)-\tilde{k}(W, Y)\tilde{k}(V, X)\},$$

$$\begin{aligned}&\bar{K}(BW, BV, BY, C)=((\nabla_W h)V-(\nabla_V h)W-\tilde{l}(W)kV+\tilde{l}(V)kW, Y),\\ &\bar{K}(BW, BV, BY, D)=((\nabla_W k)V-(\nabla_V k)W+\tilde{l}(W)hV-\tilde{l}(V)hW, Y)\end{aligned} \tag{0. 12}$$

and

$$K(BW, BV, C, D)=2(d\tilde{l})(W, V)-\tilde{k}(hV, W)+\tilde{k}(hW, V).$$

In the rest of this section we assume that the enveloping manifold M is Kählerian manifold with the structure (F, G). We further assume that the surface (S, ξ) is a complex analytic hypersurface, that is, S is a complex manifold of complex dimension $n-1$ with complex structure f and the immersion ξ: $S \rightarrow M$ is complex analytic. Then the induced metric g on S given by (0. 2) is Hermitian, that is,

$$(fX, fY)=(X, Y) \tag{0. 13}$$

for any two vector fields X and Y on S, where (X, Y) denotes the inner product of X and Y with respect to g. In fact, the equation (0. 2) is equivalent to the condition that $\langle BX, BY\rangle=(X, Y)$ for any vector fields X and Y on S. On the other hand we have $\langle BX, BY\rangle=\langle FBX, FBY\rangle$, because the metric G on M is Hermitian. Taking account of (0. 1) and (0. 2), we obtain

$$\langle FBX, FBY\rangle=\langle BfX, BfY\rangle$$

and

$$\langle BfX, BfY\rangle=(fX, fY)$$

respectively. Thus we have the equation (0. 13), which shows that the structure (f, g) is Hermitian. But it is easily verified that this structure (f, g) is Kählerian. In fact $\nabla_X f=0$ for any vector field X on S, where ∇ is the induced connection given by (0. 3). (For the proof of this statement, cf. Smyth [4] and Yano [7], for example.) Thus we conclude that *a complex analytic hypersurface in a Kählerian manifold is Kählerian.*

§1. Formulae for complex hypersurfaces in a locally Fubinian manifold.

The aim of this section is to investigate complex analytic hypersurfaces in a locally Fubinian manifold M, but we consider, for the present, a little more general case where M is Kählerian. Throughout this section an immersion $\xi: S\to M$ is assumed to be complex analytic, S being a complex manifold, where M and S are of complex dimension n and $n-1$ respectively. Then S has a Kählerian structure (f, g) as was stated in §0.

We now define a 2-form $\tilde{f}$ on each tangent space of S as follows:

$$\tilde{f}(X, Y)=(fX, Y), \tag{1. 1}$$

where X and Y are arbitrary vector fields on S. We easily see that $\tilde{f}$ is anti-symmetric.

Restricting ourselves to a sufficiently small coordinate neighborhood U of S, we choose a field C of unit normal vectors to S. Then FC is another field of unit normal vectors to S which is orthogonal to C. Thus we can take FC as D introduced in §0. Then the 2-forms $\tilde{h}$ and $\tilde{k}$ defined in (0. 4) satisfy

$$\tilde{k}(X, Y)=-\tilde{h}(fX, Y), \qquad \tilde{h}(X, Y)=\tilde{k}(fX, Y), \tag{1. 2}$$

because of $\bar{\nabla}_{\bar{X}}F=0$. Moreover, from (1. 2) together with symmetry of $\tilde{h}$ and $\tilde{k}$, we can see that $\tilde{h}$ and $\tilde{k}$ are both *pure tensors*, (cf. Yano [7]) that is,

$$\tilde{h}(X, Y)+\tilde{h}(fX, fY)=0 \tag{1. 3}$$

and

$$\tilde{k}(X, Y)+\tilde{k}(fX, fY)=0. \tag{1.3)'}$$

Next let us introduce tensors h and k on S of type (1, 1) by

$$(hX, Y)=\tilde{h}(X, Y) \quad \text{and} \quad (kX, Y)=\tilde{k}(X, Y)$$

for any pair of vector fields X and Y, then we have

$$fh=-k$$

if we take account of (1. 2). We note that the second equation of (0. 5) implies the third in the case of Kählerian manifold, since $\bar{\nabla}_{\bar{X}}F=0$. We get again from (1. 2)

$$\text{Trace}\, h=\text{Trace}\, k=0,$$

which shows that any *invariant hypersurface* S *of a Kählerian manifold* M *is minimal in* M. (cf. Schouten and Yano [3], Yano [7].)

We assume, in the rest of this section, that the enveloping manifold M is locally Fubinian, so the curvature tensor $\bar{K}$ of M has the following form:

$$\begin{aligned}\bar{K}(\bar{W}, \bar{V}, \bar{Y}, \bar{X})=\frac{c}{4}\{&\langle \bar{W}, \bar{X}\rangle\langle \bar{V}, \bar{Y}\rangle-\langle \bar{W}, \bar{Y}\rangle\langle \bar{V}, \bar{X}\rangle+\langle F\bar{W}, \bar{X}\rangle\langle F\bar{V}, \bar{Y}\rangle \\ &-\langle F\bar{W}, \bar{Y}\rangle\langle F\bar{V}, \bar{X}\rangle-2\langle F\bar{W}, \bar{V}\rangle\langle F\bar{Y}, \bar{X}\rangle\}\end{aligned} \tag{1.4}$$

for any vector fields $\bar{X}, \bar{Y}, \bar{V}$ and $\bar{W}$ in M, c being a constant (Yano [7]). Equations (0. 12) of Gauss, Codazzi and Ricci for an invariant hypersurface in a locally Fubinian manifold take the following form:

$$\begin{aligned}&\frac{c}{4}\{(W, X)(V, Y)-(W, Y)(V, X)+(fW, X)(fV, Y) \\ &\qquad -(fW, Y)(fV, X)-2(fW, V)(fY, X)\} \\ &=K(W, V, Y, X)-\{\tilde{h}(W, X)\tilde{h}(V, Y)-\tilde{h}(W, Y)\tilde{h}(V, X) \\ &\qquad +\tilde{k}(W, X)\tilde{k}(V, Y)-\tilde{k}(W, Y)\tilde{k}(V, X)\}\end{aligned} \tag{1.5}$$

(Equation of Gauss),

$$(\nabla_W h)V-(\nabla_V h)W-\tilde{l}(W)kV+\tilde{l}(V)kW=0, \quad \text{(Equation of Codazzi)} \tag{1.6}$$

and

$$2(d\tilde{l})(W, V)-\tilde{k}(W, hV)+\tilde{k}(V, hW)+\frac{c}{2}\tilde{f}(W, V)=0 \quad \text{(Equation of Ricci)}. \tag{1.7}$$

Next let us denote the Ricci tensor of M and that of S by $\bar{R}$ and R respectively. Then we have, from (1. 4),

$$\bar{R}(BX, BY)=R(X, Y)-\frac{c}{2}(X, Y)+(hX, hY)+(kX, kY). \tag{1.8}$$

We further get, if we take account of (1. 7),

$$\bar{R}(BX, BY)=R(X, fY)+2(d\tilde{l})(X, Y).$$

The formulations in § 0 and § 1 will be reformulated by using local coordinate expressions or the so-called tensor calculus.

