ON THE BIAS OF A SIMPLIFIED ESTIMATE OF CORRELOGRAM

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§1. Introduction.

Let $X(n)$ be a real-valued weakly stationary process with discrete time parameter n. For simplicity, we assume $EX(n)=0$.

We shall denote

$$EX(n)^2=\sigma^2 \qquad \text{and} \qquad EX(n)X(n+h)=\sigma^2\rho_h$$

and consider to estimate the correlogram ρ_h when σ^2 is known. We assume $X(n)$ to be observed at $n=1, 2, 3, \cdots, N, \cdots, N+h$. Usually, we use the estimate

$$\tilde{\gamma}_h=\frac{1}{\sigma^2}\frac{1}{N}\sum_{n=1}^{N} X(n)X(n+h)$$

for the estimation of ρ_h. $\tilde{\gamma}_h$ is an unbiased estimate of ρ_h.

We have shown that when $X(n)$ is a Gaussian process,

$$\gamma_h=\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\frac{1}{N}\sum_{n=1}^{N} X(n)\,\mathrm{sgn}\,(X(n+h))$$

is also an unbiased estimate of ρ_h, where $\mathrm{sgn}(y)$ means $1, 0, -1$ correspondingly as $y>0$, $y=0$, $y<0$, and we have evaluated the variance of γ_h ([3], [4]).

In this paper, we discuss the bias of the estimate γ_h when the assumption that $X(n)$ is a Gaussian process is not satisfied. For a class of stationary processes, which are not Gaussian, we shall show the bias of γ_h and its properties.

§2. Stationary processes which deviate from a Gaussian process.

In this paper, we shall assume a stationary process $X(n)$ which deviates from a Gaussian process to be as follows.

Let $X(n)$ be, furthermore, a strictly stationary process and $\tilde{f}(x, y)$ denote the probability density of the joint distribution of the variables $X(n)$ and $X(n+h)$. Clearly, $\tilde{f}(x, y)$ does not depend on n. We have

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$$EX(n)=EX(n+h)=0, \qquad EX(n)^2=EX(n+h)^2=\sigma^2$$

and

$$EX(n)X(n+h)=\sigma^2\rho_h.$$

Let $\Phi_2(x, y; \sigma^2, \sigma^2\rho_h)$ denote the probability density function of the two-dimensional Gaussian distribution function with the mean vector

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and the variance-covariance matrix

$$\begin{pmatrix} \sigma^2 & \sigma^2\rho_h \\ \sigma^2\rho_h & \sigma^2 \end{pmatrix}.$$

Now, we shall assume that $\tilde{f}(x, y)$ satisfies

(1) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\tilde{f}^2(x, y)}{\Phi_2(x, y; \sigma^2, \sigma^2\rho_h)} dxdy < +\infty.$$

Let us use the notations

$$L_2(R)=\left\{g(x);\ \int_{-\infty}^{\infty} g^2(x)dx < +\infty\right\}$$

and

$$L_2(R^2)=\left\{h(x, y); \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h^2(x, y)dxdy < +\infty\right\}.$$

Then the condition (1) can be written as

$$\frac{\tilde{f}(x, y)}{\sqrt{\Phi_2(x, y; \sigma^2, \sigma^2\rho_h)}} \in L_2(R^2).$$

Now we shall make two random variables

$$U(n)=X(n)-\rho_h X(n+h),$$

$$V(n+h)=X(n+h)$$

and treat these random variables $U(n)$ and $V(n+h)$ instead of $X(n)$ and $X(n+h)$. Clearly we have

$$EU(n)V(n+h)=0.$$

Corresponding to the above transformation, we change the variables as follows:

$$u=x-\rho_h y, \qquad v=y.$$

By this transformation, we assume $\tilde{f}(x, y)$ is transformed into $f(u, v)$.

Let us denote

$$\Phi_1(x, \sigma^2)=\frac{1}{\sqrt{2\pi}\sigma}e^{-x^2/2\sigma^2}.$$

Then we find

$$\Phi_2(x, y;\ \sigma^2, \sigma^2\rho_h)=\Phi_1(u, \sigma^2(1-\rho_h^2))\Phi_1(v, \sigma^2)$$

and the condition (1) can be written as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f^2(u, v)}{\Phi_1(u, \sigma^2(1-\rho_h^2))\Phi_1(v, \sigma^2)}dudv<+\infty, \tag{2}$$

that is

$$\frac{f(u, v)}{\sqrt{\Phi_1(u, \sigma^2(1-\rho_h^2))}\sqrt{\Phi_1(v, \sigma^2)}}\in L_2(R^2).$$

§3. A complete orthonormal system of $L_2(R^2)$.

