MINIMAL HARMONIC FUNCTIONS ON A RIEMANN SURFACE

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The aim of the present paper is to give another proof of the well known fact that if there exists an HD-minimal or HB-minimal function on a Riemann surface then there exists no non-constant analytic function with a finite Dirichlet integral or no non-constant bounded analytic function respectively. This fact was proved by several authors by using the universal covering surfaces or the compactification of the original Riemann surfaces; [2], [3], [5], [7], [9], [10]. In the present paper we prove this without using the universal covesing surface or the compactification.

We denote by S an arbitrary Riemann surface and by G an arbitrary open set with the relative boundary ∂G consisting of at most enumerable number of piecewise analytic Jordan curves which does not cluster on S. It is sufficient for our purpose to consider only such an open set G satisfying this condition.

Let us denote by $HP_0(G)$ the class of all non-negative harmonic functions defined on G which vanish on ∂G. If $u \in HP_0(G)$, we define u^* on S by $u^*=u$ on G and $u^*=0$ on $S-G$. We denote by $E_G u$ for $u \in HP_0(G)$ the extremisation of u over G and by $I_G v$ the inextremisation of v for $v \in HP(S)$, where $HP(S)$ is the class of all non-negative harmonic functions on S. If there occurs no confusion, we write merely Eu, Iv instead of $E_G u$, $I_G v$.

Eu and Iv are defined by

$$Eu=\inf w[w \geqq u \text{ on } G,\ w \in HP(S)], \tag{1}$$

$$Iv=\sup w[w \leqq v \text{ on } G,\ w \in HP_0(G)]. \tag{2}$$

If there exists no hamonic majorant of u^*, then we put $Eu=\infty$. We can also define Eu and Iv by using a normal exhaustion $\{S_n\}$ of S. Let $E_n u$ be a harmonic function on S_n which takes boundary value 0 on $\partial S_n \cap G^c$ and u on $\partial S_n \cap G$. Let $I_n v$ be a harmonic function on $G \cap S_n$ which takes boundary value 0 on $\partial G \cap S_n$ and v on $\partial S_n \cap G$. Then Eu and Iv are the limits of $E_n u$ and $I_n v$:

$$Eu=\lim_{n\to\infty} E_n u, \tag{3}$$

$$Iv=\lim_{n\to\infty} I_n v. \tag{4}$$

Eu and Iv have the following properties:

(1.1) $Eu \geqq u$, $v \geqq Iv$ and $Ecu=cEu$, $Icv=cIv$ for $u \in HP_0(G)$ and $v \in HP(S)$, and for a constant $c>0$;

(1.2) If $u_i \in HP_0(G)$ $(i=1, 2, \cdots, n)$ $n \leqq \infty$, $v_i \in HP(S)$ $(i=1, 2, \cdots, m)$ $m \leqq \infty$ and

$$\sum_{i=1}^{n} u_i \in HP_0(G), \qquad \sum_{i=1}^{m} v_i \in HP(S),$$

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then

$$E(\sum_{i=1}^{n} u_i) = \sum_{i=1}^{m} Eu_i, \qquad I(\sum_{i=1}^{m} v_i) = \sum_{i=1}^{m} Iv_i;$$

(1.3) If $u_1, u_2 \in HP_0(G)$, $v_1, v_2 \in HP(S)$ and $u_1 \leqq u_2$, $v_1 \leqq v_2$ then $Eu_1 \leqq Eu_2$, $Iv_1 \leqq Iv_2$;
(1.4) If $v \in HP(S)$ and $v \geqq u$ on G for a $u \in HP_0(G)$, then $Eu \leqq v$ holds on S; if $u \in HP_0(G)$ and $v \geqq u$ on G for a $v \in HP(S)$, then $Iv \geqq u$ holds on G;
(1.5) If $u \in HP_0(G)$ and $Eu < \infty$, and if $v \in EP(S)$, then $IEu = u$ holds on G; if $v \in HP(S)$ then $EIv \leqq v$ holds on S;
(1.6) If there exists a $u \in HP_0(G)$ for a $v \in HP(S)$ and satisfies $v \leqq Eu$, then $EIv = v$ holds on S;
(1.7) $Eu_1 \wedge Eu_2 = E(u_1 \wedge u_2)$ holds for $u_1, u_2 \in HP_0(G)$ with $Eu_1, Eu_2 < \infty$, where $\wedge$ means the greatest harmonic minorant;
(1.8) Let G_i $(i=1,2)$ be arbitrary open sets such that $G = G_1 \cup G_2$, $G_1 \cap G_2 = \phi$ and let $u_i \in HP_0(G)$ $(i=1,2)$ be hamonic functions such that $u_i = 0$ on G_i, then $Eu_1 \cap Eu_2 = 0$ holds.