Let M be a Riemannian manifold of dimension N and S a differentiable manifold of dimension N' where $N-N'=2$. The immersion ξ: $S\to M$ is expressed by equations

$$\xi^h=\xi^h(\eta^a) \tag{1.9}$$

in local coordinates (ξ^h) of M and (η^a) of S. The equations (1. 9) are regarded as equations defining the surface S in M with respect to local coordinates (ξ^h) and (η^a). The differential B of ξ is represented by a matrix

$$(B_a{}^h)=\partial_a\xi^h, \qquad \partial_a=\partial/\partial\eta^a. \tag{1.10}$$

On the other hand, for any fixed index a, $(B_a{}^h)$ are regarded as components of a vector field B_a tangent to S and the $(n-1)$ tangent vector fields B_a span the tangent space $T_p(S)$ at each point p belonging to S. Thus, any vector $X=(X^a)$ tangent to S can be mapped to a vector $\bar{X}=BX$ on M by the differential B of ξ and $\bar{X}$ has components of the form

$$\bar{X}^h=B_a{}^hX^a.$$

We explain here the so-called van der Wearden-Bortolotti derivative along a surface. We denote by $\mathfrak{T}(M)$ the tangent bundle of M and by $\mathfrak{T}(S)$ the restriction of $\mathfrak{T}(M)$ to S. We define $\mathfrak{T}^{r_1}_{r_2}$ by

$$\mathfrak{T}^{r_1}_{r_2}=\underbrace{\mathfrak{T}(S)\otimes\cdots\otimes\mathfrak{T}(S)}_{r_1\text{-times}}\otimes\underbrace{\mathfrak{T}^*(S)\otimes\cdots\otimes\mathfrak{T}^*(S)}_{r_2\text{-times}},$$

where the vector bundle $\mathfrak{T}^*(S)$ is dual to $\mathfrak{T}(S)$. A *$\mathfrak{T}^{r_1}_{r_2}$-valued tensor field T of type* (t_1, t_2) is defined as follows: Let $X_1, \cdots, X_{t_2}$ be vector fields on S and $\tilde{Y}_1, \cdots, \tilde{Y}_{t_1}$ covector fields on S, all being arbitrary chosen. Then, $T(X_1, \cdots, X_{t_2}; \tilde{Y}_1, \cdots, \tilde{Y}_{t_1})$ is a cross-section of the tensor bundle $\mathfrak{T}^{r_1}_{r_2}$. If we fix a system of local coordinate (ξ^h) of M and (η^a) of S, we have the local representation of a $\mathfrak{T}^{r_1}_{r_2}$-valued tensor field T of type (t_1, t_2), that is T has the components of the form

$$T^{b_1\cdots b_{t_1}}{}_{a_1\cdots a_{t_2}}{}^{i_1\cdots i_{r_1}}{}_{h_1\cdots h_{r_2}}$$

with two kinds of indices $a, b, c, \cdots$ and $h, i, j, \cdots$ which run respectively over the range $1, 2, \cdots, N'$ and $1, 2, \cdots, N$. Indices $(a, b, c, \cdots)$ are called *indices of the first kind* and $(h, i, j, \cdots)$ are called *indices of the second kind*. It can be seen that the transformation law of T under a coordinate transformation of (ξ^h) is same as that of a tensor which is of the same type of T with respect to indices of the second kind. $B_a{}^h$ defined by (1. 10) is an example of such tensors. In fact, if $B_a{}^h$ is transformed to $'B_a{}^h$ and $''B_a{}^h$ under coordinate transformations $(\eta^a)\rightarrow(\bar{\eta}^a)$ and $(\xi^h)\rightarrow(\bar{\xi}^h)$ respectively, then we have

$$'B_a{}^h=\frac{\partial \xi^h}{\partial \bar{\eta}^a}=\frac{\partial \xi^h}{\partial \eta^b}\frac{\partial \eta^b}{\partial \bar{\eta}^a}=\frac{\partial \eta^b}{\partial \bar{\eta}^a}B_b{}^h$$

and

$$''B_a{}^h=\frac{\partial \bar{\xi}^h}{\partial \eta^a}=\frac{\partial \bar{\xi}^h}{\partial \xi^j}\frac{\partial \xi^j}{\partial \eta^a}=\frac{\partial \bar{\xi}^h}{\partial \xi^j}B_a{}^j,$$

which mean that $B_a{}^h$ is a $\mathfrak{T}^1_0$-valued tensor of type $(0, 1)$. *Van der Wearden-Bortolotti derivative* is now defined as an assignment of a $\mathfrak{T}^{r_1}_{r_2}$-valued tensor of type (t_1, t_2+1) to a $\mathfrak{T}^{r_1}_{r_2}$-valued tensor of type (t_1, t_2). The assignment $'\nabla_c$ defining the van der Wearden-Bortolotti differentiation is characterized by the following three conditions:

(1. 11)

(i) $'\nabla_c$ is a derivation

(ii) $'\nabla_c \bar{T}=B_c{}^j\bar{\nabla}_j\bar{T}$, if $\bar{T}$ is a $\mathfrak{T}^{r_1}_{r_2}$-valued tensor of type $(0, 0)$

and

(iii) $'\nabla_c T=\nabla_c T$, if T is a $\mathfrak{T}^0_0$-valued tensor of type (t_1, t_2),

where $\bar{\nabla}$ is the covariant differentiation with respect to G and ∇ with respect to g respectively. If we take a $\mathfrak{T}^1_2$-valued tensor of type $(1, 1)$ having components $T_b{}^j{}^a{}_{ih}$, for example, we find directly from the definitions of van der Wearden-Bortolotti differentiation $'\nabla_c$

$$(1.\,12)\qquad \begin{aligned} '\nabla_c T_b{}^{ja}{}_{ih}=&\frac{\partial T_b{}^{ja}{}_{ih}}{\partial \eta^c}+\left[\overline{\left\{{j \atop l\ k}\right\}}T_b{}^{ka}{}_{ih}-\overline{\left\{{k \atop l\ i}\right\}}T_b{}^{ja}{}_{kh}-\overline{\left\{{k \atop l\ h}\right\}}T_b{}^{ja}{}_{ik}\right]B_c{}^l\\ &+\left\{{a \atop c\ d}\right\}T_b{}^{jd}{}_{ih}-\left\{{d \atop c\ b}\right\}T_d{}^{ja}{}_{ih}, \end{aligned}$$

where $\overline{\{{}_j{}^h{}_i\}}$ denote the Christoffel symbols constructed with the metric G on M and $\{{}_c{}^a{}_b\}$ the Christoffel symbols constructed with the induced metric g on S. One can easily verify

$$(1.\,13)\qquad '\nabla_c G_{ji}=B_c{}^k\bar{\nabla}_k G_{ji}=0, \qquad '\nabla_c g_{ba}=\nabla_c g_{ba}=0$$

and that $'\nabla_c$ commutes with all contractions with respect to indices of the first and of the second kind respectively.

As is well known, the relation between $\overline{\{_j{}^h{}_i\}}$ and $\{_c{}^a{}_b\}$ is given by

$$\left\{ {a \atop c\ b} \right\} = B^a{}_h \left[\overline{\left\{ {h \atop k\ j} \right\}} B_c{}^k B_b{}^j + \frac{\partial B_b{}^h}{\partial \eta^c} \right], \tag{1.14}$$

where $B^a{}_h$ is defined in such a way that $(B^a{}_h, G_{hl}C^l, G_{hl}D^l)$ is the inverse matrix of $(B_a{}^h, C^h, D^h)$ (Yano [7]).