Here we shall prepare for an orthogonal development of the function which belongs to $L_2(R^2)$.

We assume that $H_n(x)$ represents the Hermite polynomial defined by the relation

$$\left(\frac{d}{dx}\right)^n e^{-x^2/2}=(-1)^n H_n(x)e^{-x^2/2} \qquad (n=0, 1, 2, \cdots).$$

$H_n(x)$ is a polynomial of degree n, and we have

$$H_0(x)=1,\quad H_1(x)=x,\quad H_2(x)=x^2-1,$$

$$H_3(x)=x^3-3x, \qquad H_4(x)=x^4-6x^2+3,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots.$$

Then, as is generally known, the system

$$\left\{\frac{1}{\sqrt{n!}}\frac{1}{(2\pi)^{1/4}}H_n(x)e^{-x^2/4}\right\}$$

is a complete orthonormal system on $(-\infty, \infty)$:

$$\frac{1}{\sqrt{m!}\sqrt{n!}}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}H_n(x)H_m(x)e^{-x^2/2}\,dx=\begin{cases}1 \text{ for } m=n,\\ 0 \text{ for } m\neq n\end{cases} \quad (m, n=0, 1, 2, \cdots).$$

We write

$$\varphi_n(x, 1)=\frac{1}{\sqrt{n!}}H_n(x)\sqrt{\Phi_1(x, 1)} \qquad (n=0, 1, 2, \cdots).$$

Some properties of the Hermite polynomials are as follows:

(a) $H_{2k}(x)$ is an even function of x for $k=0, 1, 2, \cdots$.

(b) $H_{2k+1}(x)$ is an odd function of x for $k=0, 1, 2, \cdots$.

(c) $H_{k+1}(x)-xH_k(x)+kH_{k-1}(x)=0$.

Now let us define $\psi_{m,n}(x, 1; y, 1)$ by

$$\psi_{m,n}(x, 1; y, 1)=\varphi_m(x, 1)\varphi_n(y; 1) \qquad (m, n=0, 1, 2, \cdots).$$

Then the system

$$\{\psi_{m,n}(x, 1; y, 1)\}$$

is a complete orthonormal system of $L_2(R^2)$.

§4. An orthogonal expansion of $f(u, v)$ derived from the two-dimensional Gaussian distribution.

In this section, we shall discuss an expansion of $f(u, v)$ by orthogonal functions which are induced in §3. The two-dimensional Gaussian distribution plays a leading part in this expansion. We consider $f(u, v)$ to be slightly different from the two-dimensional Gaussian distribution function, that is, $\Phi_1(u, \sigma^2(1-\rho_h^2))\Phi_1(v, \sigma^2)$.

In accordance with the section 3, we define $\psi_{p,q}(u, \sigma\sqrt{1-\rho_h^2}; v, \sigma)$ by

$$\psi_{p,q}(u, \sigma\sqrt{1-\rho_h^2}; v, \sigma)=\frac{1}{\sqrt{p!}}H_p\left(\frac{u}{\sigma\sqrt{1-\rho_h^2}}\right)\frac{1}{\sqrt{q!}}H_q\left(\frac{v}{\sigma}\right)\sqrt{\Phi_1(u, \sigma^2(1-\rho_h^2))\Phi_1(v, \sigma^2)}.$$

Then $\{\psi_{p,q}(u, \sigma\sqrt{1-\rho_h^2}; v, \sigma)\}$ is a complete orthonormal system of $L_2(R^2)$.

Now, by the condition (2), we have

$$\frac{f(u, v)}{\sqrt{\Phi_1(u, \sigma^2(1-\rho_h^2))\Phi_1(v, \sigma^2)}}\in L_2(R^2),$$

so we can find the expansion such that

$$\frac{f(u, v)}{\sqrt{\Phi_1(u, \sigma^2(1-\rho_h^2))\Phi_1(v, \sigma^2)}}=\underset{P,Q\to\infty}{\mathrm{l.i.m.}}\sum_{p,q=0}^{P,Q} a_{p,q}\psi_{p,q}(u, \sigma\sqrt{1-\rho_h^2}; v, \sigma),$$

where

$$a_{p,q}=\iint\frac{f(u, v)}{\sqrt{\Phi_1(u, \sigma^2(1-\rho_h^2))\Phi_1(v, \sigma^2)}}\psi_{p,q}(u, \sigma\sqrt{1-\rho_h^2}; v, \sigma)dudv$$

$$=\frac{1}{\sqrt{p!}\sqrt{q!}}\iint H_p\left(\frac{u}{\sigma\sqrt{1-\rho_h^2}}\right)H_q\left(\frac{v}{\sigma}\right)f(u, v)dudv.$$