We refer all these results to Heins's paper [4].

We assume that S is a Riemann surface and Ω is an arbitrary domain on S. Let $HD(S)$ and $HD_0(\Omega)$ be classes of all non-negative harmonic functions with finite Dirichlet integrals which belong to $HP(S)$ and $HP_0(\Omega)$ respectively. A harmonic function $u \in HD(S)$ [or $u \in HD_0(\Omega)$] which is not identically equal to zero is said to be an HD-minimal if it satisfies the condition: for all $v \in HD(S)$ [or $v \in HD_0(\Omega)$] such that $v \leqq u$, there exists a constant c for which $v = cu$ holds. We denote by $HB(S)$ and $HB_0(\Omega)$ the classes of all non-negative bounded harmonic functions which belong to $HP(S)$ and $HP_0(\Omega)$ respectively. A harmonic function $u \in HB(S)$ [or $u \in HB_0(\Omega)$] is said to be an HB-minimal if it satisfies the condition: for all $v \in HB(S)$ [or $v \in HB_0(\Omega)$] such that $v \leqq u$, $u \not\equiv 0$ there exists a constant c for which $v = cu$ holds.

We remark that all HD-minimal founctions must necessarily be bounded.

In fact, to show this, it is sufficient to show that there exists a bounded positive harmonic minorant of u with a finite Dirichlet integral. Let us take a point p_0 on S [or on Ω] and consider the set $G_1 = \{p | u(p) > u(p_0)\}$. Take also a point p_1 from the set G_1 and consider the set $G_2 = \{p | u(p) > u(p_1)\}$. The function $[u - u(p_0)] \wedge [u(p_1) - u(p_0)]$ on G_1 is clearly a non-constant bounded harmonic function with finite Dirichlet integral by the Dirichlet principle and by the fact that it is a majorant of $[u(p_1) - u(p_0)]\omega$ on G_2 where ω is the harmonic measure of the ideal boundary of G_2; $\omega \not\equiv 0$ by Mori's lemma [8]. The extremisation of the above function to S [or to Ω] is a desired function. Thus all HD-minimal functions must be bounded.

We can see immediately by the Dirichlet principle and (3), that if $u \in HD_0(\Omega)$ and $Eu < \infty$ then Eu also belongs to the class $HD(S)$. Under these preparations we prove the following lemma.

LEMMA 1. *If a non-negative harmonic function u on S or on a domain on S is HD-minimal [or HB-minimal] then for an arbitrary constant c such that $c < \sup u$, the set $\{p | u(p) > c\}$ is connected.*

Proof. It is sufficient to prove the lemma when u is HD-minimal. In the case when u is HB-minimal, we can prove it by a quite similar method. If the set $\{p|u(p)>c\}$ is not connected, then there exist two open disjoint sets G_1 and G_2 such that $G_1 \vee G_2=\{p|u(p)>c\}$. We define two functions u_i $(i=1,2)$ on G so that $u_i=u-c$ on G_i and $u_i=0$ on $G-G_i$, where $G=\{p|u(p)>c\}$. By the property (1.8) we obtain $Eu_1 \wedge Eu_2=0$. But on the other hand $Eu_i=c_i u$ for suitable constants $c_i>0$ $(i=1,2)$ by HD-minimality of u. This is sbsurd and thus the set G must be connected.

We denote by $u_1 \wedge u_2$ and $u_1 \vee u_2$ for harmonic functions u_1 and u_2 on S, the greatest harmonic minorant and the least harmonic majorant of u_1, u_2 respectively. Let $\{S_n\}$ be a normal exhaustion of S, we can see easily the following relations:

$$u_1 \vee u_2=\lim_{n\to\infty} H_{S_n}^{\max(u_1,u_2)}, \tag{5}$$

$$u_1 \wedge u_2=\lim_{n\to\infty} H_{S_n}^{\min(u_1,u_2)}. \tag{6}$$

(5) or (6) is valid if there exists a limit

$$\lim_{n\to\infty} H_{S_n}^{\max(u_1,u_2)} \text{ or } \lim_{n\to\infty} H_{S_n}^{\min(u_1,u_2)}$$

for an arbitrary exhaustion $\{S_n\}$ or it is known that there exists a harmonic majorant or minorant of u_1, u_2 respectively. For a boundary function f, defined on the relative boundary of G, we denote by H_G^f the solution of the Dirichlet problem which takes zero on the ideal boundary of G provided it does exist.