The vector field $\bar{\nabla}_{\bar{X}}\bar{Y}$ has components of the form

$$\bar{X}^j \bar{\nabla}_j \bar{Y}^h = X^b B_b{}^j \bar{\nabla}_j (B_a{}^h Y^a).$$

By making use of van der Wearden-Bortolotti derivative and taking account of (ii) and (iii) given in (1.11), we have

$$\begin{aligned} \bar{X}^j \nabla_j \bar{Y}^h &= X^b{}' \nabla_b (B_a{}^h Y^a) \\ &= X^b \{ ('\nabla_b B_a{}^h) Y^a + B_a{}^h{}' \nabla_b Y^a \} \\ &= (X^b \nabla_b Y^a) B_a{}^h + X^b Y^a ('\nabla_b B_a{}^h). \end{aligned}$$

On the other hand we have, from (1.12)

$$'\nabla_b B_a{}^h = \frac{\partial B_a{}^h}{\partial \eta^b} + \overline{\left\{ {h \atop k\ j} \right\}} B_b{}^k B_a{}^j - \left\{ {c \atop b\ a} \right\} B_c{}^h.$$

Relation (1.14) shows that $'\nabla_b B_a{}^h$ is a linear combination of C and D, so we may put

$$'\nabla_b B_a{}^h = h_{ba} C^h + k_{ba} D^h,$$

where

$$h_{ba} = G_{ih} \left[\frac{\partial B_a{}^i}{\partial \eta^b} + \overline{\left\{ {i \atop k\ j} \right\}} B_b{}^k B_a{}^j \right] C^h$$

and

$$k_{ba} = G_{ih} \left[\frac{\partial B_a{}^i}{\partial \eta^b} + \overline{\left\{ {i \atop k\ j} \right\}} B_b{}^k B_a{}^j \right] D^h.$$

These equalities show that h_{ba} and k_{ba} are both symmetric tensors and they give local expressions of $\tilde{h}$ and k in (0.4) respectively.

Thus we can see that the tangential part $\nabla_X Y$ and the normal part $N_X Y$ of $\bar{\nabla}_{\bar{X}}\bar{Y}$ are given by

$$B_a{}^h(\nabla_X Y)^a$$

and

$$(h_{ba}X^b Y^a)C^h+(k_{ba}X^b Y^a)D^h.$$

The equations of Weingarten (0. 5) are now written as, using van der Wearden-Bortolotti derivative,

$$(1.15)\qquad \begin{cases} '\nabla_b B_a{}^h=h_{ba}C^h+k_{ba}D^h, \\ '\nabla_b C^h=-h_b{}^a B_a{}^h+l_b D^h, \\ '\nabla_b D^h=-k_b{}^a B_a{}^h-l_b C^h. \end{cases}$$

In fact,

$$0={}'\nabla_b(G_{ih}C^i B_a{}^h)=G_{ih}\{('\nabla_b C^i)B_a{}^h+C^i{}'\nabla_b B_a{}^h\}$$
$$=G_{ih}('\nabla_b C^i)B_a{}^h+h_{ba},$$

which shows that the tangential part of $'\nabla_b C^i$ is $-h_b{}^a B_a{}^i$. By the similar method, we can verify that the tangential part of $'\nabla_b D^i$ is $-k_b{}^a B_a{}^i$. On the other hand we have

$$0={}'\nabla_b(G_{ih}C^i D^h)=G_{ih}\{('\nabla_b C^i)D^h+C^i{}'\nabla_b D^h\},$$

from which we can see that if we denote by $l_b D^h$ the normal part of $'\nabla_b C^h$, then the normal part of $'\nabla_b D^h$ is $-l_b C^h$.

Now we assume that the enveloping manifold M is Kählerian and S is a complex analytic hypersurface of M. Since S is assumed to be invariant under F, FB_a must be also tangent to S and then we can put

$$(1.16)\qquad F_i{}^h B_a{}^i=f_a{}^b B_b{}^h,$$

where $F_i{}^h$ are components of F and $f_b{}^a$ define a tensor field of type (1,1) on S. Comparing equations (0. 1) and (1. 16), we can easily see that the tensor $(f_b{}^a)$ coincides with the tensor f defining the complex structure of S.

The components g_{ba} of the induced metric g on S are given by

$$(1.17)\qquad g_{ba}=G_{ji}B_b{}^j B_a{}^i=\langle B_b, B_a\rangle,$$

by virtue of $\langle BX, BY\rangle=(X, Y)$. The components f_{ba} of $\tilde{f}$ are given by

$$f_{ba}=f_b{}^c g_{ca},$$

which are also written as

$$f_{ba}=F_{ji}B_b{}^jB_a{}^i=\langle FB_b, B_a\rangle, \tag{1.18}$$

where F_{ji} are components of fundamental 2-form $\tilde{F}$ on M and are given by

$$F_{ji}=F_j{}^kG_{ki}.$$

We shall here prove the following Proposition 1.1 by using method of local coordinate expressions mentioned above.

PROPOSITION 1.1. ([4]) *Let M be a locally Fubinian manifold and S be an invariant hypersurface of M. If S is Einstein, then it is locally symmetric.*

Proof. By our assumption for M and S, M can be covered by a system of complex coordinate neighborhoods

$$z^\kappa=\xi^\kappa+i\xi^{\bar\kappa}.$$[4]

Then we can introduce a complex coordinate system (u^α) on S defined by

$$u^\alpha=\eta^\alpha+i\eta^{\bar\alpha},$$

such that the equation of S can be written as

$$z^\kappa=z^\kappa(u^\alpha),$$

where $z^\kappa(u^\alpha)$ are complex analytic functions of complex variables u^α. Since S is a complex analytic hypersurface in a locally Fubinian manifold, the almost complex structure $f=(f_a{}^b)$ on S is reduced to the numerical components $\begin{pmatrix} i\delta_\beta^\alpha & 0 \\ 0 & -i\delta_{\bar\beta}^{\bar\alpha}\end{pmatrix}$ with respect to local complex coordinate system on S. It is well known that non-vanishing components of the curvature tensor are $K_{\delta\bar\gamma\beta\bar\alpha}$ (Yano [7]). We have

$$K_{\delta\bar\gamma\beta\bar\alpha}=-2h_{\bar\gamma\bar\alpha}\,h_{\delta\beta}+\frac{c}{4}(g_{\delta\bar\alpha}g_{\bar\gamma\beta}+f_{\delta\bar\alpha}f_{\bar\gamma\beta}-2f_{\delta\bar\gamma}f_{\beta\bar\alpha}) \tag{1.19}$$

if we take account of (1.5) and pureness of h_{ba} and hybridness[5] of g_{ba} and f_{ba}. The equations (1.6) in the complex coordinate system are written as

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4) Greek indices $\kappa, \lambda, \mu, \cdots$ run over the range $1, 2, \cdots, n$, while $\alpha, \beta, \gamma, \cdots$ the range $1, 2, \cdots, n-1$. We give the value $1, 2, \cdots, n$; $\bar{1}, \bar{2}, \cdots, \bar{n}$ to $i, j, k, \cdots$ and the value $\bar{1}, \bar{2}, \cdots, \overline{n-1}$; $1, 2, \cdots, n-1$ to $a, b, c, \cdots$ in the complex coordinate system.

