# A REMARK ON THE SPACE OF CLOSED RIEMANN SURFACES WITH ORDINARY WEIERSTRASS POINTS

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It is known that the space of hyperelliptic Riemann surfaces of genus $g(\geqq 2)$ forms a $(2g-1)$-dimensional complex-analytic submanifold of the Teichmüller space $T_g$ (cf. [2], [9]). In the paper [10], Rauch generalized this result as follows: the space of closed Riemann surfaces of genus $g(\geqq 2)$ having Weierstrass points where the first non-gap values is $n(\leqq g)$ forms an $(n+2g-3)$-dimensional complex-analytic submanifold of $T_g$. For the proof he used the Garabedian deformation.

We want to discuss the same problem by making use of deformations by Beltrami differentials introduced by Bers [6] (see also [3]). We have succeeded for the case $n=g$.

We remark that a related problem has been discussed by Bers [7] by using quasi-Fuchsian groups.

**1.** We begin with the statement of our result. In the present paper we consider only closed Riemann surfaces of a given genus $g(\geqq 2)$. Let $S_0$ be such a surface fixed once for all. We denote by $\sigma$ a homotopy class of sense-preserving homeomorphisms of $S_0$ onto another $S$, and call the pair $(S, \sigma)$ a marked Riemann surface. Two marked Riemann surfaces $(S, \sigma)$ and $(S', \sigma')$ are said to be conformally equivalent if the homotopy class $\sigma'\sigma^{-1}$ contains a conformal mapping of $S$ onto $S'$. We denote by $<S, \sigma>$ the conformal equivalence class of $(S, \sigma)$, and call the set of all $<S, \sigma>$ the Teichmüller space, which will be denoted by $T_g$. For given $<S_1, \sigma_1>$ and $<S_2, \sigma_2>$, there exists only one quasiconformal mapping $f$ of $S_1$ onto $S_2$ which minimizes the maximal dilatation in the homotopy class $\sigma_2\sigma_1^{-1}$ (cf. [1], [5], [11]). We define the distance between two elements by

$$d(<S_1, \sigma_1>, <S_2, \sigma_2>)=\log K(f),$$

where $K(f)$ is the maximal dilatation of the mapping $f$. A topology on $T_g$ is induced by this metric.

Let $n$ be a positive integer and $P_0$ be a point of $S$. If no meromorphic function exists on $S$ having as its only singularity a pole of order $n$ at $P_0$, we say that $n$ is a gap value at $P_0$, or that the point $P_0$ has a gap value $n$. It is known that there exist exactly $g$ gap values at each point $P_0$. A point $P_0$ is called a Weierstrass point of $S$ if it has a gap value $n$ greater than $g$. There are only a finite number of Weierstrass points on $S$. Consequently, except a finite number of points on $S$,

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each point has gap values 1, 2, $\cdots$, $g$. A Weierstrass point with gap values 1, 2, $\cdots$, $g-1$, $g+1$ will be called an ordinary Weierstrass point.

The purpose of this paper is to prove the following theorem.

THEOREM. *The set of all the conformal equivalence classes of closed marked Riemann surfaces of genus $g(\geqq 2)$ having ordinary Weierstrass points is an open subset of the Teichmüller space $T_g$.*

**2.** By a differential of type $(p, q)$ on a Riemann surface $S$ we mean an invariant form $\lambda(z)dz^p d\bar{z}^q$ on $S$. We call a differential of type $(-1, 1)$ a Beltrami differential, a differential of type $(-1, 0)$ an inverse differential, and a differential of type $(2, 0)$ a quadratic differential. In this paper we consider only Beltrami differentials $\beta=\mu(z)d\bar{z}/dz$ with bounded $|\beta|=|\mu(z)|$. When such a Beltrami differential $\beta$ is given on $S$, we denote by $S^\beta$ the new Riemann surface which has the conformal structure on $S$ defined by the conformal metric $ds=|dz+\mu d\bar{z}|$. More precisely, over the topological space $S$, $S^\beta$ is defined by a local parameter $\zeta(z)$ which is a homeomorphic solution of the Beltrami equation $\zeta_{\bar{z}}=\zeta_z$ for each local parameter $z$ on $S$.

