FREDHOLM EIGEN VALUE PROBLEM FOR GENERAL DOMAINS

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1. Formulation of the problem.

Let D be a general planar domain and $\{D_n\}$ be its exhaustion in the usual sense. Let $L_2(D_n)$ be a class of single-valued, square integrable, analytic functions having a single-valued indefinite integral in D_n. In $L_2(D_n)$ we shall, as usual, introduce the notion of the inner product $(\varphi, \psi)_{D_n}$ by an integral

$$\iint_{D_n} \varphi(z)\overline{\psi(z)}\, d\tau_z,$$

then $L_2(D_n)$ forms a complete Hilbert space. In this space $L_2(D_n)$, there are the so-called reproducing kernel $K_n(z, \bar{u})$ and its adjoint l-kernel $l_n(z, u)$ which satisfy the following identities: for any $f(z) \in L_2(D_n)$,

$$(f(z), K_n(z, \bar{u}))_{D_n} = f(u), \qquad K_n(z, \bar{u}) = \overline{K_n(u, \bar{z})}$$

and

$$l_n(z, u) = l_n(u, z), \qquad (l_n(z, u), l_n(u, w))_{D_n} = K_n(z, \overline{w}) - \Gamma_n(z, \overline{w}),$$

$$\Gamma_n(z, \overline{w}) = \frac{1}{\pi^2}\iint_{D_n^c} \frac{d\tau_\zeta}{(\zeta - z)^2\overline{(\zeta - w)^2}},$$

where D_n^c denotes the complementary set of D_n. The kernels $K_n(z, \bar{u})$, $\Gamma_n(z, \bar{u})$ and $K_n(z, \bar{u}) - \Gamma_n(z, \bar{u})$ are all positive definite and hermitian. For these, see [2] and [3]. $K_n(z, \bar{u})$ and $l_n(z, u)$ converge strongly and hence uniformly in the wider sense to $K(z, \bar{u})$ and $l(z, u)$, respectively, when n tends to the infinity. For these, see [5] and [8]. Therefore we have the corresponding identities:

$$(f(z), K(z, \bar{u}))_D = f(u)$$

for any $f(z) \in L_2(D)$ and

$$(l(z, u), l(u, w))_D = K(z, \overline{w}) - \Gamma(z, \overline{w}), \qquad l(z, u) = l(u, z),$$

where we put

$$\Gamma(z, \overline{w}) = \frac{1}{\pi^2}\iint_{D^c} \frac{d\tau_\zeta}{(\zeta - z)^2\overline{(\zeta - w)^2}} = \lim_{n\to\infty} \Gamma_n(z, \overline{w}).$$

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And $\Gamma(z, \bar{w})$ belongs to the class $L_2(D)$. In fact, we have

$$0 \leqq \iint_{D_n} |\Gamma_n(z, \bar{w})|^2 d\tau_z \leqq K_n(w, \bar{w})$$

for any n by the eigenfunction expansion of $\Gamma_n(z, \bar{w})$ and a fact that each eigen value $\lambda_\nu^{(n)2}$ of the Fredholm eigen value problem

$$\lambda_\nu^{(n)2}(\varphi_\nu(u), K_n(u, \bar{z}) - \Gamma_n(u, \bar{z}))_{D_n} = \varphi_\nu(z), \qquad \varphi_\nu(z) \in L_2(D_n)$$

for D_n is greater than 1. For these, see [2], [3], [6] and [7]. By Fatou's theorem we have

$$0 \leqq \iint_D |\Gamma(z, \bar{w})|^2 d\tau_z \leqq \overline{\lim_{n\to\infty}} \iint_{D_n} |\Gamma_n(z, \bar{w})|^2 d\tau_z \leqq K(w, \bar{w}),$$

which shows that $\Gamma(z, \bar{w}) \in L_2(D)$.