LEMMA 2. *Let R be a proper subsurface of a Riemann surface S with the relative boundary which consists of at most enumerable number of piecewise analytic Jordan curves. If u_i $(i=1,2)$ are harmonic functions with finite Dirichlet integrals which vanish on the relative boundary of R, then $u_1 \vee u_2$ and $u_1 \wedge u_2$ have also finite Dirichlet integrals.*

Proof. We obtain from (5) and (6) the following relations:

$$u_1 \vee u_2+u_1 \wedge u_2=u_1+u_2, \tag{7}$$

$$u_1 \wedge u_2=u_1+(u_2-u_1) \wedge 0. \tag{8}$$

Therefore it is sufficient to prove that if u belongs to the class $HD_0(R)$, then $u \wedge 0$ also has a finite Dirichlet integral. We put $G=\{p|u(p)\leqq 0\}$ and define f by $f=u$ on G and $f=0$ on $R-G$. Then we obtain by (6)

$$u \wedge 0=\lim_{n\to\infty} H_{S_n \wedge R}^f$$

where $\{S_n\}$ is a normal exhaustion of S provided that $\lim_{n\to\infty} H_{S_n \wedge R}^f$ exists for a suitable exhaustion $\{S_n\}$. Denoting by $D_G(u)$ the Dirichlet integral of u over G, we get by the Dirichlet principle $D_{S_n}(H_{S_n \wedge R}^f) \leqq D_{G \wedge S_n}(u)<\infty$. By our assumption, we can take a fixed point p_0 on an analytic part of the relative boundary of R. Then there is a one-to-one conformal mapping $g(p)$ from a suitably chosen neighborhood $V(p_0)$ of p_0 to the unit circle such that $g(p_0)=0$ and $V(p_0) \cap R$ corresponds to the upper half of the unit circle while the analytic part of the relative boundary of R corresponds to the part of real axis in the circle. The composed function of $g(p)$ and $H^f_{S_n\cap R}$ can be extended to the unit circle and the extended function has a finite Dirichlet integral on the unit circle which is uniformly bounded from above by $2D_R(u)$ and takes the value zero at the origin. Therefore $\{H^f_{S_n\wedge R}\}$ is a normal family and we can select $\{S_n\}$ so that $\lim_{n\to\infty} H^f_{S_n\cap R}$ exists. Thus we see that $H^f_{S_n\cap R}$ coverges monotonely and the limit function has a finite Dirichlet integral on R. Therefore $u\wedge 0$ has a finite Dirichlet integral.

We generalize the lemma 2 as follows.

LEMMA 3 (Constantinescu—Cornea [2]). *If u_i $(i=1,2)$ on a Riemann surface S have finite Dirichlet integrals, then $u_1\vee u_2$ and $u_1\wedge u_2$ have also finite Dirichlet integrals.*

Proof. As above it is sufficient to prove only that if u has a finite Dirichlet integral on a Riemann surface S, then $u\vee 0$ also has a finite Dirichlet integral. Let R_0 be a parametric disc. Then $D_{S-R_0}(H^u_{S-R_0})<\infty$ and therefore $D_{S-R_0}(u-H^u_{S-R_0})<\infty$, and by above lemma 2, $u-H^u_{S-R_0}=V_1+V_2$ with $V_1=(u-H^u_{S-R_0})\vee 0$, $V_2=(u-H^u_{S-R_0})\wedge 0$. Let us put $A=\sup H^u_{S-R_0}$ and $B=\sup(V_1-V_2)$, where the supremum is taken on the relative boundary of R_0, then we have $u\leqq V_1-V_2+A$ on $S-R_0$. Because the relative boundary of R_0 is compact, there is a harmonic majorant of V_1-V_2+A defined on S. In fact, the function f defined by $f=A+B\vee(V_1-V_2)+N-N\omega$ on $S-R_0$ and $f=A+B+N$ on R_0 is a positive superharmonic function for a sufficiently large N, where ω is the harmonic measure of the ideal boundary of $S-R_0$ defined on $S-R_0$. We assume that W is a harmonic majorant of V_1-V_2+A defined on S, then $u\vee 0=\lim_{n\to\infty} H^{f^*_n}_{S_n}\leqq W$, where f^*_n is a boundary function on the relative boundary of S_n which is equal to zero on $\partial S_n\cap\{p|u(p)<0\}$ and equal to u otherwise. Thus $u\vee 0$ really exists and by the Dirichlet principle it has a finite Dirichlet integral.