In the above expression, we find

$$a_{0,0}=\iint f(u, v)dudv=1,$$

$$a_{1,0}=\iint H_1\left(\frac{u}{\sigma\sqrt{1-\rho_h^2}}\right)f(u, v)dudv=\frac{EU(n)}{\sigma\sqrt{1-\rho_h^2}}=0,$$

$$a_{0,1}=\iint H_1\left(\frac{v}{\sigma}\right)f(u, v)dudv=\frac{EV(n+h)}{\sigma}=0,$$

$$a_{2,0}=\frac{1}{\sqrt{2}}\iint H_2\left(\frac{u}{\sigma\sqrt{1-\rho_h^2}}\right)f(u, v)dudv=\frac{1}{\sqrt{2}}\left(\frac{EU(n)^2}{\sigma^2(1-\rho_h^2)}-1\right)=0,$$

$$a_{1,1}=\iint H_1\left(\frac{u}{\sigma\sqrt{1-\rho_h^2}}\right)H_1\left(\frac{v}{\sigma}\right)f(u, v)dudv=\frac{EU(n)V(n+h)}{\sigma^2\sqrt{1-\rho_h^2}}=0,$$

$$a_{0,2}=\frac{1}{\sqrt{2}}\iint H_2\left(\frac{v}{\sigma}\right)f(u, v)dudv=\frac{1}{\sqrt{2}}\left(\frac{EV(n+h)^2}{\sigma^2}-1\right)=0.$$

So we have

$$\frac{f(u, v)}{\sqrt{\Phi(u, \sigma^2(1-\rho_h^2))\Phi_1(v, \sigma^2)}}$$

$$=\underset{P,Q\to\infty}{\text{l.i.m.}}\left[\sqrt{\Phi_1(u, \sigma^2(1-\rho_h^2))\Phi_1(v, \sigma^2)}+\sum_{\substack{p,q=0\\p+q\geqq 3}}^{P,Q}a_{p,q}\phi_{p,q}(u, \sigma\sqrt{1-\rho_h^2}; v, \sigma)\right].$$

§5. An orthogonal expansion of $(u+\rho_h v)\,\mathrm{sgn}\,(v)$.

At the beginning, let us arrange our discussion. The essential point of our discussion is to evaluate the value of $EX(n)\,\mathrm{sgn}\,(X(n+h))$. Now, the value of $EX(n)\mathrm{sgn}(X(n+h))$ is as follows:

$$EX(n)\,\mathrm{sgn}\,(X(n+h))=\iint x\,\mathrm{sgn}\,(y)\tilde{f}(x, y)dxdy$$

$$=\iint(u+\rho_h v)\,\mathrm{sgn}\,(v)f(u, v)dudv.$$

The function $(u+\rho_h v)\,\mathrm{sgn}\,(v)$ does not belong to $L_2(R^2)$. But by the condition (2),

$$\frac{f(u, v)}{\sqrt{\Phi_1(u, \sigma^2(1-\rho_h^2))}\sqrt{\Phi_1(v, \sigma^2)}}$$

belongs to $L_2(R^2)$. So, let us express the above value as follows:

$$EX(n)\,\mathrm{sgn}\,(X(n+h))=\iint(u+\rho_h v)\,\mathrm{sgn}\,(v)f(u,\ v)dudv$$

$$=\iint(u+\rho_h v)\,\mathrm{sgn}\,(v)\sqrt{\Phi_1(u,\ \sigma^2(1-\rho_h{}^2))\Phi_1(v,\ \sigma^2)}\cdot\frac{f(u,\ v)}{\sqrt{\Phi_1(u,\ \sigma^2(1-\rho_h{}^2))\Phi_1(v,\ \sigma^2)}}dudv.$$

Then both

$$(u+\rho_h v)\,\mathrm{sgn}\,(v)\sqrt{\Phi_1(u,\ \sigma^2(1-\rho_h{}^2))\Phi_1(v,\ \sigma^2)}\ \text{and}\ \frac{f(u,\ v)}{\sqrt{\Phi_1(u,\ \sigma^2(1-\rho_h{}^2))\Phi_1(v,\ \sigma^2)}}$$

belong to $L_2(R^2)$.

