A REMARK ON THE GENERALIZATION OF HARNACK'S FIRST THEOREM

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1. In the previous papers [1], [2], we gave some uniqueness conditions for the solution of the Dirichlet problem concerning semi-linear elliptic equations of the second order

$$L(u) \equiv \sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = f(x, u, \nabla u), \tag{1. 1}$$

and under one of those uniqueness conditions, Harnack's first theorem was extended to the solution of the equation (1. 1). It was the case where the function $f(x, u, p)$ was non-decreasing with respect to u. In the present paper, we consider the case where the function $f(x, u, p)$ has not necessarily the above-mentioned property, and since Harnack's first theorem for solutions of the elliptic differential equation is really based on the *continuous dependence* of solutions upon the boundary data, we will here treat of this dependence.

Regarding the notations used in the present paper, confer the above-cited papers.

2. Let D be a bounded domain in the m-dimensional Euclidean space and let the differential operator $L(u)$ be of elliptic type in the domain D. In the present paper, we always suppose that the function $f(x, u, p)$ is defined in the domain

$$\mathfrak{D} = \{(x, u, p);\ x \in D,\ |u| < +\infty,\ |p| < +\infty\}.$$

For the sake of comparison with the later discussion, we first mention:

THEOREM 1. *Let the function $f(x, u, p)$ fulfill the following condition: For $\bar{u} > u$ and any p, q, we have*

$$f(x, \bar{u}, q) - f(x, u, p) \geqq -\alpha_0(x)(\bar{u} - u) - \alpha_1(x)\,|q - p|, \tag{2. 1}$$

where $\alpha_0(x)$ and $\alpha_1(x)$ are functions defined in D. And suppose further that there exists a function $\omega(x)$ belonging to $C^2[D] \cap C[\bar{D}]$, which is positive in $\bar{D}$ and satisfies the inequality

$$\alpha_0(x)\omega(x) + \alpha_1(x)\,|\nabla\omega(x)| + L(\omega(x)) < 0. \tag{2. 2}$$

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Then there exists at most one solution of the equation (1. 1) *which attains prescribed boundary values on the boundary* $\dot{D}$ *of* D.

Proof. Let $u_1(x)$ and $u_2(x)$ be solutions of the equation (1. 1) which attain the same boundary values. To prove the theorem, it is sufficient to show that a contradiction arises if these solutions are not identically equal with each other.

Suppose that $u_1(x) \not\equiv u_2(x)$ in D, then, without loss of generality we can assume that there exists a point $\bar{x} \in D$, such that

$$u_2(\bar{x}) - u_1(\bar{x}) > 0.$$

If we put

$$\operatorname*{Sup}_{D} \frac{u_2(x) - u_1(x)}{\omega(x)} = k(>0),$$

then there exists a point $\xi \in D$, such that

$$u_2(\xi) - u_1(\xi) = k\omega(\xi), \tag{2. 3}$$

and for any $x \in D$,

$$u_2(x) - u_1(x) \leqq k\omega(x).$$

Hence we have

$$\nabla u_2(\xi) - \nabla u_1(\xi) = k\nabla\omega(\xi) \tag{2. 4}$$

and

$$L(u_2(\xi)) - L(u_1(\xi)) \leqq kL(\omega(\xi)). \tag{2. 5}$$

On the other hand, by (2. 1), (2. 2), (2. 3) and (2. 4), we obtain

$$\begin{aligned} L(u_2(\xi)) - L(u_1(\xi)) &= f(\xi, u_2(\xi), \nabla u_2(\xi)) - f(\xi, u_1(\xi), \nabla u_1(\xi)) \\ &> -\alpha_0(\xi)(u_2(\xi) - u_1(\xi)) - \alpha_1(\xi)\,|\nabla u_2(\xi) - \nabla u_1(\xi)| \\ &= -k\{\alpha_0(\xi)\omega(\xi) + \alpha_1(\xi)\,|\nabla\omega(\xi)|\} > kL(\omega(\xi)), \end{aligned}$$

which contradicts (2. 5).

3. Next we prove the following:

LEMMA. *Let* $\alpha_0(x)$ *and* $\alpha_1(x)$ *be functions defined in the domain* D *and let* $\omega(x)$ *be a function belonging to* $C^2[D]$, *such that* $\omega(x) \geqq 0$ *in* D *and*

$$\alpha_0(x)\omega(x) + \alpha_1(x)\,|\nabla\omega(x)| + L(\omega(x)) < 0 \text{ in } D. \tag{3. 1}$$

Then we have $\omega(x) > 0$ *in* D.