5) A tensor $T_{\cdots b\cdots a\cdots}$ is said to be *hybrid* with respect to b and a if

$$T_{\cdots b\cdots a\cdots}-f_b{}^df_a{}^cT_{\cdots d\cdots c\cdots}=0.$$

(For detail, see Yano [7].) The curvature tensor K_{dcba} is hybrid with respect to d and c (or b and a), that is $K(fW, fV, Y, X)-K(W, V, Y, X)=0$.

(1. 20) $$\nabla_\delta h_{\bar\gamma\bar\alpha} - il_\delta h_{\bar\gamma\bar\alpha} = 0, \qquad \text{conj.} \quad (\text{i.e. } \nabla_{\bar\delta} h_{\gamma\alpha} + il_{\bar\delta} h_{\gamma\alpha} = 0)$$

because of (1. 2).

From the assumption that S is Einstein, we get, by transvecting $g^{\delta\bar\alpha}$ to (1. 19),

(1. 21) $$h_{\bar\gamma}{}^{\delta} h_{\delta\beta} = A g_{\bar\gamma\beta},$$

where A is a non-zero constant. Transvecting $h_\varepsilon{}^{\bar\alpha}$ to (1. 20), we have

$$h_{\bar\gamma\alpha} \nabla_\delta h_\varepsilon{}^{\bar\alpha} + il_\delta A g_{\varepsilon\bar\gamma} = 0,$$

because of (1. 21). Transvecting $h_\beta{}^{\bar\gamma}$ to these equations we get

(1. 22) $$\nabla_\delta h_{\varepsilon\alpha} + il_\delta h_{\alpha\varepsilon} = 0, \qquad \text{conj.}$$

On the other hand (1. 6) gives the equation

(1. 23) $$\nabla_\varepsilon h_{\bar\gamma\bar\alpha} - il_\varepsilon h_{\bar\gamma\bar\alpha} = 0, \qquad \text{conj.}$$

Differentiating (1. 19) covariantly along S we get

$$\nabla_\varepsilon K_{\delta\bar\gamma\beta\bar\alpha} = -2[(\nabla_\varepsilon h_{\bar\gamma\bar\alpha}) h_{\delta\beta} + h_{\bar\gamma\bar\alpha}(\nabla_\varepsilon h_{\delta\beta})].$$

The right hand member is zero by virtue of (1. 22) and (1. 23) which is obtained by the assumption that S is Einstein. By the same method, we can get

$$\nabla_{\bar\varepsilon} K_{\delta\bar\gamma\beta\bar\alpha} = 0,$$

which shows, together with $\nabla_\varepsilon K_{\delta\bar\gamma\beta\bar\alpha} = 0$, that S is locally symmetric. q.e.d.

§ 2. Realization of a Kählerian manifold in a Fubinian manifold.

As we referred at the beginning of this paper a *Fubinian manifold* is a homogeneous Kählerian manifold with constant holomorphic sectional curvature which is simply connected. A Fubinian manifold M of n complex dimensions is identified with the coset space G/H, G being the group of all complex $(n+1)\times(n+1)$ matrices operating on a complex vector space L of $n+1$ complex dimensions and preserving Hermitian form $\langle X, Y\rangle$ (i.e. $\langle \bar X, \bar Y\rangle = \overline{\langle X, Y\rangle}$) of signature 0 (or $n-2$) if M is of positive (or negative) holomorphic sectional curvature, and H is a subgroup of G which consists of all matrices leaving a particular vector having magnitude 1 (or -1) in L. That is, a Fubinian manifold $M=G/H$ is regarded as a sort of Klein space having G as its fundamental group of motions.

Let M be a Fubinian manifold of complex dimension n and S be a Kählerian manifold of complex dimension $n-1$ having a structure (f, g). The question arises if S can be realized as an complex analytic hypersurface in M. In other words, when can a complex analytic immersion $\xi: S\rightarrow M$ exist in the sense of §0? To answer this question, we must, first of all, determine $2n-1$ vector fields $B_a=(B_a{}^h)$ and $C=(C^h)$ in M mentioned in the previous section, where

$$B_a{}^h=\frac{\partial \xi^h}{\partial \eta^a} \tag{2.1}$$

(η^a) being a parameter on S. The field C being determined we take FC as another vector field which is orthogonal to C and B_a. The vector field B_a and C should satisfy the conditions

$$\begin{cases}\langle B_b, B_a\rangle-g_{ba}=0,\\ \langle C, C\rangle-1=0,\\ \langle B_a, C\rangle=0\end{cases} \tag{2.2}$$

and

$$\langle B_a, FC\rangle=0, \tag{2.3}$$

where $\langle\ ,\ \rangle$ denotes the inner product with respect to G, the metric of M.

Now we look for the system of partial differential equations which B_a and C should satisfy and the integrability conditions of these equations.

In order that S is immersed in M, B_a and C must satisfy (1.13) and (2.3). Thus a necessary and sufficient condition for S to be complex analytically immersed in M is that $B_a{}^h$ and C^h are solutions of the system of partial differential equations

$$\begin{cases}\dfrac{\partial B_a{}^h}{\partial \eta^b}=B_c{}^h\left\{{c \atop b\ a}\right\}-B_b{}^m B_a{}^l\left\{\overline{{h \atop m\ l}}\right\}+h_{ba}C^h+k_{ba}F_i{}^hC^i,\\ \dfrac{\partial C^h}{\partial \eta^b}=h_b{}^a B_a{}^h-\left\{\overline{{h \atop m\ l}}\right\}B_b{}^m C^l+l_b F_i{}^h C^i\end{cases} \tag{2.4}$$

with an algebraic additional condition

$$G_{ji}F_h{}^j B_a{}^i C^h=0, \tag{2.5}$$

and satisfy identically the conditions (2.2). In (2.4), $\left\{\overline{{h \atop m\ l}}\right\}$ are the Christoffel symbols constructed from G, $\left\{{c \atop b\ a}\right\}$ the Christoffel symbols constructed from g, h_{ba} and $k_{ba}=-f_b{}^c h_{ca}$ the second fundamental tensors and l_a is the third fundamental tensor of S, $f_b{}^c$ being the complex structure of S.

Let there be given differentiable functions h_{ba}, l_a on a Kählerian manifold S whose structure is denoted by (f, g). These functions are assumed to satisfy the following conditions.