Here we shall discuss an orthogonal expansion of the function

$$(u+\rho_h v)\,\mathrm{sgn}\,(v)\sqrt{\Phi_1(u,\sigma^2(1-\rho_h{}^2))\Phi_1(v,\ \sigma^2)}.$$

As this function belongs to $L_2(R^2)$, we can expand this function by the orthogonal system

$$\{\psi_{k,\,l}(u,\ \sigma\sqrt{1-\rho_h{}^2};\ v,\ \sigma)\}.$$

We consider that this expansion is

$$(u+\rho_h v)\,\mathrm{sgn}\,(v)\sqrt{\Phi_1(u,\ \sigma^2(1-\rho_h{}^2))\Phi_1(v,\ \sigma^2)}=\underset{K,L\to\infty}{\mathrm{l.i.m.}}\sum_{k=0}^{K}\sum_{l=0}^{L}c_{k,\,l}\psi_{k,\,l}(u,\ \sigma\sqrt{1-\rho_h{}^2};\ v,\ \sigma).$$

Now we have

$$c_{k,\,l}=\iint(u+\rho_h v)\,\mathrm{sgn}\,(v)\sqrt{\Phi_1(u,\ \sigma^2(1-\rho_h{}^2))\Phi_1(v,\ \sigma^2)}\,\psi_{k,\,l}(u,\ \sigma\sqrt{1-\rho_h{}^2};\ v,\ \sigma)dudv$$

$$=\frac{1}{\sqrt{k\,!}\sqrt{l\,!}}\iint(u+\rho_h v)\,\mathrm{sgn}\,(v)H_k\Big(\frac{u}{\sigma\sqrt{1-\rho_h{}^2}}\Big)H_l\Big(\frac{v}{\sigma}\Big)\Phi_1(u,\ \sigma^2(1-\rho_h{}^2))\Phi_1(v,\ \sigma^2)dudv$$

$$=\frac{1}{\sqrt{k\,!}\sqrt{l\,!}}\iint u\,\mathrm{sgn}\,(v)H_k\Big(\frac{u}{\sigma\sqrt{1-\rho_h{}^2}}\Big)H_l\Big(\frac{v}{\sigma}\Big)\Phi_1(u,\ \sigma^2(1-\rho_h{}^2))\Phi_1(v,\ \sigma^2)dudv$$

$$+\frac{\rho_h}{\sqrt{k\,!}\sqrt{l\,!}}\iint v\,\mathrm{sgn}\,(v)H_k\Big(\frac{u}{\sigma\sqrt{1-\rho_h{}^2}}\Big)H_l\Big(\frac{v}{\sigma}\Big)\Phi_1(u,\ \sigma^2(1-\rho_h{}^2))\Phi_1(v,\ \sigma^2)dudv.$$

The first term of the above expression is

$$\frac{1}{\sqrt{k\,!}\sqrt{l\,!}}\iint u\,\mathrm{sgn}\,(v)H_k\Big(\frac{u}{\sigma\sqrt{1-\rho_h{}^2}}\Big)H_l\Big(\frac{v}{\sigma}\Big)\Phi_1(u,\ \sigma^2(1-\rho_h{}^2))\Phi_1(v,\ \sigma^2)dudv$$

$$=\left\{\frac{1}{\sqrt{k!}}\int uH_k\left(\frac{u}{\sigma\sqrt{1-\rho_h^2}}\right)\Phi_1(u,\ \sigma^2(1-\rho_h^2))du\right\}\left\{\frac{1}{\sqrt{l!}}\int \mathrm{sgn}\,(v)H_l\left(\frac{v}{\sigma}\right)\Phi_1(v,\ \sigma^2)dv\right\}$$

$$=\begin{cases}\sigma\sqrt{1-\rho_h^2}\cdot\dfrac{1}{\sqrt{(2i+1)!}}\displaystyle\int \mathrm{sgn}\,(v)H_{2i+1}\left(\frac{v}{\sigma}\right)\Phi_1(v,\ \sigma^2)dv, & k=1,\ l=2i+1\ \ (i=0,1,2,\cdots),\\ 0, & \text{otherwise.}\end{cases}$$

The second term is as follows. As stated in § 3, it holds

$$\frac{v}{\sigma}H_l\left(\frac{v}{\sigma}\right)=H_{l+1}\left(\frac{v}{\sigma}\right)+lH_{l-1}\left(\frac{u}{\sigma}\right).$$

Using this relation, we have

$$\frac{\rho_h}{\sqrt{k!}\sqrt{l!}}\iint v\,\mathrm{sgn}\,(v)H_k\left(\frac{u}{\sigma\sqrt{1-\rho_h^2}}\right)H_l\left(\frac{v}{\sigma}\right)\Phi_1(u,\ \sigma^2(1-\rho_h^2))\Phi_1(v,\ \sigma^2)dudv$$

$$=\rho_h\left\{\frac{1}{\sqrt{k!}}\int H_k\left(\frac{u}{\sigma\sqrt{1-\rho_h^2}}\right)\Phi_1(u,\ \sigma^2(1-\rho_h^2))du\right\}\left\{\frac{1}{\sqrt{l!}}\int v\ \mathrm{sgn}\,(v)H_l\left(\frac{v}{\sigma}\right)\Phi_1(v,\ \sigma^2)dv\right\}$$

