SOME THEOREMS IN AN EXTENDED RENEWAL THEORY, IV

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1. Consider a sequence of non-negative independent random variables X_1, $X_2, \cdots$ and let $N(t)$ be the number of sums $X_1, X_1+X_2, \cdots$ which are less than t. In other words $N(t)$ is the random variable such that

$$\sum_{k=1}^{N(t)} X_k < t \leqq \sum_{k=1}^{N(t)+1} X_k. \tag{1.1}$$

The existence of

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} E(X_k) = m, \tag{1.2}$$

with some conditions on $X_1, X_2, \cdots$, ensures

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T [H(t+h)-H(t)]dt = \frac{h}{m}, \tag{1.3}$$

where

$$H(t) = E\{N(t)\} = \sum_{n=1}^{\infty} P(X_1+\cdots+X_n<t). \tag{1.4}$$

This problem is a renewal theorem in a wide sense and has been treated by Kawata [6] and extended by the author [3], [4]. Moreover, we have as an extension of (1.3) above that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T dt \int_0^t \psi(t-u)\,dH(u) = \frac{1}{m} \int_0^{\infty} \psi(u)\,du \tag{1.5}$$

where $\psi(u)$ is a Baire function integrable over $(0, \infty)$. This fact has been demonstrated by the author [5] in a somewhat extended form. By the similar way as in Theorem 1 of [5], we can prove the following

Theorem 1. *Let X_k, $k=1,2,\cdots$, be non-negative, independent random variables having finite mean values. If there exist positive constants L and K such that*

$$E(X_k) \geqq L, \quad \mathrm{Var}\,(X_k) \leqq K \qquad \textit{for } k=1,2,\cdots$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} E(X_k) = m, \qquad m>0$$

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exists, then

$$\int_0^T dt \int_0^t \psi(t-u)\, dN(u)$$

is a random variable, i. e. a measurable function defined on the probability space and it follows that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T dt \int_0^t \psi(t-u)\, dN(u) = \frac{1}{m} \int_0^\infty \psi(u)\, du \tag{1.6}$$

with probability 1, *where* $\psi(u)$ *is a Baire function integrable over* $(0, \infty)$.

Proof. $\psi(x)$ being a continuous function,

$$\int_0^T dt \int_0^t \psi(t-u)\, dN(u)$$

is measurable, because this double integral may be considered as the limit of the Riemann sum $\sum \psi(t-u) \Delta N(u) \Delta t$. Using the transfinite induction, we can see the measurability of the integral in which $\psi(u)$ is any Baire function integrable over $(0, \infty)$. Applying Theorem 1 in [2] to the sequence X_k, $k=1, 2, \cdots$, we get

$$\lim_{T\to\infty} \frac{N(T)}{T} = \frac{1}{m} \qquad \text{(a. s.)},$$

from which (1.6) is obtained by the similar way as in Theorem 1 of [5].

In this Theorem 1, if $\psi(u)$ is the characteristic function of the semi-closed interval $(0, h]$, (1.6) becomes

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T [N(t) - N(t-h)] dt = \frac{h}{m} \qquad \text{(a. s.)}, \tag{1.7}$$

where $N(t-h) = 0$ for $t - h < 0$. So (1.6) may be considered to show an asymptotic property of $N(t+h) - N(t)$, i. e. the renewal number in $(t, t+h]$ as $t \to \infty$. In the next section, we assume the law of iterated logarithm on X_k, $k = 1, 2, \cdots$, to make Theorem 1 more complete.

2. In this section we make the following assumptions:

(2.1) X_k, $k = 1, 2, \cdots$, are non-negative independent random variables,

(2.2) $0 < m < \infty$ and $\frac{1}{n} \sum_{k=1}^n E(X_k) = m + o\left(\sqrt{\frac{\log\log n}{n}}\right) \qquad (n \to \infty)$,

(2.3) $B_n \equiv \sum_{k=1}^n \operatorname{Var}(X_k) \to \infty \qquad (n \to \infty)$,

(2.4) $X_k - E(X_k)$, $k = 1, 2, \cdots$, obey the law of iterated logarithm,

and

(2.5) $\psi(x)$ is a Baire function integrable over $(0, \infty)$ and

$$\int_x^\infty |\psi(u)|\, du = O\left(\frac{\log\log x}{x}\right) \qquad (x \to \infty).$$

In the first place, we shall prepare the following Lemma which is found as Theorem 4 in [2].