We have the following lemma 4.

LEMMA 4. *If u belongs to $HP(\Omega)$ for a domain Ω on S and is bounded by a certain v, $v=H^v_\Omega$, $v\in HP(\Omega)$ i.e., $0\leqq u\leqq v=H^v_\Omega$ on Ω, then $u=H^u_\Omega$ holds.*

Proof. By the property of the solution of the Dirichlet problem we have clearly $u\geqq H^u_\Omega$ and $v-u\geqq H^{v-u}_\Omega=H^v_\Omega-H^u_\Omega=v-H^u_\Omega\geqq 0$, and therefore $u=H^u_\Omega$ holds. Let R be an arbitrary subsurface of a Riemann surface with the relative boundary, if it does exist, which consists of at most enumerable number of piecewise analytic Jordan curves. We assume that R has an HD-minimal [or an HB-minimal] function v. Note that when we consider a minimal function on a subsurface, we always assume that it vanishes on the relative boundary.

By lemma 1 the set $\{p|v(p)>c\}$ is a domain for any positive constant c with $c<M=\sup v$. We put $V_m=\{p|v(p)>M-1/m\}$ for an arbitrary positive integer m, then V_m has the following properties:

(2.1) V_m is a non-void domain on R;

(2.2) $V_m\subseteqq V_n$ for $m\geqq n$;

(2.3) The intersection of V_m for all positive integers m is void;
(2.4) The family (V_m) makes a basis of a filter.

We call an arbitrary open set U to be a neighborhood of an HD-minimal [or HB-minimal] point v, if there exists a V_m such that $V_m \subset U$.

For an arbitrary single-valued complex function f defined on a neighborhood of a minimal point v, we define a cluster set $C_v(f)$ at the minimal point v by

$$C_v(f) = \bigcap_{m=1}^{\infty} \overline{f(V_m)}$$

where we take the closure on the Riemann sphere. Clearly $C_v(f)$ is a continuum or a point when f is continuous, by above properties (2.1) and (2.4).

If the cluster set $C_v(f)$ is a single point, we say that f coverges at the HD-minimal [or HB-minimal] point v.

We now define the harmonic measure of sets in R at the minimal point v. Suppose that a set A with relative boundary which consists of at most enumerable number of piecewise analytic Jordan curves meets all V_m, then we first define $H_n(A, v)$ by $H^1_{(\overline{A \cap V_n})^c}$, where we note that $H^1_{(\overline{A \cap V_n})^c}$ is the solution of the Dirichlet problem which is defined on $R - A \cap V_n$ and takes the boundary value 1 on the relative boundary of $A \cap V_n$ and 0 on the relative boundary of R and the ideal boundary of $R - \overline{A \cap V_n}$. Thus we have by definition

$$H_n(A, v) = H^1_{(\overline{A \cap V_n})^c}. \tag{9}$$

We can see easily that $\{H_n(A, v)\}$ coverges and it limit is a non-negative harmonic function on R. We denote this limit by $H(A, v)$, i.e.