A Beltrami differential $\beta=\mu(z)d\bar{z}/dz$ on $S$ is called locally trivial or stationary if there exists an inverse differential $h(z)/dz$ such that $h(z)$ is continuous and

$$\mu=\frac{\partial h}{\partial \bar{z}}$$

is satisfied for each local parameter $z$ on $S$ with respect to the generalized derivative.[1] The following lemma is important.

TEICHMÜLLER'S LEMMA. *A Beltrami differential $\mu d\bar{z}/dz$ on a closed Riemann surface $S$ is locally trivial if and only if*

$$\iint_S \mu f dxdy=0$$

*for all regular quadratic differentials $fdz^2$ on $S$ (cf.* [3], [6], [11]).

The dimension of the complex factor space $B$ of all Beltrami differentials on $S$ modulo the locally trivial ones is known to be $3g-3$ (cf. [3], [6], [11]). We call a basis of $B$ a (complex) Beltrami basis on $S$. Bers introduced a complex-analytic structure in $T_g$ by the following method (cf. [6]). Let $<S, \sigma>$ be an arbitrary element of $T_g$ and $b=(\beta_1, \beta_2, \cdots, \beta_{3g-3})$ be a Beltrami basis on $S$. For complex vectors $e=(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{3g-3})\in C_{3g-3}$ sufficiently near the origin, we define the mapping

$$e \longrightarrow <S^{e\cdot b}, \tau\sigma> \in T_g,$$

where

$$e\cdot b=\varepsilon_1\beta_1+\varepsilon_2\beta_2+\cdots+\varepsilon_{3g-3}\beta_{3g-3}$$

and each $\tau$ is the homotopy class of homeomorphisms of $S$ onto $S^{e\cdot b}$ containing the identity mapping. It is shown that this mapping is a homeomorphism of a neigh-

1) As for the generalized derivative, see Bers [4].

borhood of the origin of $C_{3g-3}$ onto a neighborhood of $<S, \sigma>$, and that $e=(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{3g-3})$ are complex-analytic co-ordinates on $T_g$ (see also [3]).

**3.** Let $\theta_i=\varphi_i(z)dz$ $(i=1, 2, \cdots, g)$ be a basis of the space of all the abelian differentials of the first kind on $S$. We denote by $W(z)$ the Wronskian of functions $\varphi_1(z), \varphi_2(z), \cdots, \varphi_g(z)$ for a local parameter $z$ on $S$. It is well known that a point of $S$ is a Weierstrass point if and only if it is a zero of the differential $\Omega=W(z)dz^{g(g+1)/2}$ of type $(g(g+1)/2, 0)$. Furthermore, a point $P_0 \in S$ is an ordinary Weierstrass point if and only if $\Omega$ has a simple zero at $P_0$ (cf. [8]). For $S^{e,b}$ we define the Wronskian $W^e(\zeta)$ in the same way.

As a consequence, in order to prove our theorem, it suffices to show the following: if $W(z)$ has a simple zero at $z(P_0)$, then $W^e(\zeta)$ also has a simple zero at $\zeta(P_0)$ whenever $e=(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{3g-3})$ is sufficiently near the origin, for a suitable Beltrami basis $b=(\beta_1, \beta_2, \cdots, \beta_{3g-3})$ on $S$ and a certain local parameter $\zeta$ on $S^{e,b}$.

**4.** Let $P_0$ be an ordinary Weierstrass point of $S$ and let $U$ be a fixed parametric disk about $P_0$. We shall show that there exists a Beltrami basis $(\mu_1 d\bar{z}/dz, \mu_2 d\bar{z}/dz, \cdots, \mu_{3g-3} d\bar{z}/dz)$ on $S$ satisfying

$$\mu_j=0 \text{ in } U \ (j=1, 2, \cdots, 3g-3). \tag{1}$$

For this purpose we define an inner product in the space $Q$ of regular quadratic differentials on $S$ by setting

$$(f dz^2, g dz^2)=\iint_{S-U} \frac{f\bar{g}}{\rho}\, dxdy,$$

where $\rho|dz|$ is a metric on $S$ such that $\rho>0$. The dimension of $Q$ is $3g-3$ (cf. [11]). An orthonormal basis of $Q$ relative to this inner product will be denoted by $f_1 dz^2, f_2 dz^2, \cdots, f_{3g-3} dz^2$. By setting

$$\mu_j=\begin{cases} 0 & \text{in } U, \\ \bar{f}_j/\rho & \text{in } S-U, \end{cases}$$