LEMMA. *Under the conditions* (2.1)–(2.4), *we have*

$$\overline{\lim_{T\to\infty}} \frac{\left| N(T) - \dfrac{T}{m} \right|}{\sqrt{2T \log\log T}} \leqq \frac{\sqrt{B}}{\sqrt{m^3}} \qquad \text{(a. s.)}, \tag{2.6}$$

where

$$B = \overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{k=1}^{n} \operatorname{Var}(X_k).$$

THEOREM 2. *Under the conditions* (2.1)–(2.5), *we have*

$$\overline{\lim_{T\to\infty}} \frac{\left| \displaystyle\int_0^T dt \int_0^t \psi(t-u)\, dN(u) - \frac{T}{m} \int_0^\infty \psi(u)\, du \right|}{\sqrt{2T \log\log T}} \leqq \frac{3\sqrt{B}}{\sqrt{m^3}} \| \psi \| + \frac{\|\psi\| + C/2}{m} \tag{2.7}$$

with probability 1, *where*

$$B = \overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{k=1}^{n} \operatorname{Var}(X_k), \quad \|\psi\| = \int_0^\infty |\psi(u)|\, du$$

and

$$C = \overline{\lim_{x\to\infty}} \frac{x}{\log\log x} \int_x^\infty |\psi(u)|\, du.$$

Proof. We have

$$\begin{aligned} &\int_0^T dt \int_0^t \psi(t-u)\, dN(u) - \frac{T}{m} \int_0^\infty \psi(u)\, du \\ &= \left[N(T) - \frac{T}{m} \right] \int_0^\infty \psi(v)\, dv - \int_0^{T - \sqrt{2T\log\log T}} dN(u) \int_{T-u}^\infty \psi(v)\, dv \\ &\quad - \int_{T-\sqrt{2T\log\log T}}^{T} dN(u) \int_{T-u}^\infty \psi(v)\, dv \\ &= P + Q + R, \qquad \text{say.} \end{aligned} \tag{2.8}$$

Then, we have

$$\overline{\lim_{T\to\infty}} \frac{|P|}{\sqrt{2T\log\log T}} \leqq \frac{\sqrt{B}}{\sqrt{m^3}} \|\psi\| \tag{2.9}$$

by (2.6), because $|P| \leqq |N(T) - T/m| \cdot \|\psi\|$.

In the next place, we have for an arbitrary small $\varepsilon > 0$ with probabitity 1 that

$$\frac{|Q|}{\sqrt{2T \log\log T}} \leqq \frac{N(T)}{T} \sqrt{\frac{T}{2\log\log T}} \int_{\sqrt{2T\log\log T}}^{\infty} |\psi(v)|\, dv$$
$$< (C+\varepsilon) \cdot \frac{N(T)}{T} \cdot \frac{\log\log\sqrt{2T\log\log T}}{2\log\log T} < \frac{C+2\varepsilon}{2} \cdot \frac{N(T)}{T}$$

for sufficiently large T. So we have

$$\overline{\lim_{T\to\infty}} \frac{|Q|}{\sqrt{2T\log\log T}} \leqq \frac{C}{2m} \qquad \text{(a. s.)}. \tag{2.10}$$

By (2.6), we have for an arbitrary small $\varepsilon > 0$ with probability 1 that

$$N(T) < \frac{T}{m} + \left(\frac{\sqrt{B}}{\sqrt{m^3}} + \varepsilon\right)\sqrt{2T\log\log T}$$

and

$$\frac{1}{m}(T - \sqrt{2T\log\log T}) - \left(\frac{\sqrt{B}}{\sqrt{m^3}} + \varepsilon\right)\sqrt{2T\log\log T} < N(T - \sqrt{2T\log\log T})$$

for sufficiently large T, so that we get with probability 1 that

$$|R| \leqq \|\psi\| [N(T) - N(T - \sqrt{2T\log\log T})]$$
$$\leqq \|\psi\| \left[\frac{1}{m} + 2\left(\frac{\sqrt{B}}{\sqrt{m^3}} + \varepsilon\right)\right]\sqrt{2T\log\log T}$$

for sufficiently large T, which implies

$$\overline{\lim_{T\to\infty}} \frac{|R|}{\sqrt{2T\log\log T}} \leqq \left(\frac{1}{m} + \frac{2\sqrt{B}}{\sqrt{m^3}}\right)\|\psi\|. \tag{2.11}$$

Summing up (2.8), (2.9), (2.10) and (2.11), we get (2.7) with probability 1.