(C1) $$h_{ba}=h_{ab},$$

(C2) $$g_{a[d}g_{c]b}-f_{a[d}f_{c]b}-f_{dc}f_{ba}-\frac{1}{2}K_{dcba}=h_{a[d}h_{c]b}+f_a{}^e h_{e[d}h_{c]f}f_b{}^f,$$

(C3) $$\nabla_{[d}h_{c]b}-l_{[d}f_{c]}{}^a h_{ab}=0$$

and

(C4) $$\partial_{[d}l_{c]}-h_{[c}{}^a f_{d]}{}^b h_{ba}+\frac{c}{4}f_{dc}=0,$$

where K_{dcba} is the curvature tensor of S and $n(n+1)c$ is the curvature scalar of M. (C2), (C3) and (C4) are nothing but the equations of Gauss, Codazzi and Ricci (1. 5), (1. 6) and (1. 7), respectively. We shall now show that the system of partial differential equations (2. 4) with additional condition (2. 5) is completely integrable, if h_{ba} and l_a satisfy the conditions (C1), (C2), (C3) and (C4) mentioned above. Differentiating the right hand side of the first equation given in (2. 4) and taking skew-symmetric parts, we have

$$\begin{aligned}\frac{\partial^2 B_a{}^h}{\partial\eta^c\partial\eta^b}-\frac{\partial^2 B_a{}^h}{\partial\eta^b\partial\eta^c}=&-B_{cba}^{kji}\bar{K}_{kji}{}^h+B_d{}^h[K_{cba}{}^d+h_{ac}h_b{}^d-h_{ab}h_c{}^d+k_{ca}k_b{}^d-k_{ab}k_c{}^d]\\&+C^h[\nabla_c h_{ba}-\nabla_b h_{ca}-l_c k_{ba}+l_b k_{ca}]\\&+D^h[\nabla_c k_{ba}-\nabla_b k_{ca}+l_c h_{ba}-l_b h_{ca}],\end{aligned}$$

where $B_{cba}^{kji}=B_c{}^k B_b{}^j B_a{}^i$. The coefficients of C^h and D^h are zero, because h_{ba} and l_a satisfy (C3) and $k_{ba}=-f_b{}^c h_{ca}$. On the other hand, the curvature tensor of a Fubinian manifold is given by (1. 4) which can be written as

(2. 6) $$\bar{K}_{kji}{}^h=\frac{c}{4}(\delta_k^h G_{ji}-\delta_j^h G_{ki}+F_k{}^h F_{ji}-F_j{}^h F_{ki}-2F_i{}^h F_{kj}).$$

Thus, if we take account of the additional condition (2. 5), we have

$$B_{cba}^{kji}K_{kji}{}^h=\frac{c}{4}B_d{}^h(\delta_c^d g_{ba}-\delta_b^d g_{ca}+f_c{}^d f_{ba}-f_b{}^d f_{ca}-2f_a{}^d f_{cb}).$$

So the coefficient of $B_d{}^h$ must be zero, since (C2) is valid. Therefore we have

(2. 7) $$\frac{\partial^2 B_a{}^h}{\partial\eta^c\partial\eta^b}-\frac{\partial^2 B_a{}^h}{\partial\eta^b\partial\eta^c}=0.$$

In the next step, differentiating the right hand side of the second equation given in (2. 4) and taking skew-symmetric parts, we obtain

$$\frac{\partial^2 C^h}{\partial\eta^c\partial\eta^b}-\frac{\partial^2 C^h}{\partial\eta^b\partial\eta^c}=-B_{cb}^{kj}C^i\bar{K}_{kji}{}^h$$
$$+B_a{}^h[\nabla_b h_c{}^a-\nabla_c h_b{}^a-l_b k_c{}^a+l_c k_b{}^a]$$
$$+F_i{}^h C^i[\partial_c l_b-\partial_b l_c+h_c{}^a k_{ba}-h_b{}^a k_{ca}]$$

because of the additional condition (2. 5), where $B_{cb}^{kj}=B_c{}^k B_b{}^j$. The coefficient of $B_a{}^h$ is zero because of (C3). (2. 6) shows that

$$B_{cb}^{kj}C^i\bar{K}_{kji}{}^h=-\frac{c}{2}f_{cb}F_i{}^h C^i.$$

Therefore the coefficient of $F_i{}^h C^i$ in

$$\frac{\partial^2 C^h}{\partial\eta^c\partial\eta^b}-\frac{\partial^2 C^h}{\partial\eta^b\partial\eta^c}$$

is zero because of (C4) and then we have

$$\frac{\partial^2 C^h}{\partial\eta^c\partial\eta^b}-\frac{\partial^2 C^h}{\partial\eta^b\partial\eta^c}=0. \tag{2. 8}$$

Summing up (2. 7) and (2. 8), we can now conclude that the system of partial differential equations (2. 4) with additional condition (2. 5) are completely integrable.

Now we examine the initial conditions (2. 2). Let $B_a{}^h$ and C^h be the system of solutions of the system of the partial differential equations (2. 4) with additional condition (2. 5). Then we have

$$\frac{\partial}{\partial\eta^c}\{\langle B_b, B_a\rangle-g_{ba}\}=h_{cb}\langle B_a, C\rangle+h_{ca}\langle B_b, C\rangle$$
$$+k_{cb}\langle B_a, FC\rangle+k_{ca}\langle B_b, FC\rangle,$$

$$\frac{\partial}{\partial\eta^c}\langle B_a, C\rangle=\left\{\begin{matrix} c \\ b\ a\end{matrix}\right\}\langle B_c, C\rangle+h_{ba}\{\langle C, C\rangle-1\}$$
$$-h_b{}^c\{\langle B_a, B_c\rangle-g_{ac}\}+l_b\langle B_a, FC\rangle$$

and

$$\frac{\partial}{\partial\eta^c}\{\langle C, C\rangle-1\}=-2h_c{}^a\langle B_a, C\rangle.$$

The system of these three equations is regarded as a system of homogeneous partial differential equations with unknown functions $\langle B_b, B_a\rangle-g_{ba}$, $\langle B_a, C\rangle$ and $\langle C, C\rangle-1$. Therefore, if

$$\langle B_b, B_a\rangle - g_{ba} = 0, \qquad \langle B_a, C\rangle = 0$$

and

$$\langle C, C\rangle - 1 = 0$$

are satisfied at a point (η^a) for a system of solutions $B_a{}^h$ and C^h of (2.4) with additional condition (2.5), then they are satisfied identically also for the solutions $B_a{}^h$ and C^h. That is to say, if we choose the solutions $B_a{}^h$ and C^h of (2.4) with additional condition (2.5) which satisfy (2.2) at an initial point, then the system of solutions $B_a{}^h$ and C^h satisfies identically (2.2) at each point.

Since the number of unknown functions $B_a{}^h$ and C^h is $2n(2n-1)$, while the number of the relations in (2.2) is $2n^2-n$, so we can choose a system of solutions of (2.4) under the initial conditions (2.2) with $(2n^2-n)$ arbitrary constants. After B_a are determined, we can get $\xi^h(\eta^a)$ by solving

$$\frac{\partial \xi^h}{\partial \eta^a} = B_a{}^h,$$

which include $2n$ arbitrary constants. Thus we conclude:

In order that a complex $(n-1)$-dimensional Kählerian manifold S with Kählerian structure $(f_b{}^a, g_{ba})$ be immersed as a complex analytic hypersurface of a complex n-dimensional Fubinian manifold M with Kählerian structure $(F_j{}^i, G_{ji})$, it is necessary and sufficient that functions h_{ba} and l_b given on S satisfy equations of Gauss, Codazzi and Ricci (C2), (C3) and (C4). Then the 1-st fundamental tensor of S, as a submanifold of M, is g_{ba}, the 2-nd fundamental tensors are h_{ba} and k_{ba} and the third fundamental tensor is l_d and the solutions include $2n^2+n$ arbitrary constants.

On the other hand it is well known that a Fubinian manifold M admits the Lie group of analytic motions of dimension $2n^2+n$.

Thus we have

THEOREM 2.1. *Let S be a complex $(n-1)$-dimensional simply connected Kählerian manifold with Kählerian structure (f, g) and M a complex n-dimensional Fubinian manifold whose curvature scalar is given by $n(n+1)c$, c being given in* (C4). *We assume that there are given differentiable functions h_{ba} and l_a satisfying* (C1)–(C4), *then there exists a complex analytic immersion ξ from S to M which makes S a complex analytic hypersurface of M having the induced structure (f, g), the second fundamental tensors h and $-fh$ and the third fundamental tensor l. Moreover, S is uniquely determined up to analytic motions of M.*

§ 3. Totally geodesic submanifold in a locally Fubinian manifold.