we obtain Beltrami differentials $\beta_j=\mu_j d\bar{z}/dz$ $(j=1, 2, \cdots, 3g-3)$ on $S$, which are linearly independent as elements of the factor space $B$. In fact, if $c_1\beta_1+c_2\beta_2+\cdots+c_{3g-3}\beta_{3g-3}$ is locally trivial, then, by Teichmüller's lemma,

$$0=\sum_{j=1}^{3g-3} c_j \iint_S \mu_j f_i dxdy=\sum_{j=1}^{3g-3} c_j \iint_{S-U} \frac{f_i \bar{f}_j}{\rho}\, dxdy$$
$$=c_i \quad (i=1, 2, \cdots, 3g-3).$$

Therefore $(\beta_1, \beta_2, \cdots, \beta_{3g-3})$ is a Beltrami basis on $S$ satisfying (1).

Now we fix a Beltrami basis $b=(\beta_1, \beta_2, \cdots, \beta_{3g-3})$ satisfying (1) and

$$|\mu_j|\leqq k<1 \qquad (j=1, 2, \cdots, g),$$

where $\beta_j=\mu_j d\bar{z}/dz$ and $k$ is a constant. Let $A_1, B_1, A_2, B_2, \cdots, A_g, B_g$ be a canonical homology basis on $S$ which does not meet $U$. We denote by $\theta_i=\varphi_i(z)dz$ and $\theta_i^e$

$=\varphi_i^e(\zeta)d\zeta$ $(i=1, 2, \cdots, g)$ the normalized bases of abelian differentials of the first kind belonging to this homology basis on $S$ and $S^{e\cdot b}$, respectively. That is, they satisfy

$$(2) \qquad \int_{A_j}\theta_i=\delta_{ij}, \qquad \int_{A_j}\theta_i^e=\delta_{ij} \qquad (i, j=1, 2, \cdots, g).$$

Note that $A_j$, $B_j$ $(j=1, 2, \cdots, g)$ is a homology basis on both $S$ and $S^{e\cdot b}$. We fix a local parameter $z=z(\mathrm{P})$ in $U$, where $z(\mathrm{P}_0)=0$ and $z(U)=$unit disk. By the assumption, $W(z)$, the Wronskian of $\varphi_1(z), \varphi_2(z), \cdots, \varphi_g(z)$, has a simple zero at $z=0$. By condition (1) the parameter $z$ is also a local parameter on $S^{e\cdot b}$. We take this parameter $z$ as a local parameter $\zeta$ in $U\,(\subset S^{e\cdot b})$ which we mentioned at the end of Section 3.

Then the proof of our theorem is reduced to show that, if $|e|\to 0$,

$$(3) \qquad W^e(z)\to W(z) \qquad \text{uniformly in the wider sense in } |z|<1,$$

where $W^e(z)$ is the Wronskian of $\varphi_1^e(z), \varphi_2^e(z), \cdots, \varphi_g^e(z)$. Indeed, by Hurwitz' theorem, (3) implies that for an arbitrary neighborhood $U_0$ of $z=0$ $W^e(z)$ has only one simple zero in $U_0$ if $|e|$ is sufficiently small.

**5.** In order to prove (3) we shall represent $W(z)$ and $W^e(z)$ by periods of certain abelian differentials of the second kind. Let P be an arbitrary point of $U$. We denote by $\omega_{\mathrm{P},n}=\phi_{\mathrm{P},n}(z)dz$ the abelian differential of the second kind on $S$ which has, as its only singularity, a pole at P with the principal part

$$(4) \qquad (z-z(\mathrm{P}))^{-n}dz \qquad (n\text{: integer, } n\geqq 2)$$

for the previously fixed parameter $z$ in $U$, and which satisfies

$$(5) \qquad \int_{A_j}\omega_{\mathrm{P},n}=0 \qquad (j=1, 2, \cdots, g).$$

Similarly, we denote by $\omega_{\mathrm{P},n}^e=\phi_{\mathrm{P},n}^e(\zeta)d\zeta$ the abelian differential of the second kind on $S^{e\cdot b}$ which has, as its only singularity, a pole at P with the principal part (4) for the same parameter $z$ on $S^{e\cdot b}$ and satisfies

$$\int_{A_j}\omega_{\mathrm{P},n}^e=0 \qquad (j=1, 2, \cdots, g).$$

Furthermore we set

$$\pi_{\mathrm{P},n,j}=\int_{B_j}\omega_{\mathrm{P},n}, \qquad \pi_{\mathrm{P},n,j}^e=\int_{B_j}\omega_{\mathrm{P},n}^e.$$