Proof. We will show first that both the equalities $\omega(x)=0$ and $|\nabla\omega(x)|=0$ can not occur at the same point of D. Suppose that $\omega(x)=0$ and $|\nabla\omega(x)|=0$ hold at a point ξ of D, then we have $L(\omega(x))<0$ at the point ξ by the inequality (2. 1). Hence we see $\omega(x)<0$ at some points lying in a neighborhood of the point ξ. This fact contradicts the hypothesis that the function $\omega(x)$ is non-negative in D.

Therefore, if $\omega(x)$ vanishes at a point ξ of D, then $|\nabla\omega(x)|$ does not vanish at this point ξ. But this situation shall be shown also not to occur.

Suppose that at a point ξ of D, we have

$$\omega(\xi)=0, \qquad |\nabla\omega(\xi)|\neq 0,$$

then there exists a direction ν issuing from the point ξ, such that the differential coefficient $\partial_\nu\omega(x)$ with respect to the direction ν has the same value as $|\nabla\omega(x)|$ at the point ξ. We can therefore find out a point ζ in a neighborhood of the point ξ such that $\omega(x)<0$ holds at the point ζ, which contradicts the hypothesis that the function $\omega(x)$ is non-negative in D.

By the above reasoning, we have $\omega(x)>0$ in D.

4. For any positive number ε, we denote by $\mathfrak{F}_\varepsilon$ the family of all couples $(u_1(x), u_2(x))$ of solutions of the equation (1. 1), which satisfy the inequality

$$\overline{\lim_{x\to\dot{x}}}\,|u_2(x)-u_1(x)|\leqq\varepsilon$$

for any boundary point $\dot{x}\in\dot{D}$. Furthermore we put

$$M(u_1, u_2)=\operatorname*{Sup}_{D}|u_2(x)-u_1(x)|$$

and

$$M(\varepsilon)=\operatorname*{Sup}_{\mathfrak{F}_\varepsilon} M(u_1, u_2).$$

Then we can prove the following:

THEOREM 2. *Let the function $f(x, u, p)$ satisfy the following condition:*

$$f(x, \bar{u}, q)-f(x, u, p)\geqq-\alpha_0(x)(\bar{u}-u)-\alpha_1(x)|q-p| \tag{4. 1}$$

for any $x\in D$, and any p, q, and $\bar{u}>u$, where $\alpha_0(x)$ and $\alpha_1(x)$ are defined in D, and $\alpha_0(x)$ is bounded in D.

Suppose further that there exists a non-negative function $\omega(x)\in C^2[D]\cap C[\bar{D}]$, such that

$$\alpha_0(x)\omega(x)+\alpha_1(x)|\nabla\omega(x)|+L(\omega(x))<-\eta<0 \text{ in } D, \tag{4. 2}$$

for some positive number η.

Then, under the above-mentioned conditions, we have

(4. 3) $$M(\varepsilon)\to 0 \quad \textit{for}\ \varepsilon\to 0.$$

Proof. It is sufficient to show that a contradiction arises, if (4. 3) is false. Suppose that (4. 3) does not hold, then there exists a positive number M_0 such that, for all $\varepsilon>0$,

(4. 4) $$M(\varepsilon)>M_0>0.$$

Let ε_1 and M_1 be positive numbers such that $M_0>2\varepsilon_1$, and

(4. 5) $$\frac{M_0-2\varepsilon_1}{K}\geqq M_1>0,$$

where $K=\operatorname{Max}_{\bar{D}}\omega(x)$. Furthermore let ε_2 be a positive number such that

(4. 6) $$-2\alpha_0(x)\varepsilon+M_1\eta\geqq 0$$

for any $x\in D$ and any positive number $\varepsilon\leqq\varepsilon_2$. The existence of such a number ε_2 may be verified by the boundedness of the function $\alpha_0(x)$ in D.

We put

(4. 7) $$\varepsilon_0=\operatorname{Min}\{\varepsilon_1,\ \varepsilon_2\},$$

then, the inequality (4. 4) implies that there exists a couple $(u_1(x),\ u_2(x))\in\mathfrak{F}_{\varepsilon_0}$ such that

$$\operatorname*{Sup}_{D}|u_2(x)-u_1(x)|>M_0,$$

and without loss of generality, we can assume

$$\operatorname*{Sup}_{D}\{u_2(x)-u_1(x)\}>M_0.$$

Thus the function

$$v(x)=u_2(x)-u_1(x)-2\varepsilon_0$$

assumes positive values in D and we have

$$\varlimsup_{x\to\dot{x}} v(x)\leqq-\varepsilon_0<0$$

for any boundary point $\dot{x}\in\dot{D}$. Furthermore, by Lemma in §3, we see $\omega(x)>0$ in D.