Evidently the kernels $K(z, \bar{u})$, $\Gamma(z, \bar{u})$ and $K(z, \bar{u}) - \Gamma(z, \bar{u})$ are all hermitian positive definite. We shall now consider the Fredholm eigen value problem *for D defined as follows: To seek for any constant ρ and the corresponding* function $\varphi(z)$ satisfying a homogeneous integral equation of the Fredholm type

$$(1) \qquad (\varphi(u), K(u, \bar{z}) - \Gamma(u, \bar{z})) = \rho^2 \varphi(z).$$

When ρ^2 and $\varphi(z)$ satisfy the equation (1), then we call φ the eigenfunction to *a spectrum ρ^2 or an eigen value $1/\rho^2$. And any non-trivial eigenfunction can* be normalized by the normalization

$$\|\varphi\|_D{}^2 = (\varphi(z), \varphi(z))_D = 1.$$

Let γ be a transformation of $L_2(D)$ into $L_2(D)$ defined by the left hand side of the equation (1). This transformation γ is hermitian self-adjoint and positive unless the kernel $K(z, \bar{u}) - \Gamma(z, \bar{u})$ vanishes identically. Moreover γ satisfies the half-boundedness

$$(\gamma\varphi, \varphi) \leqq (\varphi, \varphi)$$

for any $\varphi \in L_2(D)$, which is obtained by the reproducing property of K and the positive definiteness of the kernel $\Gamma(z, \bar{u})$. Therefore by Neumann's theory of hermitian operators in Hilbert space we have a unique spectral decomposition

$$\gamma = \int_{-0}^{1} \rho^2 dE(\rho),$$

where $E(\rho)$ is a resolution of the identity corresponding uniquely to the γ.

If there are only the point spectra, then we have an orthonormal complete system $\{\varphi_\nu\}$ of the eigenfunctions of (1) such that for any element $\varphi \in L_2(D)$

$$\varphi(z)=\sum_{\nu=1}^{\infty}a_\nu\varphi_\nu(z),\qquad a_\nu=(\varphi,\varphi_\nu).$$

Evidently we have

$$K(z,\overline{w})=\sum_{\nu=1}^{\infty}\varphi_\nu(z)\overline{\varphi_\nu(w)}$$

by its reproducing property. Let $\lambda_\nu^2=1/\rho_\nu^2$ be the eigen value for $\varphi_\nu(z)$, then we have

$$\Gamma(z,\overline{w})=\sum_{\nu=1}^{\infty}\left(1-\frac{1}{\lambda_\nu^2}\right)\varphi_\nu(z)\overline{\varphi_\nu(w)}.$$

These relations are formally equivalent to those in [2]. However we can recognize that several differences lie between theirs and ours. For example, the first eigen value λ_1^2 may be equal 1 in our case. And secondly, the sum

$$\sum_{\nu=1}^{\infty}\frac{1}{\lambda_\nu^4}$$

does not converge in our case. A simple example illustrating these phenomena is a domain D excluding a straight line segment $[-2, 2]$ from the whole complex plane. For this domain D we have

$$\Gamma(z,\overline{w})\equiv 0,\qquad K(z,\overline{w})=\sum_{\nu=1}^{\infty}\psi_\nu(z)\overline{\psi_\nu(w)},\qquad \psi_\nu(z)=i\sqrt{\frac{\nu}{\pi}}\,z^{-\nu+1}(z^2-1)^{-1},$$

which shows that all the eigen values are equal to 1. In Bergman-Schiffer's case [2], the above domain D is excluded by their analyticity assumption for the boundary curves.

2. Fredholm eigen values and the class $N_{\mathfrak{D}}$.

Let us now define a notion of the Fredholm null-set. Let E be the complementary closed set of D, that is, $E=D^c$.

DEFINITION. $E\in N_F$ means that all spectra of the Fredholm eigen value problem (1) for D concentrate on any non-negative number.

THEOREM 1. $$N_F\equiv N_{\mathfrak{D}},$$
where $E\in N_{\mathfrak{D}}$ *means* $D\in O_{AD}$.