3. In this section, we shall give some results on the process $X(t)$ defined as follows:

$$\begin{aligned} X(t) &= t - \sum_{k=1}^{N(t)} X_k \qquad &&\text{if } N(t) \geqq 1, \\ &= X_0 + t &&\text{if } N(t) = 0 \end{aligned} \tag{3.1}$$

where X_0 is a non-negative random variable. Doob [1] has shown that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T X(t)\, dt = \frac{E(X_k^2)}{2E(X_k)} \qquad \text{(a. s.)}, \tag{3.2}$$

when X_k, $k=1,2,\cdots$, are non-negative, mutually independent, identically distributed random variables.

Even in the case where X_k, $k=1,2,\cdots$, do not necessarily have the same distribution, we have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T X(t)\,dt=\frac{b}{2m}\qquad\text{(a. s.)}\tag{3.3}$$

by assuming the existence of the limits

$$m=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}E(X_k)\quad\text{and}\quad b=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}E(X_k^2)$$

with some additional conditions on X_1, X_2, $\cdots$. This fact has been noticed by the author [2]. In the following, we shall try to make (3.3) above more complete. In addition to the assumptions (2.1)–(2.4), we assume the following conditions:

(3.4) $\quad X_k^2-E(X_k^2)$, $k=1,2,\cdots$, obey the law of iterated logarithm

and

$$\frac{1}{n}\sum_{k=1}^{n}E(X_k^2)=b+o\left(\sqrt{\frac{\log\log n}{n}}\right)\qquad(n\to\infty).\tag{3.5}$$

THEOREM 3. *Under the conditions* (2.1)–(2.4), (3.4) *and* (3.5), *we have*

$$\overline{\lim_{T\to\infty}}\frac{\left|\int_0^T X(t)\,dt-\frac{b}{2m}T\right|}{\sqrt{2T\log\log T}}\leqq\frac{b\sqrt{B}}{2\sqrt{m^3}}+\frac{\sqrt{V}}{2\sqrt{m}}\tag{3.6}$$

with probability 1, *where*

$$B=\overline{\lim_{n\to\infty}}\frac{1}{n}\sum_{k=1}^{n}\operatorname{Var}(X_k)\quad\textit{and}\quad V=\overline{\lim_{n\to\infty}}\frac{1}{n}\sum_{k=1}^{n}\operatorname{Var}(X_k^2).$$

Proof. Since

$$\begin{aligned}\int_0^T X(t)\,dt&=\int_0^{X_1}(X_0+t)\,dt+\int_{X_1}^{X_1+X_2}(t-X_1)\,dt+\cdots+\int_{S_{N(T)}}^{T}(t-S_{N(T)})\,dt\\&=X_0X_1+\frac{1}{2}\sum_{k=1}^{N(T)}X_k^2+\frac{1}{2}(T-S_{N(T)})^2,\end{aligned}$$

we have

$$X_0X_1+\frac{1}{2}\sum_{k=1}^{N(T)}X_k^2<\int_0^T X(t)\,dt\leqq X_0X_1+\frac{1}{2}\sum_{k=1}^{N(T)+1}X_k^2.\tag{3.7}$$

By (3.4), we have for an arbitrary small $\varepsilon>0$ with probability 1 that

$$
\begin{aligned}
&\sum_{k=1}^{N(T)} E(X_k^2) - (1+\varepsilon)\sqrt{2V_{N(T)} \log\log V_{N(T)}} \\
(3.8)\qquad &< \sum_{k=1}^{N(T)} X_k^2 \\
&< \sum_{k=1}^{N(T)} E(X_k^2) + (1+\varepsilon)\sqrt{2V_{N(T)} \log\log V_{N(T)}}
\end{aligned}
$$

for sufficiently large T, where $V_n = \sum_{k=1}^n \mathrm{Var}\,(X_k^2)$. On the other hand, (3.5) gives