$$\lim_{n \to \infty} H_n(A, v) = H(A, v). \tag{10}$$

We call $H(A, v)$ the harmonic measure of the set A at the minimal point v. The notion of the harmonic measure of the set A was first introduced by Kuramochi [6]. We shall show that $H_n(A, v)$ and $H(A, v)$ have the following properties:
(3.1) $0 \leqq H(A, v) \leqq 1$ on R;
(3.2) For any open set G such that $G \subset (\overline{A \cap V_n})^c$ for a V_n, it holds $H(A, v) = H_G^{H(A,v)}$ on G;
(3.3) $H(A, v)$ is the greatest non-negative harmonic function among the harmonic functions w which satisfy $0 \leqq w \leqq 1$ and $w = H_G^W$ for any G in (3.2);
(3.4) For any matually disjoint open sets A_i $(i = 1, 2)$ such that A_i meets all V_m, it holds $H(A_1, v) + H(A_2, v) \geqq H(A_1 \cup A_2, v)$ on R;
(3.5) For two open sets A, B such that $A \subset B$, it holds, $H(A, v) \leqq H(B, v)$ on R;
(3.6) $H_n(A, v) \leqq H_m(A, v)$ holds on $(A \cap V_m)^c$ for $n \geqq m$;
(3.7) If $H(A, v) \not\equiv 0$, then $\sup H(A, v) = 1$.

Proof. All these properties can be proved by the similar ones valid for $H_n(A, v)$ instead of $H(A, v)$, though we must change the scope of the variable domain G and make other slight modifications. The properties for $H_n(A, v)$ can be proved by the properties of the solution of the Dirichlet problem.

Proof of (3.2) can be made by using the lemma 4.

Proof of (3.3). This is immediate by the property (3.3) for $H_n(A, v)$ and (10).

Proof of (3.7). If we assume $H(A, v)\neq 0$, then $\sup H(A, v)=c>0$. $(1/c)H(A, v)$ satisfies the condition of (3.3) for w, therefore $(1/c)H(A, v)\leqq H(A, v)$ holds, and this implies that $\sup H(A, v)=1$. The remaining properties are clear.

If $\sup v=1$ we shall show that $H(v)=v$ holds on R where we put briefly

$$H(R, v)=H(v) \text{ on } R. \tag{11}$$

Let us denote by $E_{V_m}u$ for $u\in HP_0(V_m)$ the extremisation of u over V_m to R. Clearly $v-(1-1/m)\in HD_0(V_m)$ [or $\in HB_0(V_m)$] and by the Dirichlet principle [or by the boundedness of the function], we obtain the relation $D_R(E_{V_m}(v-(1-1/m)))<\infty$, [or the boundedness of $E_{V_m}(v-(1-1/m))$]. Therefore we obtain the relation $E_{V_m}(v-(1-1/m))=cv$ for a constant c with $0<c\leqq 1$ by the property (1.4) and the HD-minimality [or HB-minimality] of v.

Now we determine the constant c. By the property (1.3) of the extremisation, we get $c\leqq 1/m$ and by (1.1) and the assumption that $\sup v=1$, we obtain $c\geqq 1/m$. Hence $c=1/m$. Thus we obtained the relation

$$E_{V_m}\left(v-\left(1-\frac{1}{m}\right)\right)=\frac{1}{m}v \text{ on } R. \tag{12}$$

By (1), (3) and $v-(1-1/m)<1$, it holds the relation $E_{V_m}(v-(1-1/m))<H^f_{R-\bar{V}_m}$ on $R-\bar{V}_m$, where the function f is the boundary function on the relative boundary of $R-V_m$ such that $f=0$ on ∂R if it does exist and $f=1$ on ∂V_m. Thus we obtain, by lemma 4,

$$v=H^v_{R-V_m} \text{ on } R-V_m. \tag{13}$$

Here we note that when we consider an HD-minimal [or HB-minimal] function v on a domain with the relative boundary, we always assume that v vanishes on the relative boundary of the domain. By (13) and the definition (9) of $H_n(A, v)$, it follows

$$v=\left(1-\frac{1}{m}\right)H_m(R, v) \text{ on } R-V_m. \tag{14}$$

Taking the limit of (14), we obtain the desired result $H(v)=H(R, v)=v$.

If a Riemann surface has an HD-minimal [or HB-minimal] function v with supremum 1, then $v-(1-1/m)\in HD_0(V_m)$ [or $v-(1-1/m)\in HB_0(V_m)$], is also an HD-minimal [or HB-minimal] function on V_m. By $\{p|m(v(p)-(1-1/m))>1-1/n\}=\{p|v(p)>1-1/nm\}$ we obtain the following relation for an arbitrary function f defined on a neighorhood of an HD-minimal [or HB-minimal] point v:

$$C_v(f)=C_{m(v-(1-1/m))}(f) \text{ for sufficiently large } m. \tag{15}$$

The above fact (15) will be often used. We prove the following theorem.