$$=\begin{cases}\rho_h\left\{\dfrac{1}{\sqrt{k!}}\displaystyle\int H_k\left(\frac{u}{\sigma\sqrt{1-\rho_h^2}}\right)\Phi_1(u,\ \sigma^2(1-\rho_h^2))du\right\} & \\ \quad\times\left\{\dfrac{\sigma}{\sqrt{l!}}\displaystyle\int \mathrm{sgn}\,(v)\,H_{l+1}\left(\frac{v}{\sigma}\right)\Phi_1(v,\ \sigma^2)dv+\dfrac{l\sigma}{\sqrt{l!}}\int \mathrm{sgn}\,(v)H_{l-1}\left(\frac{v}{\sigma}\right)\Phi_1(v,\ \sigma^2)dv\right\}, & l\geqq 1,\\ \rho_h\left\{\dfrac{1}{\sqrt{k!}}\displaystyle\int H_k\left(\frac{u}{\sigma\sqrt{1-\rho_h^2}}\right)\Phi_1(u,\ \sigma^2(1-\rho_h^2))du\right\}\left\{\int |v|\Phi_1(v,\ \sigma^2)dv\right\}, & l=0,\end{cases}$$

$$=\begin{cases}\rho_h\displaystyle\int |v|\Phi_1(v,\ \sigma^2)dv, & k=0,\quad l=0,\\ \rho_h\sigma\dfrac{1}{\sqrt{(2j)!}}\left\{\displaystyle\int \mathrm{sgn}(v)H_{2j+1}\left(\frac{v}{\sigma}\right)\Phi_1(v,\ \sigma^2)dv\right. & \\ \quad\left.+(2j)\displaystyle\int \mathrm{sgn}(v)H_{2j-1}\left(\frac{v}{\sigma}\right)\Phi_1(v,\ \sigma^2)dv\right\} & k=0,\quad l=2j\quad (j\geqq 1),\\ 0, & \text{otherwise.}\end{cases}$$

Therefore we find

$$c_{k,l}=\begin{cases}\rho_h\int|v|\Phi_1(v,\sigma^2)dv, & k=0,\quad l=0,\\ \rho_h\sigma\dfrac{1}{\sqrt{(2j)!}}\Big\{\int\operatorname{sgn}(v)H_{2j+1}\Big(\dfrac{v}{\sigma}\Big)\Phi_1(v,\sigma^2)dv \\ \qquad +(2j)\int\operatorname{sgn}(v)H_{2j-1}\Big(\dfrac{v}{\sigma}\Big)\Phi_1(v,\sigma^2)dv\Big\}, & k=0,\quad l=2j\quad(j\geqq 1),\\ \sigma\sqrt{1-\rho_h^2}\dfrac{1}{\sqrt{(2i+1)!}}\int\operatorname{sgn}(v)H_{2i+1}\Big(\dfrac{v}{\sigma}\Big)\Phi_1(v,\sigma^2)dv, & k=1,\quad l=2i+1\ (i\geqq 0),\\ 0, & \text{ortherwise.}\end{cases}$$

Consequently we have

$$(u+\rho_h v)\operatorname{sgn}(v)\sqrt{\Phi_1(u,\sigma^2(1-\rho_h^2))\Phi_1(v,\sigma^2)}$$

$$=\underset{K,L\to\infty}{\text{l.i.m.}}\Big\{\sqrt{\frac{2}{\pi}}\sigma\rho_h\phi_{0,0}(u,\sigma\sqrt{1-\rho_h^2};v,\sigma)$$

$$+\sum_{i=1}^{K}c_{0,2i}\phi_{0,2i}(u,\sigma\sqrt{1-\sigma_h^2};v,\sigma)+\sum_{i=0}^{L}c_{1,2i+1}\phi_{1,2i+1}(u,\sigma\sqrt{1-\rho_h^2};v,\sigma)\Big\}.$$

§6. Evaluation of the bias of the estimate γ_h.