$$
(3.9)\qquad \sum_{k=1}^{N(T)} E(X_k^2) = bN(T) + o(\sqrt{N(T) \log\log N(T)}) \qquad (T \to \infty) \qquad \text{(a. s.)}
$$

which implies with (3.7) and (3.8) with probability 1 that

$$
\begin{aligned}
&\frac{b}{2}\left[N(T) - \frac{T}{m}\right] - \frac{1}{2}(1+\varepsilon)\sqrt{2V_{N(T)} \log\log V_{N(T)}} + o(\sqrt{N(T) \log\log N(T)}) \\
(3.10)\qquad &< \int_0^T X(t)\,dt - \frac{b}{2m}T \\
&< \frac{b}{2}\left[N(T) - \frac{T}{m}\right] + X_0X_1 + \frac{b}{2} + \frac{1}{2}(1+\varepsilon)\sqrt{2V_{N(T)+1} \log\log V_{N(T)+1}} \\
&\quad + o(\sqrt{N(T) \log\log N(T)})
\end{aligned}
$$

for sufficiently large T. Noticing that we have

$$
\lim_{T\to\infty} \frac{N(T) \log\log N(T)}{T \log\log T} = \frac{1}{m} \qquad \text{(a. s.)}
$$

and

$$
\overline{\lim_{T\to\infty}} \frac{V_{N(T)} \log\log V_{N(T)}}{T \log\log T} \leqq \frac{V}{m} \qquad \text{(a. s.)}
$$

by the similar method as in the proof of Theorem 4 of [2], we can show (3.6) from (3.10) and (2.6).

REMARK. Assuming the weaker condition instead (3.5) that

$$
(3.5')\qquad \frac{1}{n}\sum_{k=1}^{n} E(X_k^2) = b + O\left(\sqrt{\frac{\log\log n}{n}}\right) \qquad (n \to \infty),
$$

we can maintain Theorem 3 by only adding

$$
C_1 = \frac{1}{2}\overline{\lim_{n\to\infty}}\sqrt{\frac{n}{\log\log n}}\left|\frac{1}{n}\sum_{k=1}^{n} E(X_k^2) - b\right|
$$

to the right side of (3.6).

4. In this section, we make the following assumptions:

(4.1) $X_k,\ Y_k,\ \cdots,\ Z_k,\quad k=1,2,\cdots,$ are non-negative independent random variables,

(4.2) $0<m_x,\ m_y,\ \cdots,\ m_z<\infty$ and

$$\frac{1}{n}\sum_{k=1}^{n}E(X_k)=m_x+o\left(\sqrt{\frac{\log\log n}{n}}\right),$$
$$\frac{1}{n}\sum_{k=1}^{n}E(Y_k)=m_y+o\left(\sqrt{\frac{\log\log n}{n}}\right),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$\frac{1}{n}\sum_{k=1}^{n}E(Z_k)=m_z+o\left(\sqrt{\frac{\log\log n}{n}}\right)\qquad (n\to\infty),$$

(4.3)
$$B_{x,n}\equiv\sum_{k=1}^{n}\mathrm{Var}\,(X_k)\to\infty,$$
$$B_{y,n}\equiv\sum_{k=1}^{n}\mathrm{Var}\,(Y_k)\to\infty,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$B_{z,n}\equiv\sum_{k=1}^{n}\mathrm{Var}\,(Z_k)\to\infty\qquad (n\to\infty),$$

(4.4) $X_k-E(X_k),\ k=1,2,\cdots;\quad Y_k-E(Y_k),\ k=1,2,\cdots;\quad \cdots;$
$Z_k-E(Z_k),\ k=1,2,\cdots,$ obey the law of iterated logarithm respectively,

(4.5) $\phi(x,y,\cdots,z)$ is a non-decreasing function defined on the domain: $x>0,\ y>0,\ \cdots,\ z>0$ and satisfies Lipschitz condition, i. e. that there exist positive constants $A,\ B,\ \cdots,\ C$ such that

$$0\leqq\phi(x+h,\ y+k,\ \cdots,\ z+l)-\phi(x,y,\cdots,z)\leqq Ah+Bk+\cdots+Cl$$
for all positive $x,\ y,\ \cdots,\ z,\ h,\ k,\ \cdots,\ l$

and

(4.6) $\phi(x,y,\cdots,z)$ is positive homogeneous and there exists a positive constant γ such that

$$\phi(x,y,\cdots,z)\geqq\gamma\cdot\min\,(x,y,\cdots,z).$$

As examples of the function ϕ satisfying the conditions (4.5) and (4.6), we can give $\max(x,y,\cdots,z)$, $\min(x,y,\cdots,z)$ and $\sqrt[p]{x^p+y^p+\cdots+z^p}$ with $p\geqq 1$.