THEOREM 1. *If a Riemann surface S or its subsurface R has an HD-minimal [or HB-minimal] function v with supremum* 1, *then there exists no non-constant meromorphic function defined on a open set containing a set $\{p|v(p)>1-1/m^*\}$ for $m^*>0$, such that its cluster set consists of only a single point.*

Proof. If we suppose that the given non-constant meromorphic function $f(p)$ coverges at the minimal point v, then, by considering the composed function of the given function $f(p)$ and a suitably chosen linear transformation if necessary, we can suppose that the limit of the function $f(p)$ itself at the minimal point v is infinite. By this assumption and the relation (15), we can assume without any loss of generality that $f(p)$ is defined on R itself and satisfies the condition $|f(p)|\geqq 1$ on R. As f coverges to the point at infinity at the minimal point v, there exists a positive integer $n(N)$ for a large positive integer N for which $|f(p)|\geqq N$ holds on $V_{n(N)}$. Now the function $\log|f(p)|/\log N$ is a positive superharmonic function on R which is not smaller than 1 on the relative boundary of $V_{n(N)}$. Hence we have $\log|f(p)|/\log N \geqq H_{n(N)}(R, v)\geqq H(v)$ on $R-V_{n(N)}$ and therefore

$$\frac{\log|f(p)|}{\log N}\geqq H(v)=v \text{ on } R. \tag{16}$$

Since we can take N arbitrarily large, (16) means that the harmonic measure $H(v)$ must vanish identically. This is a contradiction. Thus there exists no non-constant meromorophic function having a limit at the minimal point v.

The method of the proof also shows that there exists no non-negative superharmonic function which has the limit ∞ at the minimal point v.

Next in order to prove the well known fact that if there exists an HD-minimal [or HB-minimal] function on a Riemann surface, then there exists no non-constant analytic function with a finite Dirichlet integral [or no non-constant bounded analytic function], we prove the following theorem.

THEOREM 2. *If a Riemann surface or its subsurface R has an HD-minimal [or HB-minimal] function v, then any bounded harmonic function u with a finite Dirichlet integral [or any bounded harmonic function u] on a neighborhood of a minimal point v with supremum* 1 *has a limit at the minimal point v.*

Proof. We shall show a contradiction if we assume that u does not coverge at the minimal point v. By (15) we can assume without any loss of generality that u is defined on R itself and bounded and has a finite Dirichlet integral there. By considering $c_1u+c_2v+c_3$ instead of v for suitably chosen constants c_i $(i=1, 2, 3)$ with $c_1\neq 0$, we can assume that $U=c_1u+c_2v+c_3$ satisfies $0\leqq U\leqq 2$ and has a cluster set which contains both values smaller and larger than 1. By the assumption made above, the sets $A=\{p|U(p)\geqq 1\}$ and $B=\{p|U(p)\leqq 1\}$ meet the set V_m for any positive integer m. Then by the fact $H(v)=v$, (11) and (3.4) at least one of $H(A, v)$ and $H(B, v)$ cannot vanish identically. By considering $2-U$ instead of U, if necessary, we suppose that $H(A, v)\not\equiv 0$. We obtain the following relation for a certain constant c by HD-minimality [or HB-minimality] of v and lemma 3: $U\wedge v=cv$.

Since v coverges to 1 at the minimal point v, there exist V_n for an arbitrary positive number ε for which the following holds:

$$\frac{1}{1-\varepsilon}v\geqq H_n(A, v)\geqq H(A, v)>0 \quad \text{on } R-V_n, \tag{17}$$

$$U \geqq H_n(A, v) \geqq H(A, v) > 0 \quad \text{on } R - V_n. \tag{18}$$

It follows immediately from (10), (17), (18)

$$cv = U \wedge v \geqq H(A, v) \quad \text{on } R. \tag{19}$$

By the property (3.7) and by the assumption $\sup v = 1$, the constant c in (19) must be equal to 1. This shows the following relation:

$$U \geqq v \quad \text{on } R. \tag{20}$$

(20) contradicts our assumption that $C_v(U)$ contains a number smaller than 1. Thus the theorem 2 has been proved.

Theorem 2 corresponds to the fact that any harmonic function with a finite Dirichlet integral has the same radial limit almost everywhere on the *HD*-indivisible set and the Riemann surface with an *HD*-minimal function has an *HD*-indivisible set. As an immediate consequence of theorem 1 and theorem 2, we obtain the following corollary.