Proof. Assume that $E\in N_{\mathfrak{D}}$. Any function $\varphi(z)\in L_2(D)$ and its indefinite integral $\Psi(z)$ can be continued analytically onto E, and hence $\Psi(z)\equiv$ const. or equivalently $\varphi\equiv 0$. Thus the equation (1) is satisfied by any real non-negative number, which shows $E\in N_F$. Conversely we assume that $E\in N_F$. Let a be a real non-negative number on which all spectra concentrate. Then there is an orthonormal complete system $\{\varphi_\nu\}$ of eigenfunctions of the equation (1). And hence we have

$$K(z, \overline{w}) - \Gamma(z, \overline{w}) = a^2 \sum_{\nu=1}^{\infty} \varphi_\nu(z)\overline{\varphi_\nu(w)}$$

and

$$\Gamma(z, \overline{w}) = (1 - a^2) \sum_{\nu=1}^{\infty} \varphi_\nu(z)\overline{\varphi_\nu(w)}.$$

However, since a is arbitrary, $\Gamma(z, \overline{w}) \equiv 0$ putting $a = 1$ and hence $K(z, \overline{w}) \equiv 0$ putting $a = 0$. On the other hand, it is well known that $K(z, \overline{z}) = 0$ is equivalent to $E \in N_{\mathfrak{D}}$ [1]. Thus $E \in N_F$ implies that $E \in N_{\mathfrak{D}}$.

THEOREM 2. *If all the spectra are equal to zero for D and the two-dimensional measure of E is equal to zero, that is, $m(E) = 0$, then $E \in N_{\mathfrak{D}}$.*

Proof. Since all the spectra are concentrated at zero, we have

$$K(z, \overline{w}) \equiv \Gamma(z, \overline{w}).$$

And $m(E) = 0$ implies $\Gamma(z, \overline{w}) \equiv 0$, whence follows $K(z, \overline{w}) \equiv 0$, that is, $E \in N_{\mathfrak{D}}$.

In the theorem 2 the assumption $m(E) = 0$ cannot be excluded, since we have $l(z, w) \equiv 0$ for the exterior of the unit circle, which does not belong to the class $N_{\mathfrak{D}}$.

LEMMA. *Let D' be a domain and $K_{D'}(z, \overline{w})$ be the reproducing kernel of $L_2(D')$. Assume that $K_{D'}(z, \overline{w})$ has the local expansion*

$$\sum_{\mu,\nu=0}^{\infty} k_{\mu\nu}(z - z_0)^\mu \overline{(w - z_0)}^\nu$$

around (z_0, z_0) and $\varphi(z)$ is an analytic function around z_0 having the local expansion

$$\sum_{\mu=1}^{\infty} \mu c_\mu (z - z_0)^{\mu-1}.$$

If there holds a system of inequalities

$$\left| \sum_{\mu=1}^{N} \mu c_\mu x_\mu \right|^2 \leqq M \sum_{\mu,\nu=0}^{N-1} k_{\mu\nu} x_{\nu+1} \overline{x}_{\mu+1}$$

for any complex number x_μ and any integer N, then $\varphi(z) \in L_2(D')$, and vice versa.

Proof. This lemma has already been proved in our previous paper [4] in a somewhat restricted case, that is, in a case of finitely connected domain D'

with analytic boundaries. Since, however, the proof carried previously has been quite formal, we can extend our lemma to the general case.

THEOREM 3. *Let U be a circular disc contained in D. If all the spectra of the equation* (1) *for the domain $D-U$ are concentrated at zero and the two-dimensional measure $m(E)$ of E is equal to zero, then $E \in N_{\mathfrak{D}}$.*

Proof. Since all the spectra are concentrated at zero, we can choose an orthonormal complete system $\{\varphi_\nu\}$ of eigenfunctions of the problem (1) for $D-U$. Therefore we have $\gamma\varphi \equiv 0$ for any $\varphi \in L_2(D-U)$, that is, $K_{D-U}(z, \overline{w}) - \Gamma_{D-U}(z, \overline{w})$ is orthogonal to the space $L_2(D-U)$. This implies an identity

$$K_{D-U}(z, \overline{w}) \equiv \Gamma_{D-U}(z, \overline{w}).$$

However Γ-term is additive by its definition, that is,

$$\Gamma_{D-U}(z, \overline{w}) = \Gamma_D(z, \overline{w}) + \Gamma_{U^c}(z, \overline{w}).$$