Hence, the function $v(x)/\omega(x)$ attains the positive maximum in the interior of D, that is, there exists a point $\xi\in D$, such that

$$\frac{v(\xi)}{\omega(\xi)}=\operatorname*{Max}_{D}\frac{v(x)}{\omega(x)}=k>0.$$

We obtain therefore

(4. 8) $$\operatorname*{Max}_{D}\{v(x)-k\omega(x)\}=v(\xi)-k\omega(\xi)=0,$$

(4. 9) $$\nabla v(\xi)=k\nabla\omega(\xi),$$

(4. 10) $$L(v(\xi))\leqq kL(\omega(\xi)).$$

On the other hand, by (4. 5) and (4. 7) we see

(4. 11) $$k=\operatorname*{Max}_{D}\frac{v(x)}{\omega(x)}>\frac{M_0-2\varepsilon_0}{K}\geqq\frac{M_0-2\varepsilon_1}{K}\geqq M_1.$$

Now, by (4. 1) we have

$$\begin{aligned}L(v(\xi))&=L(u_2(\xi))-L(u_1(\xi))\\&=f(\xi,\,u_2(\xi),\,\nabla u_2(\xi))-f(\xi,\,u_1(\xi),\,\nabla u_1(\xi))\\&\geqq-\alpha_0(\xi)(u_2(\xi)-u_1(\xi))-\alpha_1(\xi)\,|\,\nabla u_2(\xi)-\nabla u_1(\xi)\,|\\&=-2\alpha_0(\xi)\varepsilon_0-\alpha_0(\xi)v(\xi)-\alpha_1(\xi)\,|\,\nabla v(\xi)\,|,\end{aligned}$$

and it follows from (4. 2), (4. 8), (4. 9) and (4. 11) that

$$\begin{aligned}L(v(\xi))&\geqq-2\alpha_0(\xi)\varepsilon_0-k\{\alpha_0(\xi)\omega(\xi)+\alpha_1(\xi)\,|\,\nabla\omega(\xi)\,|\}\\&>-2\alpha_0(\xi)\varepsilon_0+k\{\eta+L(\omega(\xi))\}\\&\geqq-2\alpha_0(\xi)\varepsilon_0+M_1\eta+kL(\omega(\xi)).\end{aligned}$$

Since $-2\alpha_0(\xi)\varepsilon_0+M_1\eta\geqq0$ by virtue of (4. 6) and (4. 7), we get

$$L(v(\xi))>kL(\omega(\xi)),$$

which contradicts (4. 10). Thus the theorem is proved completely.

REMARK: To the existence of the function $\omega(x)$ satisfying the inequality of the same sort as (2. 2) or (4. 2), a reference has been given in Nagumo's book [3], p. 134.

THEOREM 3. *Let D be a domain lying between two hyperplanes $x_i=\alpha$ and $x_i=\beta\,(-\infty<\alpha<\beta<+\infty)$. Suppose further that the function $f(x, u, p)$ satisfies the condition*

$$f(x,\,\bar{u},\,\bar{p})-f(x,\,u,\,p)\geqq-\frac{A(x)}{(x_i-\alpha)(\beta-x_i)}\,(\bar{u}-u)-\frac{B(x)}{|\,2x_i-\alpha-\beta\,|}\,|\,\bar{p}_i-p_i\,|,$$

for any $x\in D$, $p=(p_1,\,\cdots,\,p_i,\,\cdots,\,p_m)$, $\bar{p}=(p_1\,\cdots,\,\bar{p}_i,\,\cdots,\,p_m)$ and $\bar{u}>u$, where $A(x)$ and $B(x)$ are functions defined in D, such that $A(x)/(x_i-\alpha)(\beta-x_i)$ is bounded in D and

$$A(x)+B(x)+\eta<2a_{ii}(x)$$

for some positive number η.

Then we have $M(\varepsilon)\to 0$ *for* $\varepsilon\to 0$.

We can prove this theorem along the same lines of the proof of Theorem 2, by putting $\omega(x)=(x_i-\alpha)(\beta-x_i)$.

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