Then, by use of Riemann's period relation, we have

$$\left.\begin{aligned}\frac{d^{n-2}\varphi_j(z)}{dz^{n-2}}&=\frac{(n-1)!}{2\pi i}\pi_{\mathrm{P},n,j},\\ \frac{d^{n-2}\varphi_j^e(z)}{dz^{n-2}}&=\frac{(n-1)!}{2\pi i}\pi_{\mathrm{P},n,j}^e\end{aligned}\right\} \qquad (j=1, 2, \cdots, g;\ n=2, 3, \cdots),$$

where $z=z(\mathrm{P})$. Hence we obtain

$$(6)\qquad W(z)=\frac{2!\,3!\cdots g!}{(2\pi i)^g}\begin{vmatrix}\pi_{\mathrm{P},2,1} & \cdots & \pi_{\mathrm{P},2,g}\\ & \cdots\cdots\cdots & \\ \pi_{\mathrm{P},g+1,1} & \cdots & \pi_{\mathrm{P},g+1,g}\end{vmatrix},$$

$$(7)\qquad W^e(z)=\frac{2!\,3!\cdots g!}{(2\pi i)^g}\begin{vmatrix}\pi^e_{\mathrm{P},2,1} & \cdots & \pi^e_{\mathrm{P},2,g}\\ & \cdots\cdots\cdots & \\ \pi^e_{\mathrm{P},g+1,1} & \cdots & \pi^e_{\mathrm{P},g+1,g}\end{vmatrix},$$

where $z=z(\mathrm{P})$.

**6.** The proof of (3) is derived from the variational formula

$$(8)\qquad \pi^e_{\mathrm{P},n,j}-\pi_{\mathrm{P},n,j}=2i\sum_{r=1}^{3g-3}\varepsilon_r\iint_S \mu_r\varphi_j\phi_{\mathrm{P},n}dxdy+o(|e|),$$

which we shall prove in the sequel.

If we set briefly

$$e\cdot b=\mu(z)d\bar{z}/dz \quad \text{i.e.} \quad \mu(z)=\sum_{r=1}^{3g-3}\varepsilon_r\mu_r(z),$$

we have

$$|\mu|\leqq c|e| \qquad (c=k\sqrt{3g-3}).$$

The identity mapping $f^e$ of $S$ onto itself is a quasiconformal mapping of $S$ onto $S^{e\cdot b}$. For arbitrary local parameters $z$ and $\zeta$ on $S$ and $S^{e\cdot b}$, respectively, the composite function $\zeta=f^e(z)$ satisfies the Beltrami equation $\zeta_{\bar{z}}=\zeta_z$. We denote by $\alpha^e_{\mathrm{P},n}$ the transplant of $\omega^e_{\mathrm{P},n}$ by the mapping $f^e$, that is,

$$\alpha^e_{\mathrm{P},n}=\phi^e_{\mathrm{P},n}(f^e(z))(f^e_z dz+f^e_{\bar{z}}d\bar{z}).$$

Then clearly

$$(9)\qquad \int_{A_j}\alpha^e_{\mathrm{P},n}=\int_{A_j}\omega^e_{\mathrm{P},n}=0 \qquad (j=1,2,\cdots,g),$$

$$(10)\qquad \int_{B_j}\alpha^e_{\mathrm{P},n}=\int_{B_j}\omega^e_{\mathrm{P},n}=\pi^e_{\mathrm{P},n,j} \qquad (j=1,2,\cdots,g).$$

We notice that the differential $\alpha^e_{\mathrm{P},n}-\omega_{\mathrm{P},n}$ has no singularities on $S$ since $f^e(z)=z$ for the previously fixed parameter $z$ on $S$ and $S^{e\cdot b}$. By using (2), (5), (9), (10) and Riemann's period relation, we obtain

$$\pi^e_{\mathrm{P},n,j}-\pi_{\mathrm{P},n,j}=\int_{B_j}\alpha^e_{\mathrm{P},n}-\omega_{\mathrm{P},n}$$

$$=\int_{B_j}\alpha^e_{\mathrm{P},n}-\omega_{\mathrm{P},n}+\theta_j-\int_{B_j}\theta_j$$

