ON THE HAHN-BANACH TYPE THEOREM AND THE JORDAN DECOMPOSITION OF MODULE LINEAR MAPPING OVER SOME OPERATOR ALGEBRAS

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Introduction.

In [6], Nachbin has shown the real Hahn-Banach extension property of $C_R(\Omega)$, the space of all real valued continuous functions over a compact stonean space Ω, and recently Hasumi has proved the generalization of this result to complex case in [4]. While $C(\Omega)$ is considered, not only as a Banach space, but as a commutative algebra, then the extension problem of a module linear mapping over $C(\Omega)$ comes into our consideration. The same problem has been treated by Nakai in [7] which was independently presented from ours.

On the other hand, Takeda and Grothendieck have shown the Jordan decomposition of self-adjoint linear functional on an operator algebra corresponding to the Jordan decomposition of real Radon measure on a locally compact space in [3] and [9], respectively.

In the present note we shall show the extension property of $C(\Omega)$ for module linear mappings over $C(\Omega)$ and the generalization of Takeda-Grothendieck's result for self-adjoint module linear mappings over $C(\Omega)$.

1. Let $\boldsymbol{M}$ be a C^*-algebra, $\boldsymbol{M}^*$ the conjugate space of $\boldsymbol{M}$ and $\boldsymbol{M}^{**}$ the second conjugate space of $\boldsymbol{M}$. If π is a $*$-representation of $\boldsymbol{M}$ on a Hilbert space H, then π is uniquely extended to the metric homomorphism $\tilde{\pi}$ from $\boldsymbol{M}^{**}$ onto the weak closure of $\pi(\boldsymbol{M})$ which is continuous for $\sigma(\boldsymbol{M}^{**}, \boldsymbol{M}^*)$-topology and σ-weak topology of the weak closure of $\pi(\boldsymbol{M})$. Since $\boldsymbol{M}^*$ is linearly spanned by the positive part of $\boldsymbol{M}^*$ by Takeda-Grothendieck Theorem (cf. [3] and [9]), there exists the unique W^*-algebra such that it is isometric to $\boldsymbol{M}^{**}$ and its σ-weak topology coincides with $\sigma(\boldsymbol{M}^{**}, \boldsymbol{M}^*)$-topology. Therefore this W^*-algebra is called the universal enveloping algebra of $\boldsymbol{M}$ and denoted by $\tilde{\boldsymbol{M}}$ in the following (cf. [10]).

Let E be a normed linear space, it is called a *normed left* (resp. *right*) *$\boldsymbol{M}$-module* if the following conditions are satisfied:

(i) E is an algebraic left (resp. right) $\boldsymbol{M}$-module,

(ii) For every $a \in \boldsymbol{M}$ and $x \in E$

$$\|ax\| \leqq \|a\| \|x\| \qquad (\text{resp. } \|xa\| \leqq \|a\| \|x\|).$$

If E is a two-sided normed $\boldsymbol{M}$-module, we call E a *normed $\boldsymbol{M}$-module* simply.

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In the following we assume that E is a normed $\boldsymbol{M}$-module.

For any $a \in \boldsymbol{M}$ we define an operator L_a (resp. R_a) on $\boldsymbol{M}^*$ as follows:

$$\langle b, L_a\varphi\rangle = \langle ab, \varphi\rangle \qquad (\text{resp. } \langle b, R_a\varphi\rangle = \langle ba, \varphi\rangle)$$

for all $b \in \boldsymbol{M}$ and $\varphi \in \boldsymbol{M}^*$, where the inner product $\langle x, \varphi\rangle$ is the value of φ at x. Similarly an operator $\mathfrak{L}_a$ (resp. $\mathfrak{R}_a$) is defined on the conjugate space E^* of a normed $\boldsymbol{M}$-module E for every $a \in \boldsymbol{M}$ as

$$\langle x, \mathfrak{L}_a\varphi\rangle = \langle ax, \varphi\rangle \qquad (\text{resp. } \langle x, \mathfrak{R}_a\varphi\rangle = \langle xa, \varphi\rangle)$$

for all $x \in E$ and $\varphi \in E^*$. Then the following properties are easily verified:

$$\begin{aligned} &\mathfrak{L}_{(\lambda a+\mu b)} = \lambda\mathfrak{L}_a + \mu\mathfrak{L}_b && (\text{resp. } \mathfrak{R}_{(\lambda a+\mu b)} = \lambda\mathfrak{R}_a + \mu\mathfrak{R}_b),\\ &\mathfrak{L}_{(ab)} = \mathfrak{L}_b\mathfrak{L}_a && (\text{resp. } \mathfrak{R}_{(ab)} = \mathfrak{R}_a\mathfrak{R}_b),\\ &\mathfrak{L}_a\mathfrak{R}_b = \mathfrak{R}_b\mathfrak{L}_a && \end{aligned}$$

for all $a, b \in \boldsymbol{M}$ and complex numbers λ, μ.

