RELATIONS BETWEEN DOMAINS OF HOLOMORPHY AND MULTIPLE COUSIN'S PROBLEMS

BY JOJI KAJIWARA

Introduction.

Oka [12] proved that a domain D of holomorphy in C^n is a *Cousin*-I *domain*, that is, any additive Cousin's distribution in D has a solution. On the other hand from Cartan [5]-Behnke-Stein [2]'s theorem, a Cousin-I domain in C^2 is a domain of holomorphy. In this way any domain of holomorphy in C^2 can be completely characterized by additive Cousin's problems. For $n \geqq 3$, however, Cartan [6] showed that a Cousin-I domain in C^n is not necessarily a domain of holomorphy. In the previous paper [10] we tried to characterize a domain of holomorphy in a Stein manifold by additive Cousin's problems. An open set G in C^n is called *regular* if $G \cap P$ is a Cousin-I open set for any relatively compact polycylinder P in C^n. We proved that a domain in C^n is a domain of holomorphy if and only if it can be exhausted by regular domains. Moreover, we proved that a regular open set is pseudoconvex in the Cartan's sense at its continuous boundary point. Making use of the results of Oka [13] or Docquier-Grauert [7] respectively, we proved that a domain in C^n or more generally in a Stein manifold with a smooth boundary is a domain of holomorphy if and only if it is locally regular at its each boundary point.

Concerning multiple Cousin's problems the situation is more or less different. Thullen [16] gave an example of a domain in C^2 which is not a domain of holomorphy but a *Cousin*-II *domain*, that is, a domain in which any multiple Cousin's distribution has a solution. Let $\mathfrak{O}$ and $\mathfrak{O}^*$ be, respectively, the sheaves of all germs of holomorphic mappings in C and $GL(1, C)$. As we remarked in [9], Thullen's example is a Cousin-II domain D with $H^1(D, \mathfrak{O}^*) \neq 0$. In the previous paper [11] we proved that a domain (D, φ) over C^n with $H^1(D, \mathfrak{O}^*) = H^1(\varphi^{-1}(H), \mathfrak{O}^*) = 0$ for any analytic plane H in C^n is a domain of holomorphy. Especially a domain (D, φ) over C^2 satisfies $H^1(D, \mathfrak{O}^*) = 0$ if and only if (D, φ) is a domain of holomorphy with $H^2(D, Z) = 0$ where Z is the abelian group of all integers. These facts suggest that we should obtain a sufficient condition that a domain D in C^n is a domain of holomorphy, if we put a similar discussion forward as in [10] substituting a domain G with $H^1(G, \mathfrak{O}^*) = 0$ in stead of a Cousin-I domain.

As a polycylinder P does not necessarily satisfy $H^1(P, \mathfrak{O}^*) = 0$, we shall consider only simply connected polycylinders in the definition below. An open set G in C^n is called *regular** if $H^1(G \cap P, \mathfrak{O}^*) = 0$ for any relatively compact and simply connected polycylinder P in C^n. In the present paper we shall prove that a domain

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in C^n which can be exhausted by regular* domains is a domain of holomorphy and that a regular* domain in C^n is pseudoconvex in the Cartan's sense at its continuous boundary point. Making use of the affirmative solution of the Levi problem loco citato, we can prove that a domain over a Stein manifold with a simultaneously continuous and locally regular* boundary is a domain of holomorphy.

§ 1. Limit of cohomology groups.

A sequence $\{(D_n, \varphi_n);\ n=1, 2, 3, \cdots\}$ of domains (D_n, φ_n) over a Stein manifold S is called a *monotonously increasing sequence* of domains over S if there exists a holomorphic mapping τ_m^n of D_n in D_m with $\varphi_n = \varphi_m \circ \tau_m^n$ for m and n with $m \geqq n$. In the previous paper [8] we proved the existence of a domain (D, φ) over S with the following properties:

(1) There exists a holomorphic mapping τ_n of D_n in D with $\varphi_n = \varphi \circ \tau_n$ for any n.

(2) Let (D', φ') be a domain over S such that there exists a holomorphic mapping τ_n' of D_n in D' with $\varphi_n = \varphi' \circ \tau_n'$ for any n. Then there exists a holomorphic mapping τ' of D in D' with $\tau_n' = \tau' \circ \tau_n$ for any n.

(D, φ) is called the *limit of the sequence* $\{(D_n, \varphi_n);\ n=1, 2, 3, \cdots\}$. We consider the universal covering manifold $(D^\#, \varphi^\#)$ of (D, φ). Let λ be the canonical mapping of $D^\#$ in D. Then τ_m^n, τ_n and λ induce canonically homomorphisms τ_m^{n*}: $\mathrm{H}^1(D_m, \mathfrak{O}^*) \to \mathrm{H}^1(D_n, \mathfrak{O}^*)$, τ_n^*: $\mathrm{H}^1(D, \mathfrak{O}^*) \to \mathrm{H}^1(D_n, \mathfrak{O}^*)$ and λ^*: $\mathrm{H}^1(D, \mathfrak{O}^*) \to \mathrm{H}^1(D^\#, \mathfrak{O}^*)$. For the sake of brevity we put ${}^\#\alpha = \lambda^*(\alpha)$ for $\alpha \in \mathrm{H}^1(D, \mathfrak{O}^*)$ and ${}^\#\mathrm{H}^1(D, \mathfrak{O}^*) = \lambda^*(\mathrm{H}^1(D, \mathfrak{O}^*))$ and we shall use these notations frequently. $\{\mathrm{H}^1(D_n, \mathfrak{O}^*), \tau_m^{n*}\}$ is an inverse system of abelian groups over a directed set $\{n=1, 2, 3, \cdots\}$. We consider its inverse limit and denote it by $\lim \mathrm{H}^1(D_n, \mathfrak{O}^*)$. We denote the canonical homomorphism of $\mathrm{H}^1(D, \mathfrak{O}^*)$ in $\lim \mathrm{H}^1(D_n, \mathfrak{O}^*)$ by π. Unfortunately we can not yet succeed to prove that π is injective but we have the following lemma, which is sufficient for our purpose and the proof of which is quite similar to that of Proposition 2 in the previous paper [8].