COROLLARY 1. *If a Riemann surface has an HD-minimal* [*or HB-minimal*] *function, then there exists no non-constant analytic function with a finite Dirthlet integral* [*or no non-constant bounded analytic function*].

We next study the behavior of non-constant meromorphic functions in a neighborhood of an *HD*-minimal point.

THEOREM 3. *If a Riemann surface or its subsurface has an HD-minimal function, then any non-constant meromorphic function f satisfies at least one of the following conditions*: (1) *the cluster set $c_v(f)$ of f is total and the set of the points taken infinitly often times by f is dense on the Riemann sphere*; (2) *the spherical area of the image of f is infinite.*

Proof. We distinguish two cases: (i) the closure of the image of V_m by f is total for any positive integer m; (ii) otherwise. In the case (i), if we put $W_n = f(V_n)$, then W_n is open and $\overline{W}_n$ is total. In this case $\cap_{n=1}^{\infty} W_n$ is everywhere dense on the Riemann sphere. In fact, let p be a point on the Riemann sphere and U an arbitrary neighborhood of p. Since W_1 is everywhere dense and open, we can take an open set D_1 in U such that $D_1 \subset W_1$ and D_1 contains a closed set including an open set E_1. Since W_2 is everywhere dense on the Riemann sphere, we can take an open set $D_2 \subset W_2 \cap E_1$ which contains a closed set including an open set E_2, and so on. Thus we obtain a sequence satisfying (1) $D_1 \supset D_2 \supset \cdots$ and (2) $D_i \subset W_i \cap E_{i-1}$, $E_i \neq \phi$, $E_i \subset D_i$ $(i = 1, 2, \cdots)$ where D_i are open and E_i are closed. By Cantor's theorem $\cap_{i=1}^{\infty} E_i \neq \phi$. This implies that $\cap_{i=1}^{\infty} W_i$ has a point in U, and therefore $\cap_{i=1}^{\infty} W_i$ is dense on the Riemann sphere. Next in the case (ii), there is a V_n such that the closure of the image is not total and therefore $C(f(V_n))$ has an interior point q. We compose f with a suitably chosen linear transformation which corresponds to the rotation of the Riemann sphere and carries q to the point at infinity. Then the composed function is bounded and must have an infinite Dirichlet integral. Therefore it must have an infinite spherical area, by corollary 1. Thus f must have an infinite spherical area.

We next study the behavior of non-constant meromorphic functions defined on a neighborhood of an HB-minimal point. From above theorems we have the following theorem.

THEOREM 4. *If a Riemann surface has an HB-minimal function v with supremum* 1. *Then any non-constant meromorphic function defined on a set including $\{p|v(p)>1-1/n\}$ for a certain positive integer n, takes all the values on the Riemann sphere infinitely many times except at most a F_σ set of capacity zero.*

Proof. It is sufficient to prove that any non-constant meromorphic function f takes all the values except at most a closed set of capacity zero on each neighborhood of the minimal point v. Clearly the cluster set $C_v(f)$ must be total, by corollary 1. If we assume the existence of a neighborhood of minimal point v on which f excepts a closed set F of positive capacity, then there exists a non-constant bounded harmonic function $h(p)$ on F^c. The composed non-constant bounded harmonic function $h(f(p))$ must coverge at the minimal point v, by theorem 2. But this is clearly a contradiction because $C_v(f)$ is total. Thus we have proved the theorem.

As an immediate consequence of the proof of theorem 3 and theorem 4, we obtain the following corollary.

COROLLARY 2. *If a Riemann surface has an HB-minimal* [*or HD-minimal*] *function, then there exists no non-constant meromorphic function f such that $\nu_f(w)<\infty$ for all w* [*or $\nu_f(w)$ is uniformly bounded with respect to w*]. *In particular, a Riemann surface with finite genus or a Riemann surface of an algebroid function has neither HB-minimal nor HD-minimal function where $\nu_f(w)$ is the number of w-points taken by f.*

We can also prove the known fact that if there exists an HB-minimal function on a Riemann surface S, then there exists no non-constant Lindelöfian meromorphic function on S by using the universal covering surface of S, theorem 1 and the fact $H(v)=v$, where v is an HB-minimal function on S with supremum 1. Theorem 4 can also be proved by using this fact.

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