Using the results in §4 and §5, we shall, in the first place, evaluate the value of $EX(n)\operatorname{sgn}(X(n+h))$.

$$EX(n)\operatorname{sgn}(X(n+h))=\iint x\operatorname{sgn}(y)\tilde{f}(x,y)dxdy$$

$$=\iint(u+\rho_h v)\operatorname{sgn}(v)f(u,v)dudv$$

$$=\iint(u+\rho_h v)\operatorname{sgn}(v)\sqrt{\Phi_1(u,\sigma^2(1-\rho_h^2))\Phi_1(v,\sigma^2)}\frac{f(u,v)}{\sqrt{\Phi_1(u,\sigma^2(1-\rho_h^2))\Phi_1(v,\sigma^2)}}dudv$$

$$=\lim_{\substack{K,L\\P,Q}\to\infty}\iint\Big\{\sqrt{\frac{2}{\pi}}\sigma\rho_h\phi_{0,0}(u,\sigma\sqrt{1-\rho_h^2};v,\sigma)$$

$$+\sum_{i=1}^{K}c_{0,2i}\phi_{0,2i}(u,\sigma\sqrt{1-\rho_h^2};v,\sigma)+\sum_{i=0}^{L}c_{1,2i+1}\phi_{1,2i+1}(u,\sigma\sqrt{1-\rho_h^2};v,\sigma)\Big\}$$

$$\times\Big\{\phi_{0,0}(u,\sigma\sqrt{1-\rho_h^2};v,\sigma)+\sum_{\substack{p,q=0\\p+q\geqq 3}}^{P,Q}a_{p,q}\phi_{p,q}(u,\sigma\sqrt{1-\rho_h^2};v,\sigma)\Big\}dudv$$

$$=\sqrt{\frac{2}{\pi}}\sigma\rho_h+\sum_{i=2}^{\infty}c_{0,2i}a_{0,2i}+\sum_{i=1}^{\infty}c_{1,2i+1}a_{1,2i+1}.$$

So we have

$$\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}EX(n)\operatorname{sgn}(X(n+h))=\rho_h+\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\left\{\sum_{i=2}^{\infty}c_{0,2i}a_{0,2i}+\sum_{i=1}^{\infty}c_{1,2i+1}a_{1,2i+1}\right\}.$$

This means

$$E(\gamma_h)=\frac{1}{N}\sum_{n=1}^{N}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}EX(n)\operatorname{sgn}(X(n+h))$$

$$=\rho_h+\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\left\{\sum_{i=2}^{\infty}c_{0,2i}a_{0,2i}+\sum_{i=1}^{\infty}c_{1,2i+1}a_{1,2i+1}\right\}.$$

Therefore the estimate γ_h has the bias

$$\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\left\{\sum_{i=2}^{\infty}c_{0,2i}a_{0,2i}+\sum_{i=1}^{\infty}c_{1,2i+1}a_{1,2i+1}\right\}.$$

THEOREM 1. *When a strictly stationary process $X(n)$ satisfies the condition* (1), *the estimate γ_h of ρ_h has the property*:

$$E(\gamma_h)=\rho_h+b_h,$$

where b_h is the bias and

$$b_h=\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\left\{\sum_{i=2}^{\infty}c_{0,2i}a_{0,2i}+\sum_{i=1}^{\infty}c_{1,2i+1}a_{1,2i+1}\right\}.$$

§7. Some properties of $a_{p,q}$ and the relations between $a_{p,q}$ and moments.

In this section, we shall consider the relation between $a_{p,q}$ and moments, and also the relation between $a_{p,q}$ and Gaussian properties.

Now,

$$a_{p,q}=\frac{1}{\sqrt{p!}\sqrt{q!}}\iint H_p\left(\frac{u}{\sigma\sqrt{1-\rho_h^2}}\right)H_q\left(\frac{v}{\sigma}\right)f(u,v)dudv.$$

If $f(u,v)$ is the probability density of two-dimensional Gaussian distribution function, $U(n)$ is independent of $V(h+h)$. So we have clearly the following facts:

LEMMA 1. *When the joint distribution of $U(n)$ and $V(n+h)$ is two-dimensional Gaussian distribution, we have*

$$a_{p,q}=\begin{cases} 1 & for \quad p=q=0, \\ 0 & for \quad p\neq 0 \quad or \quad q\neq 0. \end{cases}$$

LEMMA 2. *If the joint distribution of $U(n)$ and $V(n+h)$ is Gaussian, the joint distribution of $X(n)$ and $X(h+h)$ is also Gaussian. And the converse is also true.*

LEMMA 3. *When $X(n)$ is a Gaussian process, we have*

$$a_{p,q}=\begin{cases} 1 & for \quad p=0 \quad and \quad q=0, \\ 0 & for \quad p\neq 0 \quad or \quad q\neq 0, \end{cases}$$

and γ_h is an unbiased estimate of ρ_h.