LEMMA 1. *$\alpha \in \mathrm{H}^1(D, \mathfrak{O}^*)$ with $\pi(\alpha)=0$ satisfies ${}^\#\alpha=0$.*

Proof. Let $(\tilde{D}_n, \tilde{\varphi}_n)$ and $(\tilde{D}, \tilde{\varphi})$ be, respectively, the envelopes of holomorphy of (D_n, φ_n) and (D, φ). Let $(D_n^\#, \varphi_n^\#)$, $(D^\#, \varphi^\#)$, $(\tilde{D}_n^\#, \tilde{\varphi}_n^\#)$ and $(\tilde{D}^\#, \tilde{\varphi}^\#)$ be, respectively, the universal covering manifolds of (D_n, φ_n), (D, φ), $(\tilde{D}_n, \tilde{\varphi}_n)$ and $(\tilde{D}, \tilde{\varphi})$. Since $(\tilde{D}_n^\#, \tilde{\varphi}_n^\#)$ and $(\tilde{D}^\#, \tilde{\varphi}^\#)$ are p_7-convex in the sense of Docquier-Grauert [7], they are domains of holomorphy from [7]. We consider canonical mappings τ_m^n: $D_n \to D_m$, τ_m: $D_m \to D$, $\tau_m^{\#n}$: $D_n^\# \to D_m^\#$, $\tau_m^\#$: $D_m^\# \to D^\#$, $\tilde{\tau}_m^{\#n}$: $\tilde{D}_n^\# \to \tilde{D}_m^\#$, $\tilde{\tau}_m^\#$: $\tilde{D}_m^\# \to \tilde{D}^\#$, λ_n: $D_n^\# \to D_n$, λ: $D^\# \to D$, μ_n: $D_n \to \tilde{D}_n$, μ: $D \to \tilde{D}$, $\tilde{\lambda}_n$: $\tilde{D}_n^\# \to \tilde{D}_n$, $\tilde{\lambda}$: $\tilde{D}^\# \to \tilde{D}$, κ_n: $D_n^\# \to \tilde{D}_n^\#$ and κ: $D^\# \to \tilde{D}^\#$ for $1 \leqq n \leqq m$. Then the commutativity holds in the following diagram:

$$\begin{tikzcd}
 & D_n \arrow[rr] \arrow[dd] & & D_m \arrow[rr] \arrow[dd] & & D \arrow[dd] \\
D_n^\# \arrow[ur] \arrow[rr] \arrow[dd] & & D_m^\# \arrow[ur] \arrow[rr] \arrow[dd] & & D^\# \arrow[ur] \arrow[dd] & \\
 & \tilde{D}_n \arrow[rr] & & \tilde{D}_m \arrow[rr] & & \tilde{D} \\
\tilde{D}_n^\# \arrow[ur] \arrow[rr] & & \tilde{D}_m^\# \arrow[ur] \arrow[rr] & & \tilde{D}^\# \arrow[ur] &
\end{tikzcd}$$

(D, φ), $(D^\#, \varphi^\#)$, $(\tilde{D}, \tilde{\varphi})$ and $(\tilde{D}^\#, \tilde{\varphi}^\#)$ are, respectively, limits of monotonously increasing sequences $\{(D_n, \varphi_n); n=1, 2, 3, \cdots\}$, $\{(D^\#_n, \varphi^\#_n); n=1, 2, 3, \cdots\}$, $\{(\tilde{D}_n, \tilde{\varphi}_n); n=1, 2, 3, \cdots\}$ and $\{(\tilde{D}^\#_n, \tilde{\varphi}^\#_n); n=1, 2, 3, \cdots\}$ of domains over S. Let $\{P_n; n=1, 2, 3, \cdots\}$, $\{Q_n; n=1, 2, 3, \cdots\}$, $\{R_n; n=1, 2, 3, \cdots\}$ and $\{S_n; n=1, 2, 3, \cdots\}$ be, respectively, sequences of relatively compact subdomains of $\tilde{D}$, D, $\tilde{D}^\#$ and $D^\#$ such that

$$P_n \Subset P_{n+1}, \quad Q_n \Subset Q_{n+1}, \quad R_n \Subset R_{n+1}, \quad S_n \Subset S_{n+1} \ (n \geqq 1),$$

$$\tilde{D} = \bigcup_{n=1}^{\infty} P_n, \quad D = \bigcup_{n=1}^{\infty} Q_n, \quad \tilde{D}^\# = \bigcup_{n=1}^{\infty} R_n, \quad D^\# = \bigcup_{n=1}^{\infty} S_n,$$

$$\mu(Q_n) \subset P_n, \quad \lambda(S_n) \subset Q_n, \quad \tilde{\lambda}(R_n) \subset P_n, \quad \kappa(S_n) \subset R_n$$