In the previous two sections we have studied complex analytic hypersurfaces in a (locally) Fubinian manifold. In this section we consider a little more general submanifold in a locally Fubinian manifold. The notations and terminologies in

§§ 1 and 2 are used also in this section.

Let M be a Kählerian manifold of real dimension $2n$ and S[6] a connected, orientable submanifold of M whose real dimension is $2n-2$. We shall restrict ourselves to a sufficiently small neighborhood in which there exist two fields of unit normal vectors to S. First, we fix two normal vector fields C and D to S which are mutually orthogonal. It is well known that a Riemannian metric g on S can be induced from the Riemannian metric G of M. We denote by $\langle\,,\rangle$ the inner product with respect to G and by $(\,,)$ the inner product with respect to g Now we put, for a tangent vector X on S

$$F(BX)=T(X)+N(X), \tag{3.1}$$

where $T(X)$ denotes the tangential part and $N(X)$ the normal part of $F(BX)$. Since $T(X)$ is tangent to S, we may put

$$\langle T(X), BY\rangle=(AX, Y),$$

A being a tensor on S of type (1, 1) and Y an arbitrary vector on S. On the other hand $N(X)$ is expressed as

$$N(X)=\tilde{\alpha}(X)C+\tilde{\beta}(X)D,$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are 1-forms on S. If we define a 2-form $\tilde{A}$ by

$$\tilde{A}(X, Y)=(AX, Y)$$

for any pair of vector fields X and Y on S, then we have, denoting by $\tilde{F}$ the fundamental 2-form of M,

$$\tilde{A}(X, Y)=\tilde{F}(BX, BY). \tag{3.2}$$

$$\tilde{\alpha}(X)=\tilde{F}(BX, C) \tag{3.3}$$

and

$$\tilde{\beta}(X)=\tilde{F}(BX, D) \tag{3.4}$$

for any vector fields X and Y on S. (3. 2) shows that $\tilde{A}$ is a skew-symmetric bilinear form. We define $||\alpha||$ and $||\beta||$ respectively by

$$||\alpha||=\sqrt{(\alpha, \alpha)} \qquad \text{and} \qquad ||\beta||=\sqrt{(\beta, \beta)}$$

where α and β are contravariant tensors of degree 1 defined by $(\alpha, X)=\tilde{\alpha}(X)$ and $(\beta, X)=\tilde{\beta}(X)$ respectively. Then we have by direct calculation

6) We here use an identification of a differentiable manifold S with $\xi(S)$, where ξ is a differentiable immersion from S into M, whose differential B: $T_p(S)\rightarrow T_{\xi(p)}(M)$ is injective.

(3. 5) $$||\alpha||^2=1-(\tilde{F}(C, D))^2$$

and

(3. 6) $$||\beta||^2=1-(\tilde{F}(C, D))^2.$$

Thus we have

$$||\alpha||^2=||\beta||^2,$$

which implies

LEMMA 3. 1. *If $\tilde{\alpha}(X)=0$ at a point p for any vector X on S, then $\tilde{\beta}(X)=0$ at p, and vice versa.*

In the case in which $\tilde{\alpha}(X)$ is identically zero for any X on S, S should be invariant under F. O'Neil [2] and Smyth [4] dealt with this case in detail.

We assume, from now on, that there is at least one point at which $||\alpha||$ is not zero and therefore $||\beta||$ is not zero either. In such a case it would be interesting to study the function $\tilde{F}(C, D)$ appearing in (3. 5) and (3. 6). $\tilde{F}(C, D)$ seems to depend upon the choice of a pair of unit normal vector fields C and D, but it is not hard to show that $\tilde{F}(C, D)$ is independent of the choice of C and D.

A straightforward computation shows that

(3. 7) $$A^2=-I+\tilde{\alpha}\otimes\alpha+\tilde{\beta}\otimes\beta$$

and

(3. 8) $$(\alpha, \beta)=0,$$

where I is the unit tensor. For the rank of A which is a $(2n-2)\times(2n-2)$ matrix, one proves

LEMMA 3. 2. *The rank of $A\geqq 2n-4$.*

REMARK. Lemma 3. 2 implies that if the rank of A is not maximal, then it is $2n-4$, because it is even.

Proof of Lemma 3. 2. In this proof we put $m=2n-2$ for simplicity. Let us denote by $T_p(S)$ the tangent space to S at p and define a subspace $T_p'(S)$ of $T_p(S)$ by $T_p'(S)=\{V\in T_p(S)|F(BV)\in T_p(S)\}$, then V must satisfy

(3. 9) $$(V, \alpha)=0 \quad\text{and}\quad (V, \beta)=0.$$

Conversely, any vector $V\in T_p(S)$ satisfying the equation (3. 9) belongs to $T_p'(S)$. On the other hand $(\alpha, \beta)=0$. Thus we have

$$\dim T_p'(S) \geqq \dim S-2,$$

where dim means the real dimension. When $(\tilde{\alpha}\wedge\tilde{\beta})(X, Y)=0$ for any pair of vectors X and Y belongs to $T_p(S)$, we have $||\alpha||=||\beta||=0$ and thus $T_p'(S)=T_p(S)$. This shows that the rank of A equals to m, because $A^2V=-V$ for any vector belonging to $T_p'(S)$.

If we suppose that the rank of A is less than m, then there exists a point p at which $||\alpha||\neq 0$. Then the orthogonal complement of $T_p'(S)$ is the linear space $T_p''(S)$ spanned by α and β, since $||\beta||\neq 0$ and $(\alpha, \beta)=0$. There exists $\gamma\in T_p(S)$ such that

$$A\gamma=0 \quad \text{and} \quad ||\gamma||\neq 0 \tag{3.10}$$

by our assumption. For such γ we have, by putting $\bar{\gamma}=B\gamma$,

$$F\bar{\gamma}=(\alpha, \gamma)C+(\beta, \gamma)D,$$

which means that γ has no part belonging to $T_p'(S)$, that is $\gamma\in T_p''(S)$. Thus γ is a linear combination of α and β. We put

$$\gamma=a\alpha+b\beta, \tag{3.11}$$

where either a or b can not be zero.

On the other hand, straightforward computations give

$$A\alpha=-\tilde{F}(C, D)\beta \tag{3.12}$$

and

$$A\beta=\tilde{F}(C, D)\alpha. \tag{3.13}$$

The equations (3.10)~(3.13) give

$$\tilde{F}(C, D)(-a\beta+b\alpha)=0,$$

from which we have

$$\tilde{F}(C, D)=0 \quad \text{at} \quad p. \tag{3.14}$$

Thus we have

$$A\alpha=A\beta=0 \quad \text{at} \quad p. \tag{3.15}$$

Since Lemma 3.1 guarantees that $||\beta||\neq 0$ at p, the equation (3.15) shows that the rank of $A\leqq m-2$. On the other hand, the equations (3.10) and (3.11) imply that an arbitrary vector γ which annihilates A is a linear combination of α and β. This concludes that the rank of A is exactly $m-2$.