On the other hand, it is well known that $l_{U^c}(z, w) \equiv 0$ and hence

$$K_{U^c}(z, \overline{w}) \equiv \Gamma_{U^c}(z, \overline{w}).$$

Since $m(E) = 0$, we have

$$\Gamma_D(z, \overline{w}) \equiv 0$$

by its definition. Therefore we have

$$K_{D-U}(z, \overline{w}) \equiv K_{U^c}(z, \overline{w}).$$

Let $\varphi(z)$ be any element of $L_2(D-U)$, then a system of inequalities

$$\left|\sum_{\mu=1}^{N} \mu c_\mu x_\mu\right|^2 \leqq M \sum_{\mu,\nu=0}^{N-1} k_{\mu\nu} x_{\nu+1} \overline{x}_{\mu+1}, \qquad M = \|\varphi\|_{D-U}^2,$$

holds for any integer N and any complex number x_ν, where we put

$$K_{D-U}(z, \overline{w}) = \sum_{\mu,\nu=0}^{\infty} k_{\mu\nu}(z-z_0)^\mu \overline{(w-z_0)^\nu}$$

and

$$\varphi(z) = \sum_{\mu=1}^{\infty} \mu c_\mu (z-z_0)^{\mu-1}.$$

By the equality $K_{U^c}(z, \overline{w}) \equiv K_{D-U}(z, \overline{w})$, we have the same local expansion of $K_{U^c}(z, \overline{w})$ as that of $K_{D-U}(z, \overline{w})$. This implies that $\varphi(z) \in L_2(U^c)$, that is, $\varphi(z)$ can be continued analytically onto E, which shows that $E \in N_{\mathfrak{D}}$.

THEOREM 4. *If $E \notin N_{\mathfrak{D}}$ and all the spectra are concentrated at* 1, *then $m(E) = 0$. Conversely, if $E \notin N_{\mathfrak{D}}$ and $m(E) = 0$, then all the eigen values* *are equal to* 1, *or all the spectra are equal to* 1. *In other words, if* $m(E) = 0$ *and if there is at least one spectrum less than* 1, *then* $E \in N_{\mathfrak{D}}$.

Proof. This theorem 4 may be regarded as a precision of theorem 2. By the assumption, we have

$$K(z, \bar{w}) = \sum_{\nu=1}^{\infty} \varphi_\nu(z)\overline{\varphi_\nu(w)} = K(z, \bar{w}) - \Gamma(z, \bar{w})$$

for an orthonormal complete system $\{\varphi_\nu\}$ of eigenfunctions. This implies that $\Gamma(z, \bar{w}) \equiv 0$, that is, $m(E) = 0$. Let φ be an eigenfunction corresponding to a real number $\lambda^2 = 1/\rho^2$, then we have $\lambda^2 \gamma \varphi = \varphi$. From this we have

$$\|\varphi\|^2 = \lambda^2(\varphi(u), (\varphi(w), K(w, \bar{u}) - \Gamma(w, \bar{u}))_D)_D.$$

By $m(E) = 0$, we have $\Gamma(w, \bar{u}) = 0$, and hence

$$\|\varphi\|^2 = \lambda^2(\varphi(u), (\varphi(w), K(w, \bar{u}))_D)_D = \lambda^2\|\varphi\|^2,$$

by the reproducing property of the kernel K. This implies the desired result $\lambda^2 = 1$. By $m(E) = 0$, we have that the $\gamma\varphi$ coincides with $(\varphi(u), K(u, \bar{z}))_D$. Thus we have $E(\rho) \equiv 0$ for $\rho < 1$ and $\equiv I$ for $\rho \geqq 1$ in the spectral decomposition of the transformation γ.

We can consider another Fredholm eigen value problem: Let $\widehat{K}(z, \bar{u})$ be the Bergman kernel function in the class $\mathfrak{L}_2(D)$, whose elements are all single-valued analytic functions square integrable on D. Then the problem to be considered concerns with the equation

$$\rho^2\varphi(u) = (\varphi(z), \widehat{K}(z, \bar{u}) - \Gamma(z, \bar{u}))_D,$$

where $\varphi(z)$ belongs to the class $\mathfrak{L}_2(D)$. This problem contains the earlier problem as its part and leads to another Fredholm null-set corresponding to a class O_{HD}. Details are omitted here.

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Added in proof. Recently, Mr. N. Suita has pointed out that our Theorem 3 can be improved. In fact, it is shown that our assumption with regard to the two-dimensional measure of E may be eliminated.