$$= \sum_{r=1}^{g}\left[\int_{A_r}\theta_j\int_{B_r}\alpha^e_{\mathrm{P},n}-\omega_{\mathrm{P},n}+\theta_j-\int_{A_r}\alpha^e_{\mathrm{P},n}-\omega_{\mathrm{P},n}+\theta_j\int_{B_r}\theta_j\right]$$

$$=\iint_S(\alpha^e_{\mathrm{P},n}-\omega_{\mathrm{P},n}+\theta_j)\wedge\theta_j$$

$$=2i\iint_S\mu\varphi_j\psi^e_{\mathrm{P},n}f^e_z dxdy$$

$$=2i\iint_S\mu\varphi_j\psi_{\mathrm{P},n}dxdy+\eta,$$

where

$$\eta=2i\iint_S\mu\varphi_j(\psi^e_{\mathrm{P},n}f^e_z-\psi_{\mathrm{P},n})dxdy.$$

By Schwarz' inequality,

$$|\eta|\leqq 2c|e|\left[\iint_S|\varphi_j|^2dxdy\right]^{1/2}\left[\iint_S|\psi^e_{\mathrm{P},n}f^e_z-\psi_{\mathrm{P},n}|^2dxdy\right]^{1/2}$$

(11)

$$\leqq 2c|e|\ ||\theta_j||\ ||\alpha^e_{\mathrm{P},n}-\omega_{\mathrm{P},n}||,$$

where $c=k\sqrt{3g-3}$. Hence it is sufficient to show

$$||\alpha^e_{\mathrm{P},n}-\omega_{\mathrm{P},n}||=O(|e|).$$

By using again (5), (9) and Riemann's period relation, we get

$$0=\iint_S(\alpha^e_{\mathrm{P},n}-\omega_{\mathrm{P},n})\wedge(\overline{\alpha^e_{\mathrm{P},n}}-\overline{\omega_{\mathrm{P},n}})$$

$$=\iint_S(|\psi^e_{\mathrm{P},n}f^e_z-\psi_{\mathrm{P},n}|^2-|\mu\psi^e_{\mathrm{P},n}f^e_z|^2)dz\wedge d\bar{z}$$

$$=-i||\alpha^e_{\mathrm{P},n}-\omega_{\mathrm{P},n}||^2+4i\iint_S|\mu\psi^e_{\mathrm{P},n}f^e_z|^2dxdy,$$

and consequently,

$$||\alpha^e_{\mathrm{P},n}-\omega_{\mathrm{P},n}||=2\left[\iint_S|\mu\psi^e_{\mathrm{P},n}f^e_z|^2dxdy\right]^{1/2}$$

$$\leqq 2c|e|\left[\iint_{S-U}|\psi^e_{\mathrm{P},n}f^e_z|^2dxdy\right]^{1/2}$$

$$\leqq 2c|e|\ ||\alpha^e_{\mathrm{P},n}||_{S-U}$$

$$\leqq 2c|e|(||\alpha^e_{\mathrm{P},n}-\omega_{\mathrm{P},n}||+||\omega_{\mathrm{P},n}||_{S-U}).$$

Therefore we obtain

(12)
$$||\alpha^e_{\mathrm{P},n}-\omega_{\mathrm{P},n}||\leqq\frac{2c|e|}{1-2c|e|}||\omega_{\mathrm{P},n}||_{S-U}=O(|e|),$$

where $c=k\sqrt{3g-3}$. Thus (8) has been proved.

**7.** By the variational formula (8) it follows that

$$\pi^{e}_{P,n,j} \longrightarrow \pi_{P,n,j} \quad \text{as} \quad |e| \longrightarrow 0. \tag{13}$$

Each integral $\iint_S \mu_r \varphi_j \phi_{P,n} dxdy$ in (8) is bounded when P varies on a closed set $F$ contained in $U$, for it is continuous with respect to P. Furthermore by (11) and (12) there exists a constant $K$, independent of $e$ and P, such that

$$|\eta| \leqq K|e|^2 ||\omega_{P,n}||_{S-U}.$$

Consequently, $\eta$ is bounded when P moves on $F$, for $||\omega_{P,n}||_{S-U}$ is bounded on $F$ by the same reason as before. Hence we can conclude by (8) that the convergence in (13) is uniform when P is restricted to a closed set in $U$. Therefore (3) follows by (6) and (7). Thus our theorem has been proved.

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