LEMMA 4. *When $X(n)$ is a strictly stationary process, $a_{p,q}$ depends only on h.*

Now let us put

$$M_{k,l}=EU(n)^k V(n+h)^l=\iint u^k v^l f(u, v)dudv$$

and

$$m_{k,l}=EX(n)^k X(n+h)^l=\iint x^k y^l \tilde{f}(x, y)dxdy.$$

Clearly we have

$$M_{0,l}=m_{0,l}.$$

Let

$$\beta_j^i(\omega_1, \omega_2, \cdots, \omega_k)$$

denote a linear combination of $\omega_1, \omega_2, \cdots, \omega_{k-1}$ and ω_k with constant coefficients. Then we have the following result.

LEMMA 5. *It holds*

$$a_{2k,2l}=a_{2l}^{2k}(M_{0,0}, M_{0,2}, \cdots, M_{0,2l}, M_{2,0}, M_{2,2}, \cdots, M_{2,2l}, \cdots, M_{2k,0}, M_{2k,2}, \cdots, M_{2k,2l}),$$

$$a_{2k,2l+1}=a_{2l+1}^{2k}(M_{0,1}, M_{0,3}, \cdots, M_{0,2l+1}, M_{2,1}, M_{2,3}, \cdots, M_{2,2l+1}, \cdots, M_{2k,1}, M_{2k,3}, \cdots, M_{2k,2l+1}),$$

$$a_{2k+1,2l}=a_{2l}^{2k+1}(M_{1,0}, M_{1,2}, \cdots, M_{1,2l}, M_{3,0}, M_{3,2}, \cdots, M_{3,2l}, \cdots, M_{2k+1,0}, M_{2k+1,2}, \cdots, M_{2k+1,2l}),$$

and

$$a_{2k+1,2l+1}=a_{2l+1}^{2k+1}(M_{1,1}, M_{1,3}, \cdots, M_{1,2l+1}, M_{3,1}, M_{3,3}, \cdots, M_{3,2l+1}, \cdots, M_{2k+1,1}, M_{2k+1,3}, \cdots, M_{2k+1,2l+1}) \qquad (k, l=0, 1, 2, \cdots).$$

As we have seen in the above, the bias of the estimate ρ_h is

$$b_h=\sqrt{\frac{\pi}{2}}\,\frac{1}{\sigma}\left\{\sum_{i=2}^{\infty} c_{0,\,2i}a_{0,\,2i}+\sum_{i=1}^{\infty} c_{1,\,2i+1}a_{1,\,2i+1}\right\}$$

and this shows that the bias is affected only by $\{a_{0,\,2i}\}$ and $\{a_{1,\,2i+1}\}$.

Now we have

$$M_{0,\,2l}=m_{0,\,2l}=m_{2l,\,0}$$

and

$$M_{1,\,2l+1}=EU(n)V(n+h)^{2l+1}=E(X(n)-\rho_h X(n+h))X(n+h)^{2l+1}$$

$$=EX(n)X(n+h)^{2l+1}-\rho_h EX(n+h)^{2l+2}=m_{1,\,2l+1}-\rho_h m_{0,\,2l+2}.$$

So we have

$$\begin{aligned} a_{0,\,2i}&=a_{2i}^0(M_{0,\,0},\,M_{0,\,2},\,\cdots,\,M_{0,\,2i})\\ &=a_{2i}^0(m_{0,\,0},\,m_{0,\,2},\,\cdots,\,m_{0,\,2i}) \end{aligned}\tag{3}$$

and

$$\begin{aligned} a_{1,\,2i+1}&=a_{2i+1}^1(M_{1,\,1},\,M_{1,3},\,\cdots,\,M_{1,\,2i+1})\\ &=\alpha_{2i+1}^1(m_{1,\,1},\,m_{1,\,3},\,\cdots,\,m_{1,\,2i+1},\,m_{0,\,2},\,m_{0,\,4},\,\cdots,\,m_{0,\,2i+2}). \end{aligned}\tag{4}$$

Examples.

$$a_{0,\,4}=\frac{1}{\sqrt{4\,!}}\left(\frac{1}{\sigma^4}M_{0,\,4}-\frac{6}{\sigma^2}M_{0,\,2}+3\right)$$

$$=\frac{1}{\sqrt{4\,!}}\left(\frac{1}{\sigma^4}m_{0,\,4}-\frac{6}{\sigma^2}m_{0,\,2}+3\right)=\frac{1}{\sqrt{4\,!}}\left(\frac{1}{\sigma^4}m_{0,\,4}-3\right),$$

$$a_{0,\,6}=\frac{1}{\sqrt{6\,!}}\left(\frac{1}{\sigma^6}M_{0,\,6}-\frac{15}{\sigma^4}M_{0,\,4}+\frac{45}{\sigma^2}M_{0,\,2}-15\right)$$