and P_n, R_n are, respectively, analytic polycylinders defined by holomorphic functions in $\tilde{D}$ and $\tilde{D}^\#$. If we take a suitable subsequence $\{\nu_n; n=1, 2, 3, \cdots\}$ of $\{1, 2, 3, \cdots\}$, the discussion below can be continued. For the sake of brevity we may assume that $\nu_n = n$ without losing generality. There exists, respectively, subdomains $'P_n$, $'Q_n$, $'R_n$ and $'S_n$ of $\tilde{D}_n$, D_n, $\tilde{D}^\#_n$ and $D^\#_n$ such that $\tilde{\tau}_n$, τ_n, $\tilde{\tau}^\#_n$ and $\tau^\#_n$ map biholomorphically $'P_n$, $'Q_n$, $'R_n$ and $'S_n$ onto P_n, Q_n, R_n and S_n and that

$$\tilde{\tau}^n_{n+1}('P_n) \subset 'P_{n+1}, \quad \tau^n_{n+1}('Q_n) \subset 'Q_{n+1}, \quad \tilde{\tau}^{\# n}_{n+1}('R_n) \subset 'R_{n+1},$$

$$\tau^{\# n}_{n+1}('S_n) \subset 'S_{n+1},$$

$$\mu_n('Q_n) \subset 'P_n, \quad \lambda_n('S_n) \subset 'Q_n, \quad \tilde{\lambda}_n('R_n) \subset 'P_n, \quad \kappa_n('S_n) \subset 'R_n$$

and the commutativity holds in the following diagram:

$$\begin{tikzcd}
 & Q_n \arrow[rr, "\mu"] & & P_n \\
'Q_n \arrow[ur, "\tau_n"] \arrow[rr, "\mu_n"] & & 'P_n \arrow[ur, "\tilde{\tau}_n"'] & \\
 & S_n \arrow[uu, "\lambda"] \arrow[rr, "\kappa"] & & R_n \arrow[uu, "\tilde{\lambda}"'] \\
'S_n \arrow[uu, "\lambda_n"] \arrow[ur, "\tau^\#_n"] \arrow[rr, "\kappa_n"'] & & 'R_n \arrow[uu, "\tilde{\lambda}_n"'] \arrow[ur, "\tilde{\tau}^\#_n"'] &
\end{tikzcd}$$

Under this preperation of the notations we shall preceed in the proof of our Lemma. Let $\mathfrak{B} = \{V_\iota; \iota \in I\}$ be any open covering of D and $\{f_{\iota j}\}$ be any element of $Z^1(\mathfrak{B}, \mathfrak{O}^*)$ such that $\{f_{\iota j} \circ \tau_n\} \in B^1(\tau_n^{-1}(\mathfrak{B}), \mathfrak{O}^*)$ for $n \geqq 1$ where $\tau_n^{-1}(\mathfrak{B}) = \{\tau_n^{-1}(V_\iota); \iota \in I\}$ is an open covering of D_n. There exists $\{f^n_\iota\} \in C^0(\tau_n^{-1}(\mathfrak{B}), \mathfrak{O}^*)$ such that

$$f_{\iota j} \circ \tau_n = f^n_\iota / f^n_j$$

in $\tau_n^{-1}(V_\iota) \cap \tau_n^{-1}(V_j) \neq \phi$ for $n \geqq 1$. If we put

$$f^n = f^n_\iota / f^{n+1}_\iota \circ \tau^n_{n+1}$$

in $\tau_n^{-1}(V_\iota)$, then f^n is well-defined and belongs to $H^0(D_n, \mathfrak{O}^*)$. Since $(\tilde{D}_n, \tilde{\varphi}_n)$ is the envelope of holomorphy of (D_n, φ_n), there exists $\tilde{f}^n \in H^0(\tilde{D}_n, \mathfrak{O}^*)$ satisfying

$$f^n = \tilde{f}^n \circ \mu_n.$$

Then $\log(\tilde{f}^n \circ \tilde{\lambda}_n) \in H^0(\tilde{D}^\#_n, \mathfrak{O})$ for any fixed branch of logarithmus ($n \geqq 1$). There holds $\log(\tilde{f}^n \circ \tilde{\lambda}_n \circ (\tilde{\tau}^\#_n | 'R_n)^{-1}) \in H^0(R_n, \mathfrak{O})$ for $n \geqq 1$ where $\tilde{\tau}^\#_n | 'R_n$ is the restriction of $\tilde{\tau}^\#_n$ to

$'R_n$. Since R_n is an analytic polycylinder defined by holomorphic functions in $\tilde{D}^{\#}$, there exists a holomorphic function $\tilde{h}^n \in \tilde{D}^{\#}$ such that

$$|\log(\tilde{f}^n \circ \tilde{\lambda}_n \circ (\tilde{\tau}_n^{\#}|'R_n)^{-1}) - \tilde{h}^n| < 2^{-n-2}$$

in R_{n-1} for $n \geqq 2$ from Behnke [1]. We put

$$H^n = \exp(\tilde{h}^n \circ \kappa) \in \mathrm{H}^0(D^{\#}, \mathfrak{O}^*).$$

There holds

$$|(f^n \circ (\tau_n|'Q^n)^{-1} \circ \lambda)/H^n - 1| < 2^{-n}$$

in S_{n-1} for $n \geqq 2$. We put

$$G^1 = 1, \quad G^n = H^1 H^2 \cdots H^{n-1} \in \mathrm{H}^0(D^{\#}, \mathfrak{O}^*)$$

and

$$F_i^n = (f_i^n \circ (\tau_n|'Q_n)^{-1} \circ \lambda) G^n \in \mathrm{H}^0(\lambda^{-1}(V_i \cap Q_n), \mathfrak{O}^*).$$