When $\|\alpha\|$ vanishes at p, then the submanifold S is invariant under F at p. Thus we call such a point an *invariant point*. The equation (3. 5) shows that $\tilde{F}(C, D)$, as a function on S, attains its maximum or minimum at an invariant point. We note that an invariant point is a critical point of the function $\tilde{F}(C, D)$. Therefore, if we assume that a submanifold S to be compact, then the maximum and the minimum of $\tilde{F}(C, D)$ are attained on S. Furthermore, if there are only two invariant points on a compact submanifold S, then S is homeomorphic to a sphere (Reeb).

On the other hand, the rank of A equals to m at an invariant point, but it is not true that a point at which $r(A)=m$ is an invariant point, where $m=\dim S$ and $r(A)$ denotes the rank of A. For one of the simplest examples, let us consider a 4-dimensional sphere S^4 in the 6-dimensional Euclidean space E^6 which is to be a Kählerian space in usual way. We assume that S^4 lies in a hyperplane of E^6. We can easily verify that invariant points of S^4 are only the southern and the northern poles and the points at which $r(A)=2$ appear along the equatorial sphere. The other points of S^4 are non-invariant points of $r(A)=4$.

Remark. If the rank of A equals to $m-2$ at a point p, then A satisfies $A^3+A=0$ at p. Furthermore, if the rank of A is constant on the whole submanifold, then S is either invariant surface of M or a so-called f-surface of rank $m-2$. (For an f-structure, c.f. Nakagawa [1] and Yano-Ishihara [6].)

Before going further with an f-structure, we observe how the properties assigned on S and on the enveloping manifold M behave for the rank of A. We assume, in the rest of this section, that M is a locally Fubinian manifold and S is totally geodesic in M. Then the next lemma is the result of direct computations.

Lemma 3. 3. *$\tilde{F}(C, D)$ is constant on S.*

We here consider the case in which there is at least one point p at which $\tilde{\alpha}(X)\neq 0$ for some X on S and $c\neq 0$. We have

Lemma 3. 4. *α defined above annihilates A and thus so does β.*

Proof. Since S is totally geodesic, equations of Codazzi, given in § 0 in the most general form, are written as

$$\tilde{\alpha}(V)(AW, Y)-(\alpha, W)(AV, Y)-2(\alpha, Y)(AV, W)=0 \tag{3. 16}$$

and

$$\tilde{\beta}(V)(AW, Y)-(\beta, W)(AV, Y)-2(\beta, Y)(AV, W)=0 \tag{3. 17}$$

for any vectors Y, V and W on S.

For a vector V which satisfies $\tilde{\alpha}(V)\neq 0$, let us define a bilinear form on S as $\tilde{L}_V(W, Y)=\tilde{\alpha}(V)(AW, Y)$, where Y and W belong to $T_p(S)$. By virtue of (3. 16), we have

$$\tilde{L}_V(W, Y)=(\alpha, W)(AV, Y)+2(\alpha, Y)(AV, W).$$

We further define, with the aid of $\tilde{L}_V$, a tensor of type (1, 1) by

$$(L_V Y, W)=\tilde{L}_V(W, Y).$$

Taking trace of L_V, we have $A\alpha=0$, since A is skew-symmetric. The same method gives $A\beta=0$, if we take account of (3.17).

We noted before that $(\alpha, \beta)=0$ and the rank of A is at least $m-2$, so there is no other γ which annihilates A and linearly independent of α and β.

By virtue of Lemmas 3.3 and 3.4 together with (3.9) and (3.11), we have the following

LEMMA 3.5. *$\tilde{F}(C, D)=0$ on S, if there is at least one point p at which $\tilde{\alpha}(X)$ does not vanish.*

We further prove

LEMMA 3.6. *Let there be at least one point p on S at which $\tilde{\alpha}(X)\neq 0$. Then $2n$, the real dimension of M, is 4 and $T_p(S)$ and $N_p(S)$, the normal space to S at p, are transformed under F into each other.*

Proof. The composition of A with L_V gives a tensor L_V' of type (1, 1) such that $(L_V' W, Y)=((L_V A)W, Y)$. This is also written as

$$(L_V' W, Y)=\tilde{\alpha}(V)\{(\beta, W)(\beta, Y)+(\alpha, W)(\alpha, Y)-(W, Y)\},$$

if we take account of (3.16) and (3.7). Let us compute the trace of L_V' taking account of (3.16), (3.7), (3.6) and (3.5). Then we have

$$\tilde{\alpha}(V)(m-2)=0.$$

Thus we have

$$m-2=0,$$

since $m=2n-2$ and $\alpha(V)\neq 0$. From this it follows that $n=2$, $||\alpha||=||\beta||=1$ at p and $FC, FD\in T_p(S)$. Consequently we see that M and S are 4-dimensional and 2-dimensional respectively and the rank of A is zero. Therefore we have

$$F(BX)=\tilde{\alpha}(X)C+\tilde{\beta}(X)D. \quad \text{q.e.d.} \tag{3.18}$$

Such a submanifolds, i.e. such that $F(BT_p(S))=N_p(S)$, is said to be *anti-holomorphic* (Yano-Ishihara [6]).

(3.18) is written as

$$F(B\alpha)=||\alpha||^2 C$$

by virtue of (3. 8). The fact that $||\alpha||^2=1$ on S shows

$$FC=-B\alpha.$$

The similar computation gives

$$FD=-B\beta.$$

Forming the inner product with itself in (3. 18), we get

$$g=\tilde{\alpha}\otimes\tilde{\alpha}+\tilde{\beta}\otimes\tilde{\beta}, \tag{3. 19}$$

if we take account of (1. 3).

Summing up results of this section we have the following

THEOREM 3. 1. *Let M be locally Fubinian manifold of complex dimension n and S a totally geodesic submanifold of M of real dimension $2n-2$. Then the case in which S is not complex analytic occurs only when $n=2$.*

§ 4. Totally umbilical submanifolds in a locally Fubinian manifold.

In this section we investigate a totally umbilical submanifold S in a locally Fubinian manifold M. We assume, throughout this section, that $\dim M-\dim S=2$. A submanifold S is said to be *umbilical at a point p*, if there is a vector H perpendicular to $T_p(S)$ for which the normal part of $\bar{\nabla}_{\bar{Y}}\bar{X}$ (see (0. 5)) at p is expressed as $(X, Y)H$ at p, where X and Y are arbitrary vector fields on S and $(\ ,\ )$ is the inner product with respect to the induced metric on S. *A totally umbilical submanifold* is a manifold which is umbilical at any point of the manifold. By the definition of a totally umbilical submanifold, there exist differentiable functions ρ and σ on S for which we have

$$(hX, Y)=(X, Y)\rho \quad \text{and} \quad (kX, Y)=(X, Y)\sigma, \tag{4. 1}$$

where X and Y are arbitrary vector fields on S. It is well known that an invariant submanifold of Kählerian manifold is minimal and then there exists no invariant submanifold which is totally umbilical except a totally geodesic submanifold. A similar result is obtained for a totally umbilical submanifold in a locally Fubinian manifold which is not invariant. We have

THEOREM 4. 1. *Let M be a locally Fubinian manifold of dimension $2n$ whose curvature scalar is not zero. Then there exists no totally umbilical submanifold of dimension $2n-2$ such that the vector field H appeared above does not vanish on the whole submanifold.*