The author [3], [5] has discussed some properties of the renewal type on W_n, where

$$W_n=\phi\left(\sum_{k=1}^{n}X_k,\ \sum_{k=1}^{n}Y_k,\ \cdots,\ \sum_{k=1}^{n}Z_k\right),\tag{4.7}$$

that is, asymptotic properties of $M(t)$ defined as follows:

$$W_{M(t)} < t \leqq W_{M(t)+1}. \tag{4.8}$$

It can be shown that $M(t)$ is uniquely defined with probability 1 for each $t > 0$. We shall derive more complete properties on $M(t)$ than those in [3], [5].

THEOREM 4. *Under the conditions* (4.1)–(4.6), *we have*

$$\varlimsup_{T\to\infty} \frac{\left| M(T) - \dfrac{T}{\phi(m_x, m_y, \cdots, m_z)} \right|}{\sqrt{2T \log\log T}} \leqq \frac{A\sqrt{B_x} + B\sqrt{B_y} + \cdots + C\sqrt{B_z}}{\sqrt{\phi(m_x, m_y, \cdots, m_z)^3}} \tag{4.9}$$

with probability 1, *where*

$$B_x = \varlimsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathrm{Var}\,(X_k), \quad B_y = \varlimsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathrm{Var}\,(Y_k), \quad \cdots, \quad B_z = \varlimsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathrm{Var}\,(Z_k).$$

Proof. By (4.2)–(4.6), we have for an arbitrary small $\varepsilon > 0$ with probability 1 that

$$\begin{aligned}
& n\phi(m_x, m_y, \cdots, m_z) + o(\sqrt{n \log\log n}\,) \\
& \quad - (1+\varepsilon)(A\sqrt{2B_{x,n} \log\log B_{x,n}} + B\sqrt{2B_{y,n} \log\log B_{y,n}} + \cdots \\
& \qquad + C\sqrt{2B_{z,n} \log\log B_{z,n}}) \\
& < W_n \\
& < n\phi(m_x, m_y, \cdots, m_z) + o(\sqrt{n \log\log n}\,) \\
& \quad + (1+\varepsilon)(A\sqrt{2B_{x,n} \log\log B_{x,n}} + B\sqrt{2B_{y,n} \log\log B_{y,n}} + \cdots \\
& \quad + C\sqrt{2B_{z,n} \log\log B_{z,n}})
\end{aligned}$$

for sufficiently large n. On the other hand, we have

$$\lim_{T\to\infty} \frac{M(T)}{T} = \frac{1}{\phi(m_x, m_y, \cdots, m_z)} \qquad \text{(a. s.)}.$$

(See Theorem 3 in [3].) So, using the similar method as in Theorem 4 of [2], we have (4.9) which was to be proved.

COROLLARY. *Under the conditions* (4.1)–(4.6) *and* (2.5), *we have*

$$\begin{aligned}
& \varlimsup_{T\to\infty} \frac{\left| \displaystyle\int_0^T dt \int_0^t \psi(t-u)\, dM(u) - \frac{T}{\phi(m_x, m_y, \cdots, m_z)} \int_0^\infty \psi(u)\, du \right|}{\sqrt{2T \log\log T}} \\
& \leqq \frac{3(A\sqrt{B_x} + B\sqrt{B_y} + \cdots + C\sqrt{B_z})}{\sqrt{\phi(m_x, m_y, \cdots, m_z)^3}} \cdot \| \psi \| \\
& \quad + \frac{1}{m}\left(\| \psi \| + \frac{1}{2} \varlimsup_{x\to\infty} \frac{x}{\log\log x} \int_x^\infty |\psi(u)|\, du \right) \qquad \text{(a. s.)}.
\end{aligned} \tag{4.10}$$

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