Then we have $\{F_i^n\} \in \mathrm{C}^0(\lambda^{-1}(\mathfrak{V} \cap Q_n), \mathfrak{O})$ where $\lambda^{-1}(\mathfrak{V} \cap Q_n) = \{\lambda^{-1}(V_i \cap Q_n); i \in I\}$ is an open covering of $\lambda^{-1}(Q_n)$. There holds

$$|F_i^n / F_i^{n+1} - 1| < 2^{-n}$$

in $S_{n-1} \cap \lambda^{-1}(V_i)$. Hence each F_i^n converges uniformly in any compact subset of $\lambda^{-1}(V_i)$ to $F_i \in \mathrm{H}^0(\lambda^{-1}(V_i), \mathfrak{O}^*)$. Since there holds

$$f_{ij} \circ \lambda = F_i / F_j$$

in $\lambda^{-1}(V_i \cap V_j) \neq \phi$, we have proved our lemma.

COROLLARY OF LEMMA 1. *If* $\mathrm{H}^1(D_n, \mathfrak{O}^*) = 0$ *for* $n \geqq 1$, *then we have* ${}^{\#}\mathrm{H}^1(D, \mathfrak{O}^*) = 0$.

§ 2. Domains D with ${}^{\#}\mathrm{H}^1(D, \mathfrak{O}^*) = 0$.

A collection $\mathfrak{C} = \{(m_i, U_i, V_i); i \in I\}$ is called a *multiple Cousin's distribution in a complex space X with essential singularities* if the following conditions are satisfied:

(1) $\mathfrak{U} = \{U_i; i \in I\}$ is an open covering of X.

(2) Each connected component of $U_i \cap U_j \cap U_k$ contains that of $V_i \cap V_j \cap V_k$ for any $U_i \cap U_j \cap U_k \neq \phi$.

(3) m_i is a single-valued meromorphic function in an open subset V_i of U_i for any i. m_i/m_j can be analytically continued to a function belonging to $\mathrm{H}^0(U_i \cap U_j, \mathfrak{O}^*)$ for any $U_i \cap U_j \neq \phi$.

$\{m_i/m_j\}$ defines an element of $\mathrm{Z}^1(\mathfrak{U}, \mathfrak{O}^*)$ from the condition (2). Its canonical image in $\mathrm{H}^1(X, \mathfrak{O}^*)$ is denoted by α. A meromorphic function m in $X' = \cup_{i \in I} V_i$ is called a *solution of* $\mathfrak{C}$ if m/m_i can be analytically continued to a function belonging to $\mathrm{H}^0(U_i, \mathfrak{O}^*)$ for any i. Let $X^{\#}$ be a universal covering space of X and λ be the canonical mapping of $X^{\#}$ onto X. A meromorphic function M in $\lambda^{-1}(X')$ is called a *multiform solution of* $\mathfrak{C}$ if $M/m_i \circ \lambda$ can be analytically continued to a function belonging to $\mathrm{H}^0(\lambda^{-1}(U_i), \mathfrak{O}^*)$ for any i. For the canonical homomorphism

λ^*: $\mathrm{H}^1(X, \mathfrak{O}^*) \to \mathrm{H}^1(X^\sharp, \mathfrak{O}^*)$, we put ${}^\sharp\alpha = \lambda^*(\alpha)$ and ${}^\sharp\mathrm{H}^1(X, \mathfrak{O}^*) = \lambda^*(\mathrm{H}^1(X, \mathfrak{O}^*))$.

LEMMA 2. *If $\alpha=0$ in $\mathrm{H}^1(X, \mathfrak{O}^*)$, $\mathfrak{C}$ has a solution. If ${}^\sharp\alpha=0$ in ${}^\sharp\mathrm{H}^1(X, \mathfrak{O}^*)$, $\mathfrak{C}$ has a multiform solution.*

Proof. We shall prove the last half of our Lemma. If ${}^\sharp\alpha=0$ in ${}^\sharp\mathrm{H}^1(X, \mathfrak{O}^*)$, $\{m_i\circ\lambda/m_j\circ\lambda\}$ defines a coboundary of $\{F_i\}\in \mathrm{C}^0(\lambda^{-1}(\mathfrak{U}), \mathfrak{O}^*)$ as $\mathrm{H}^1(\lambda^{-1}(\mathfrak{U}), \mathfrak{O}^*) \to \mathrm{H}^1(X^\sharp, \mathfrak{O}^*)$ is injective. If we put

$$M = m_i\circ\lambda/F_i$$

in $\lambda^{-1}(V_i)$ for any $i\in I$, M is well-defined and a meromorphic function in $\lambda^{-1}(X')$ which is a multiform solution of $\mathfrak{C}$.

A complex space X is called a *Cousin*-II-E *space* (or a *multiform Cousin*-II-E *space*) if any multiple Cousin's distribution in X with essential singularities has a solution (or a multiform solution).

COROLLARY OF LEMMA 2. *If $\mathrm{H}^1(X, \mathfrak{O}^*)=0$ (or ${}^\sharp\mathrm{H}^1(X, \mathfrak{O}^*)=0$), X is a Cousin-II-E space (or a multiform Cousin-II-E space).*

A function h in a set A is called a *trace* of a function f in the superset B of A if there holds $h=f$ in A.