$$=\frac{1}{\sqrt{6\,!}}\left(\frac{1}{\sigma^6}m_{0,\,6}-\frac{15}{\sigma^4}m_{0,\,4}+\frac{45}{\sigma^2}m_{0,\,2}-15\right)$$

$$=\frac{1}{\sqrt{6\,!}}\left(\frac{1}{\sigma^6}m_{0,\,6}-\frac{15}{\sigma^4}m_{0,\,4}+30\right),$$

$$a_{1,\,3}=\frac{1}{\sqrt{3\,!}}\left(\frac{1}{\sigma^4\sqrt{1-\rho_h{}^2}}M_{1,\,3}-\frac{3}{\sigma^2\sqrt{1-\rho_h{}^2}}M_{1,\,1}\right)$$

$$=\frac{1}{\sqrt{3\,!}}\,\frac{1}{\sigma^4\sqrt{1-\rho_h{}^2}}M_{1,\,3}$$

$$=\frac{1}{\sqrt{3!}}\frac{1}{\sigma^4\sqrt{1-\rho_h^2}}(-\rho_h m_{0,4}+m_{1,3}),$$

$$a_{1,5}=\frac{1}{\sqrt{5!}}\left(\frac{1}{\sigma^6\sqrt{1-\rho_h^2}}M_{1,5}-\frac{10}{\sigma^4\sqrt{1-\rho_h^2}}M_{1,3}+\frac{15}{\sigma^2\sqrt{1-\rho_h^2}}M_{1,1}\right)$$

$$=\frac{1}{\sqrt{5!}}\left(\frac{1}{\sigma^6\sqrt{1-\rho_h^2}}M_{1,5}-\frac{10}{\sigma^4\sqrt{1-\rho_h^2}}M_{1,3}\right)$$

$$=\left(-\frac{\rho_h}{\sigma^6\sqrt{1-\rho_h^2}}m_{0,6}+\frac{10\rho_h}{\sigma^4\sqrt{1-\rho_h^2}}m_{0,4}+\frac{1}{\sigma^6\sqrt{1-\rho_h^2}}m_{1,5}-\frac{10}{\sigma^4\sqrt{1-\rho_h^2}}m_{1,3}\right).$$

When $X(n)$ is a Gaussian process, it holds

$$M_{0,2k}=(2k-1)\,!!\ M_{0,2}^k=(2k-1)\,!!\ m_{0,2}^k$$

and

$$M_{1,2k+1}=0, \quad \text{that is,} \quad m_{1,2k+1}=\rho_h m_{0,2k+2}=(2k+1)\,!!\ \rho_h m_{0,2}^{k+1}.$$

Then, we have

$$a_{2i}^0(1, M_{0,2}, \cdots, (2i-1)\,!!\ M_{0,2}^i)=a_{2i}^0(1, m_{0,2}, \cdots, (2i-1)\,!!\ m_{0,2}^i)=0$$

and

$$a_{2i+1}^1(0, 0, \cdots, 0)$$

$$=\alpha_{2i+1}^1(\rho_h m_{0,2}, 3\,!!\ \rho_h m_{0,2}^2, \cdots, (2i+1)\,!!\ \rho_h m_{0,2}^{i+1}, m_{0,2}, 3\,!!\ m_{0,2}^2, \cdots, (2i+1)\,!!\ m_{0,2}^{i+1})$$

$$=0$$

By the above results, we can say as follows:

THEOREM 2. *If $X(n)$ is a strictly stationary process satisfying the condition* (1) *and if $a_{0,2i}=0$ for $i\geqq 2$ and $a_{1,2i+1}=0$ for $i\geqq 1$, γ_h is an unbiased estimate of ρ_h. $a_{0,2i}$ and $a_{1,2i+1}$ can be expressed in the form of* (3) *and* (4) *respectively.*

If $\sum_{i=2}^{\infty}a_{0,2i}^2$ and $\sum_{i=1}^{\infty}a_{1,2i+1}^2$ are sufficiently small in comparison with $|\rho_h|$, $E\gamma_h$ is approximately equal to ρ_h. As we have stated in the above, $a_{0,4}$ is related to the coefficient of excess. Let us consider the situation in (u, v, z)-space. The value of $a_{0,4}$ gives a measure of flattening of the frequency curve on a section paprallel to the (v, z)-plane. $a_{0,2i}$ will have a meaning similar to $a_{0,4}$. On the other hand, $a_{1,2i+1}$ gives a measure of the two-dimensional asymmetry.

The other features of the frequency surface, e.g. the one-sided asymmetry, etc., do not influence the bias of the estimate γ_h.

Like the bias, will be a problem the effect on the variance of γ_h, when $X(n)$ deviates from the Gaussian process. This problem will be treated by the method similar to the above. We shall treat this subject in the future.

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