Proof. Let there be given a totally umbilical submanifold S of dimension $2n-2$

in M. We use here again (3. 1)

$$F(BX)=T(X)+N(X),$$

where X is an arbitrary vector field on S. Since S is not invariant under F, $N(X)$ is not identically zero on S. Equations of Gauss, Codazzi and Ricci for a totally umbilical submanifold in a locally Fubinian manifold are written as

$$\begin{aligned}&\frac{c}{4}\{(W,X)(V,Y)-(W,Y)(V,X)+(AW,X)(AV,Y)\\ &\qquad -(AW,Y)(AV,X)-2(AW,V)(AY,X)\}\\ &=K(W,V,Y,X)-(\rho^2+\sigma^2)\{(W,X)(V,Y)-(W,Y)(V,X)\},\end{aligned} \tag{4. 2}$$

$$\begin{aligned}&\frac{c}{4}\{\tilde{\alpha}(W)\tilde{A}(V,Y)-\tilde{\alpha}(V)\tilde{A}(W,Y)-2\tilde{A}(W,V)\tilde{\alpha}(Y)\}\\ &=(V,Y)\{W\rho-\sigma\tilde{l}(W)\}-(W,Y)\{V\rho-\sigma\tilde{l}(V)\},\end{aligned} \tag{4. 3}$$

$$\begin{aligned}&\frac{c}{4}\{\tilde{\beta}(W)\tilde{A}(V,Y)-\tilde{\beta}(V)\tilde{A}(W,Y)-2\tilde{A}(W,V)\tilde{\beta}(Y)\}\\ &=(V,Y)\{W\sigma+\rho\tilde{l}(W)\}-(W,Y)\{V\sigma+\rho\tilde{l}(V)\}\end{aligned} \tag{4. 3)$'$}$$

and

$$\frac{c}{4}\{\tilde{\beta}(W)\tilde{\alpha}(V)-\tilde{\alpha}(W)\tilde{\beta}(V)-2\phi\tilde{A}(W,V)\}=2(d\tilde{l})(W,V), \tag{4. 4}$$

respectively, where $\phi=\tilde{F}(C,D)$. By a similar method as used in Lemma 3. 4 we have respectively from (4. 3) and (4. 3)$'$

$$W\rho=\sigma\tilde{l}(W)-\frac{3c}{4(2n-3)}\phi\tilde{\beta}(W) \tag{4. 5}$$

and

$$W\sigma=-\rho\tilde{l}(W)+\frac{3c}{4(2n-3)}\phi\tilde{\alpha}(W), \tag{4. 6}$$

if we take account of (3. 9) and (3. 11). On the other hand, (3. 3) and (3. 4) show

$$W\phi=-\rho\tilde{\beta}(W)+\sigma\tilde{\alpha}(W), \tag{4. 7}$$

from which we have

$$W(\rho^2+\sigma^2)=\frac{3c}{4(2n-3)}W\phi^2. \tag{4. 8}$$

The integrability conditions of (4. 5) and (4. 6) are respectively

$$(4.9)\qquad 2(d\tilde{\rho})(W, V)=\frac{2nc\sigma}{4(2n-3)}\{-2\phi\tilde{A}(W, V)+\tilde{\alpha}(V)\tilde{\beta}(W)-\tilde{\beta}(V)\tilde{\alpha}(W)\}=0$$

and

$$(4.10)\qquad 2(d\tilde{\sigma})(W, V)=\frac{2nc\rho}{4(2n-3)}\{-2\phi\tilde{A}(W, V)+\tilde{\alpha}(V)\tilde{\beta}(W)-\tilde{\beta}(V)\tilde{\alpha}(W)\}=0,$$

where $d\tilde{\rho}$ and $d\tilde{\sigma}$ are exterior differential forms constructed with ρ and σ respectively. These are also written as

$$(4.9)'\qquad (d\tilde{\rho})(W, V)=\frac{2n\sigma}{2n-3}(d\tilde{l})(W, V)=0$$

and

$$(4.10)'\qquad (d\tilde{\sigma})(W, V)=\frac{2n\rho}{2n-3}(d\tilde{l})(W, V)=0$$

by means of (4. 4).

On the other hand $\rho^2+\sigma^2$ does not vanish by our assumption that H has no zero point. The constant c being non-zero, we have, from (4. 10) and (4. 10)′

$$\tilde{\beta}(W)\tilde{\alpha}(V)-\tilde{\alpha}(W)\tilde{\beta}(V)-2\phi\tilde{A}(W, V)=0.$$

We have, from the equation above together with (3. 5), (3. 8) and (3. 9),

$$\tilde{\beta}(W)(3\phi^2-1)=0,$$

from which we further have

$$(4.11)\qquad (1-\phi^2)(3\phi^2-1)=0.$$

There is no point at which both $1-\phi^2$ and $3\phi^2-1$ vanish. On the other hand, it can be shown that ϕ is not constant if we take account of (4. 7) and the assumption that S is not invariant under F. Therefore, (4. 11) is impossible. q.e.d.

REMARK. We see, from (4. 8), that $\rho^2+\sigma^2$ is not constant when c is not zero. However, the equation of Gauss shows that a totally umbilical submanifold S of a Fubinian manifold having zero curvature scalar is of constant curvature. The curvature scalar of S is of course positive. Thus, if we assume that M is complete, then S is compact (Bonnet). Since S is even dimensional, it is simply connected (Synge). Therefore S is isometric to a sphere. If we further assume M to be flat, then the vector field X defined by

$$(4.12)\qquad X=\xi+\frac{\rho}{\rho^2+\sigma^2}C+\frac{\rho}{\rho^2+\sigma^2}D$$

passes through some fixed point o, where ξ is the position vector of S from the origin of M. In fact, differentiating (4.12) along S we have, for any vector Y on S,

$$\nabla_Y X = 0,$$

by means of (0.5), (4.1), (4.5) and (4.6). On the other hand, distance from o to any point of S is $(\rho^2+\sigma^2)^{-1/2}$ which is constant. Thus we have a proof of a well known

THEOREM 4.2. *Let M be a flat Kählerian manifold*[7] *of dimension $2n$ and S a totally umbilical submanifold of M whose dimension is $2n-2$. Then S is a sphere of radius* $(\rho^2+\sigma^2)^{-1/2}$

The case in which ρ or σ has zero point but is not identically zero is rather complicated and we leave the discussion of this case for the future.

BIBLIOGRAPHY

[1] NAKAGAWA, H., f-structures induced on submanifolds in spaces, almost Hermitian or Kählerian. Kōdai Math. Sem. Rep. **18** (1966), 161–183.

[2] O'NEIL, B., Isotropic and Kähler immersions. Canad. J. of Math. **17** (1965), 907–915.

[3] SCHOUTEN, J. A., AND K. YANO, On invariant subspaces in the almost complex X_{2n}. Indg. Math. **17** (1955), 261–269.

[4] SMYTH, B., Differential geometry of complex hypersurfaces. Thesis, Brown Univ., June, 1966.

[5] TASHIRO, Y., AND S. TACHIBANA, On Fubinian and C-Fubinian manifolds. Kōdai Math. Sem. Rep. **15** (1963), 176–183.

[6] YANO K., AND S. ISHIHARA, The f-structure induced on submanifolds of complex and almost complex spaces. Kōdai Math. Sem. Rep. **18** (1966), 120–160.

[7] YANO, K., Differential geometry on complex and almost complex spaces. Pergamon Press, 1965.

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7) As for the case in which M is not necessarily flat, see Okumura, M., Totally umbilical submanifolds of Kaehlerian manifold, to appear in J. Math. Soc. Japan.