LEMMA 3. *Let (D, φ) be a multiform Cousin-II-E domain over C^n, $(D^\sharp, \varphi^\sharp)$ be the universal covering manifold of (D, φ) and λ be the canonical mapping of $D^\sharp$ onto D. Then for any $(n-1)$-dimensional analytic plane H in C^n and for any holomorphic function h in $\varphi^{-1}(H)$, $h\circ\lambda$ is a trace of a holomorphic function f in $D^\sharp$.*

Proof. Without loss of generality we may assume that

$$H = \{(z_1, z_2, \cdots, z_n); z_1=0\}.$$

There exists an open neighbourhood V of $\varphi^{-1}(H)$ such that h is a trace of a holomorphic function h' in V. We can take another open subset U of D such that $\mathfrak{U}=\{U, V\}$ is an open covering of D and $U\cap\varphi^{-1}(H)=\phi$. Then

$$\mathfrak{C} = \{(1, U, U), (\exp(h'/z_1\circ\varphi), V, V-\varphi^{-1}(H))\}$$

is a multiple Cousin's distribution in D with essential singularities. Hence there exists a multiform solution M of $\mathfrak{C}$. We have $M\in \mathrm{H}^0(D^\sharp - \varphi^{\sharp-1}(H), \mathfrak{O}^*)$. If we start any function element defined by $\log M$ at a point of $D^\sharp - \varphi^{\sharp-1}(H)$, it can be not only analytically continued along any curve in $D^\sharp - \varphi^{\sharp-1}(H)$ but also meromorphically continued at any point of $\varphi^{\sharp-1}(H)$. Since the fundamental group of $D^\sharp$ vanishes, it defines a meromorphic function in $D^\sharp$ which we shall denote by the same symbol $\log M$. If we put

$$f = (z_1\circ\varphi^\sharp)\log M,$$

f is a holomorphic function in $D^\sharp$ whose trace in $\varphi^{\sharp-1}(H)$ is $h\circ\lambda$.

LEMMA 4. *Let (G, φ) be a multiform Cousin-II-E domain over C^n such that*

$\varphi^{-1}(H)$ is a multiform Cousin-II-E open set over H for any m-dimensional analytic plane $H=\{z=(z_1, z_2, \cdots, z_n); z_j=c_j\ (j=s_1, s_2, \cdots, s_{n-m})\}$ where $m, s_1, s_2, \cdots$ and s_{n-m} are integers with $1\leqq m<n$, $1\leqq s_1<s_2<\cdots<s_{n-m}\leqq n$ and c_j's are complex numbers. Then G is a domain of holomorphy.

Proof. We shall prove our Lemma by induction with respect to n. For $n=1$ there is nothing to prove from Behnke-Stein [3]. Suppose that our Lemma is valid for all $n\leqq k$ and consider the case $n=k+1$. For any H $\varphi^{-1}(H)$ is an open set of holomorphy over H from the assumption of our induction. Let E be the set of all boundary point x^0 of G such that x^0 is a boundary point of $\varphi^{-1}(H)$ for some k-dimensional analytic plane $H=\{z; z_j=z_j^0\}$ where $(z_1^0, z_2^0, \cdots, z_n^0)=\varphi(x^0)$. Then E is dence in ∂G. Let x^0 be a point of E and H be a k-dimensional analytic plane satisfying the above condition for this x^0. Let $(G^\sharp, \varphi^\sharp)$ be the universal covering manifold of (G, φ) and λ be the canonical mapping of $D^\sharp$ onto D. Since $\varphi^{-1}(H)$ is an open set of holomorphy, there exists a holomorphic function h in $\varphi^{-1}(H)$ which is unbounded at x^0. Since G is a multiform Cousin-II-E domain, there exists a holomorphic function f in $G^\sharp$ such that

$$f=h\circ\lambda$$

in $\varphi^{\sharp-1}(H)$ from Lemma 3. Hence any boundary point of $(G^\sharp, \varphi^\sharp)$ belonging to $\lambda^{-1}(E)$ has the frontier property in the sense of Bochner-Martin [4]. Since E is dense in ∂G, there exists a holomorphic function g in $G^\sharp$ which is unbounded at each boundary point of $(G^\sharp, \varphi^\sharp)$ from [4]. $(G^\sharp, \varphi^\sharp)$ is a covering manifold of the domain of holomorphy of g. Hence $(G^\sharp, \varphi^\sharp)$ is a domain of holomorphy from Oka [13] or Stein [15]. (G, φ) is also a domain of holomorphy from Oka [13].

COROLLARY OF LEMMA 4. *Any multiform Cousin-II-E domain over C^2 is a domain of holomorphy.*

§ 3. Domain exhausted by regular* domains.

A domain G in C^n is called *exhausted by regular* domains* G_p if G_p's are regular* domains in C^n such that

$$G_p \Subset G_{p+1}\ (p=1, 2, 3, \cdots) \text{ and } G=\bigcup_{p=1}^{\infty} G_p.$$

LEMMA 5. *Let G be a domain in C^n exhausted by regular* domains G_p. Then ${}^\sharp\mathrm{H}^1(G, \mathfrak{O}^*)=0$. Moreover for any integers $1\leqq m<n$, $1\leqq s_1<s_2<\cdots<s_{n-m}\leqq n$ and for any complex numbers $c_j\ (j=s_1, s_2, \cdots, s_{n-m})$ the intersection $G\cap H$ of G and $H=\{z=(z_1, z_2, \cdots, z_n); z_j=c_j(j=s_1, s_2, \cdots, s_{n-m})\}$ satisfies ${}^\sharp\mathrm{H}^1(G\cap H, \mathfrak{O}^*)=0$.*

Proof. Since G_p is a relatively compact regular* domain, we have $\mathrm{H}^1(G_p, \mathfrak{O}^*)=0$ for any p. From Corollary of Lemma 1 we have ${}^\sharp\mathrm{H}^1(G, \mathfrak{O}^*)=0$.

Next we shall prove ${}^\sharp\mathrm{H}^1(G\cap H, \mathfrak{O}^*)=0$. We may assume that

$$H=\{(z, w)=(z_1, z_2, \cdots, z_m, w_1, w_2, \cdots, w_{n-m}); w_j=0\ (j=1, 2, \cdots, n-m)\}.$$

There exist $\varepsilon_p>0$ and $a_p>0$ such that

$$E_p=G_p\cap\{(z,w);\ |z_j|<a_p,\ |w_k|<\varepsilon_p\ \ (j=1,2,\cdots,m,\ k=1,2,\cdots,n-m)\}$$
$$\subset\{(z,w);\ |z_j|<a_p,\ |w_k|<\varepsilon_p,\ (z,0)\in G\cap H,\ \ (j=1,2,\cdots,m,\ k=1,2,\cdots,n-m)\},$$
$$a_p<a_{p+1}\ (p\geqq 1)\quad\text{and}\quad a_p\to\infty\ (p\to\infty),$$
$$\varepsilon_p>\varepsilon_{p+1}\ (p\geqq 1)\quad\text{and}\quad \varepsilon_p\to 0\ (p\to\infty).$$

Since G_p is regular*, we have $\mathrm{H}^1(E_p,\mathfrak{O}^*)=0$ for any p. We put

$$H_p=G_p\cap H\cap\{(z,0);\ |z_j|<a_p\ \ (j=1,2,\cdots,m)\}.$$

Then $G\cap H$ is the limit of monotonously increasing sequence of open sets H_p in H. Let $\mathfrak{V}=\{V_s;\ s\in S\}$ be an open covering of $G\cap H$. We put $V_s^p=V_s\cap H_p$ for $s\in S$. Then $\mathfrak{V}_p=\{V_s^p;\ s\in S\}$ is an open covering of H_p. We put

$$U_s^p=E_p\cap\{(z,w);\ (z,0)\in V_s\}$$

for $s\in S$. Then $\mathfrak{U}_p=\{U_s^p;\ s\in S\}$ is an open covering of E_p. Let $\{f_{st}(z)\}$ be an element of $\mathrm{Z}^1(\mathfrak{V},\mathfrak{O}^*)$. We put

$$F_{st}^p(z,w)=f_{st}(z)$$

in $U_s^p\cap U_t^p\neq\phi$. Then $\{F_{st}^p\}\in\mathrm{Z}^1(\mathfrak{U}_p,\mathfrak{O}^*)=\mathrm{B}^1(\mathfrak{U}_p,\mathfrak{O}^*)$ as $\mathrm{H}^1(\mathfrak{U}_p,\mathfrak{O}^*)\to\mathrm{H}^1(E_p,\mathfrak{O}^*)=0$ is injective. There exists $F_s^p\in\mathrm{H}^0(U_s^p,\mathfrak{O}^*)$ for any $s\in S$ such that

$$F_{st}^p=F_s^p/F_t^p$$

in $U_s^p\cap U_t^p\neq\phi$. If we put

$$f_s^p(z)=F_s^p(z,0)$$

in V_s^p for any $s\in S$, then we have

$$f_{st}=f_s^p/f_t^p$$

in $V_s^p\cap V_t^p\neq\phi$. Therefore the restriction of $\{f_{st}\}$ in any H_p is a coboundary of $\{f_s^p\}\in\mathrm{C}^0(\mathfrak{V}_p,\mathfrak{O}^*)$ for any p. From Lemma 1 $\{f_{st}\circ\lambda\}\in\mathrm{B}^1(\lambda^{-1}(\mathfrak{V}),\mathfrak{O}^*)$, $(G^\sharp,\lambda)$ being the universal covering manifold of G. Thus we have ${}^\sharp\mathrm{H}^1(G\cap H,\mathfrak{O}^*)=0$.

From Corollary of Lemma 2 and Lemmas 4 and 5 we have

PROPOSITION 1. *A domain in C^n exhausted by regular* domains is a domain of holomorphy.*

§ 4. Regular* domain with a continuous boundary.

A boundary point x^0 of an open set G in R^n is called a *continuous boundary point of G* if there exists a real-valued continuous function g of variables $x_1,x_2,\cdots,\hat{x}_j,\cdots,x_n$ in a neighbourhood V of x^0 such that

$$G\cap V=\{x=(x_1,x_2,\cdots,x_n);\ x_j=g(x_1,x_2,\cdots,\hat{x}_j,\cdots,x_n),\ x\in V\}$$

for some j. A domain (G, φ) over a complex manifold is called *pseudoconvex* at a boundary point x^0 if there exists an open neighborhood V of $\varphi(x^0)$ such that the connected component of $\varphi^{-1}(V)$ belonging to the filtre defining x^0 is holomorphically convex. A boundary point x^0 of a domain (G, φ) over a complex manifold is called a *simultaneously continuous and locally regular* boundary point* of (G, φ) if there exists a biholomorphic mapping τ of an open neighbourhood V of $\varphi(x^0)$ onto a subdomain of a complex Euclidean space such that $\tau\circ\varphi$ maps the connected component W of $\varphi^{-1}(V)$ belonging to the filtre defining x^0 biholomorphically onto $\tau(\varphi(W))$ and $\tau(\varphi(x^0))$ is a continuous boundary point of $\tau(\varphi(W))$ which is a regular* open set. If any boundary point of (G, φ) is a simultaneously continuous and locally regular* boundary point, (G, φ) is called to *have a simultaneously continuous and locally regular* boundary.*

PROPOSITION 2. *A regular* open set G in C^n is pseudoconvex at a continuous boundary point z^0 of G.*

Proof. We put $z^0=(z_1^0, z_2^0, \cdots, z_n^0)$. There exists $\varepsilon>0$ and a real-valued continuous function g of variables $z_1, z_2, \cdots, z_{j-1}, y_j, z_{j+1}, \cdots, z_n$ in a neighbourhood $V=\{z=(z_1, z_2, \cdots, z_n); |z_k-z_k^0|<\varepsilon\ (k=1, 2, \cdots, n)\}$ such that

$$\partial G\cap V=\{z; x_j=g(z_1, z_2, \cdots, z_{j-1}, y_j, z_{j+1}, \cdots, z_n), z\in V\}$$

for some j where $z_j=x_j+\sqrt{-1}y_j$. Then three cases (1), (2) and (3) may occur.

(1) $$G\cap V=\{z; x_j<g(z_1, z_2, \cdots, z_{j-1}, y_j, z_{j+1}, z_n), z\in V\}.$$

For $0\leqq t<1$ we put

$$V_t=\{z; |z_k-z_k^0|<(1-t)\varepsilon/2\ (k=1, 2, \cdots, m)\}.$$

Then we have

$$\{z; (z_1, z_2, \cdots, z_{j-1}, z_j-t\varepsilon/2, z_{j+1}, \cdots, z_n)\in V_t\}\subset V$$

for $0<t<1$. We put

$$E_t=\{z; x_j<g(z_1, z_2, \cdots, z_{j-1}, y_j, z_{j+1}, \cdots, z_n)-t\varepsilon/2, z\in V_t\}.$$

Let P be a relatively compact and simply connected polycylinder in C^n. $E_t\cap P$ is mapped onto

$$\{w; u_j<g(w_1, w_2, \cdots, w_{j-1}, v_j, w_{j+1}, \cdots, w_n), (w_1, w_2, \cdots, w_{j-1}, w_j-t\varepsilon/2, w_{j+1}, \cdots, w_n)\in V_t\cap P\}$$
$$=G\cap V\cap\{z; (z_1, z_2, \cdots, z_{j-1}, z_j-t\varepsilon/2, z_{j+1}, \cdots, z_n)\in V_t\cap P\}$$

by a biholomorphic mapping $w=(w_1, w_2, \cdots, w_n)=\gamma(z)$ defined by $w_k=z_k$ $(k\neq j)$, $w_j=z_j+t\varepsilon/2$. Since $\gamma(E_t\cap P)$ is the intersection of G and relatively compact and simply connected polycylinders, we have

$$\mathrm{H}^1(E_t\cap P, \mathfrak{O}^*)=\mathrm{H}^1(\gamma(E_t\cap P), \mathfrak{O}^*)=0.$$

Therefore E_t is a regular* open set for $0\leqq t<1$. Since E_0 is exhausted by regular* domains $\{E_t; 0<t<1\}$, $E_0=G\cap V_0$ is a domain of holomorphy from Proposition 1.

Hence G is pseudoconvex at z^0.

(2) $$G \cap V = \{z; x_j > g(z_1, z_2, \cdots, z_{j-1}, y_j, z_{j+1}, \cdots, z_n), z \in V\}.$$

In this case the situation is quite similar to the case (1).

(3) $$G \cap V = \{z; x_j \neq g(z_1, z_2, \cdots, z_{j-1}, y_j, z_{j+1}, \cdots, z_n), z \in V\}.$$

Let

$$G_1 = \{z; x_j < g(z_1, z_2, \cdots, z_{j-1}, y_j, z_{j+1}, \cdots, z_n), z \in V\}$$

and

$$G_2 = \{z; x_j > g(z_1, z_2, \cdots, z_{j-1}, y_j, z_{j+1}, \cdots, z_n), z \in V\}.$$

From the cases (1) and (2) G_1 and G_2 are pseudoconvex at z^0. Hence $G \cap V = G_1 \cup G_2$ is pseudoconvex at z^0.

§ 5. Domain with a simultaneously continuous and locally regular* boundary.

PROPOSITION 3. *A domain (G, φ) over a Stein manifold with a simultaneously continuous and locally regular* boundary is a Stein manifold.*

Proof. Let x^0 be a boundary point of (G, φ). From the assumption of our Proposition, there exists a biholomorphic mapping τ of an open neighbourhood V of $\varphi(x^0)$ onto a domain in a complex Euclidean space such that $\tau \circ \varphi$ maps the connected component W of $\varphi^{-1}(V)$ belonging to the filtre defining x^0 biholomorphically onto $\tau(\varphi(W))$ and $\tau(\varphi(x^0))$ is a continuous boundary point of $\tau(\varphi(W))$ which is a regular* open set. From Proposition 2 $\tau(\varphi(W))$ is pseudoconvex at $\tau(\varphi(x^0))$. Hence (G, φ) is pseudoconvex at x^0. Since any pseudoconvex domain over a Stein manifold is a Stein manifold from Docquier-Grauert [7], (G, φ) is a Stein manifold.

A boundary point x^0 of a domain (G, φ) over a complex manifold is called a *locally regular* boundary point* of (G, φ) if there exists a biholomorphic mapping τ of an open neighbourhood V of $\varphi(x^0)$ in a complex Euclidean space such that the image $\tau(\varphi(W))$ of the connected component W of $\varphi^{-1}(V)$ belonging to the filtre defining x^0 by the holomorphic mapping $\tau \circ \varphi$ is a regular* open set. A domain over a complex manifold is called to *have a locally regular* boundary* if its each boundary point is a locally regular* boundary point. A boundary point x^0 of a domain (G, φ) over an n-dimensional complex manifold is called a *smooth boundary point* of (G, φ) if there exists a real-valued continuously differentiable function g in an open neighbourhood V of $\varphi(x^0)$ such that $\sum_{j=1}^{2n} (\partial g/\partial t_j)^2 \neq 0$ at $\varphi(x^0)$ for real local coordinates $t_1, t_2, \cdots, t_{2n}$, φ maps the connected component W of $\varphi^{-1}(V)$ belonging to the filtre defining x^0 biholomorphically onto $\varphi(W)$ and there holds

$$\varphi(W) = \{x; g(x) < 0, x \in V\}.$$

A boundary point of a domain (G, φ) over a Stein manifold which is a smooth boundary point of (G, φ) and which is a locally regular* boundary point is a simultaneously continuous and locally regular* boundary point of (G, φ) in our sense. But a boundary point of (G, φ) which is a continuous boundary point of (G, φ) and which is a locally regular* boundary point of (G, φ) may not perhaps be a simultaneously continuous and locally regular* boundary point of (G, φ) in our sense even if we define a continuous boundary point of (G, φ) similarly. A domain over a complex manifold is called to *have a smooth boundary* if its each boundary point is a smooth boundary point. We have

COROLLARY OF PROPOSITION 3. *If a domain (G, φ) over a Stein manifold with a smooth boundary has a locally regular* boundary, then (G, φ) is a Stein manifold.*

Let G be a subdomain of a Stein manifold S with a smooth boundary. If G is not a Stein manifold, there exists a boundary point x^0 of G which is not a locally regular* boundary point of G from the above Corollary. Let V be any local coordinate neighbourhood of x^0 and τ be a biholomorphic mapping of V onto a domain in a complex Euclidean space. Then $\tau(G \cap V)$ is not a regular* open set. Hence there exists a relatively compact and simply connected polycylinder P in $\tau(V)$ such that $\mathrm{H}^1(\tau(G \cap V) \cap P, \mathfrak{O}^*) \neq 0$. Since P is a Stein manifold analytically contractible to its each point from Riemann's mapping theorem, $\tau^{-1}(P)$ is also a Stein manifold analytically contractible to its each point. We have

THEOREM 1. *Let G be a subdomain of a Stein manifold S with a smooth boundary such that $\mathrm{H}^1(G \cap D, \mathfrak{O}^*)=0$ for any subdomain D of S which is a Stein manifold analytically contractible to its each point. Then G is a Stein manifold.*

THEOREM 2. *Let G be a subdomain of a Stein manifold S with a smooth boundary. If G is not a Stein manifold, then there exists an arbitrarily small subdomain D of S which is analytically contractible to its each point such that $\mathrm{H}^1(G \cap D, \mathfrak{O}^*) \neq 0$.*

If we do not assume the smoothness of the boundary, we only have

THEOREM 3. *If a subdomain G of a Stein manifold S can be exhausted by subdomains G_p which satisfy $\mathrm{H}^1(G_p \cap D, \mathfrak{O}^*)=0$ for any subdomain D of S which is a Stein manifold analytically contractible to its each point, then G is a Stein manifold.*

Proof. Of course G_p's satisfy

$$G_p \Subset G_{p+1} \ (p \geqq 1), \qquad G = \bigcup_{p=1}^{\infty} G_p.$$

Let x^0 be a boundary point of G. We consider a biholomorphic mapping τ of an open neighbourhood V of x^0 in a complex Euclidean space C^n. There exists $\varepsilon > 0$ such that

$$Z=\{z=(z_1, z_2, \cdots, z_n);\ |z_j-z_j^0|<\varepsilon\ (j=1, 2, \cdots, n)\}\Subset\tau(V).$$

We put

$$Z_p=\{z;\ |z_j-z_j^0|<p\varepsilon/(p+1)\ (j=1, 2, \cdots, n)\}, \qquad E_p=\tau(G_p\cap V)\cap Z_p$$

for $p\geqq 1$. Let P be a relatively compact and simply connected polycylinder in C^n. Then we have

$$\tau^{-1}(E_p\cap P)=G_p\cap\tau^{-1}(Z_p\cap P).$$

Since each connected component of $\tau^{-1}(Z_p\cap P)$ is a Stein manifold analytically contractible to its each point, we have

$$\mathrm{H}^1(E_p\cap P, \mathfrak{O}^*)=\mathrm{H}^1(\tau^{-1}(E_p\cap P), \mathfrak{O}^*)=0.$$

Therefore E_p is a regular* open set for any $p\geqq 1$. Since $E=\tau(G\cap V)\cap Z$ is exhausted by regular* open sets E_p, E is a domain of holomorphy from Proposition 1. Hence G is pseudoconvex at its each boundary point and is a Stein manifold from Docquier-Grauert [7].

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COLLEGE OF GENERAL EDUCATION, NAGOYA UNIVERSITY.