Next we define an element $\omega_l(x, \varphi)$ (resp. $\omega_r(x, \varphi)$) of $\boldsymbol{M}^*$ for every $x \in E^{**}$ and $\varphi \in E^*$ as follows:

$$\langle a, \omega_l(x, \varphi)\rangle = \langle x, \mathfrak{L}_a\varphi\rangle \qquad (\text{resp. } \langle a, \omega_r(x, \varphi)\rangle = \langle x, \mathfrak{R}_a\varphi\rangle)$$

for all $a \in \boldsymbol{M}$. Then one can easily verify that the mapping $(x, \varphi) \to \omega_l(x, \varphi)$ (resp. $\omega_r(x, \varphi)$) is bilinear on $E^{**} \times E^*$ and satisfies the condition

$$\|\omega_l(x, \varphi)\| \leqq \|x\|\|\varphi\| \qquad (\text{resp. } \|\omega_r(x, \varphi)\| \leqq \|x\|\|\varphi\|).$$

For any $a \in \tilde{\boldsymbol{M}}$ and $x \in E^{**}$ a functional $\langle a, \omega_l(x, \varphi)\rangle$ of E^* determines the unique element of E^{**} which is denoted by $a \cdot x$.

Lemma 1. *E^{**} is a normed left $\tilde{\boldsymbol{M}}$-module with respect to the above product.*

Proof. For every $a \in \boldsymbol{M}$ and $x \in E^{**}$ we have easily

$$\|a \cdot x\| \leqq \|a\|\|x\|.$$

Since $\omega_l(x, \varphi)$ is binear, we get

$$\begin{aligned} (\lambda a + \mu b)\cdot x &= \lambda a \cdot x + \mu b \cdot x,\\ a \cdot (\lambda x + \mu y) &= \lambda a \cdot x + \mu b \cdot y \end{aligned}$$

for all $a, b \in \boldsymbol{M}$, $x, y \in E^{**}$ and complex numbers λ, μ. If a and b belong to $\boldsymbol{M}$, then we have

$$(ab)\cdot x = a \cdot (b \cdot x)$$

for all $x \in E^{**}$. In fact, we have

$$\begin{aligned}\langle (ab)\cdot x, \varphi\rangle &= \langle ab, \omega_l(x, \varphi)\rangle = \langle x, \mathfrak{L}_{ab}\varphi\rangle = \langle x, \mathfrak{L}_b\mathfrak{L}_a\varphi\rangle = \langle b, \omega_l(x, \mathfrak{L}_a\varphi)\rangle \\ &= \langle b\cdot x, \mathfrak{L}_a\varphi\rangle = \langle a, \omega_l(b\cdot x, \varphi)\rangle = \langle a\cdot(b\cdot x), \varphi\rangle\end{aligned}$$

for all $\varphi\in E^*$. The mapping $a\to a\cdot x$ from $\tilde{\boldsymbol{M}}$ into E^{**} is continuous for $\sigma(\tilde{\boldsymbol{M}}, \boldsymbol{M}^*)$ and $\sigma(E^{**}, E^*)$-topologies because if $\{a_\alpha\}$ is a directed sequence of $\tilde{\boldsymbol{M}}$ converging to $a\in\tilde{\boldsymbol{M}}$ for $\sigma(\tilde{\boldsymbol{M}}, \boldsymbol{M}^*)$-topology we have

$$\lim_\alpha \langle a_\alpha\cdot x, \varphi\rangle = \lim_\alpha \langle a_\alpha, \omega_l(x, \varphi)\rangle = \langle a, \omega_l(x, \varphi)\rangle = \langle a\cdot x, \varphi\rangle$$

for all $\varphi\in E^*$. Hence we have $(ab)\cdot x = a\cdot(b\cdot x)$ for all $a, b\in\tilde{\boldsymbol{M}}$ and $x\in E^{**}$. This concludes the proof.

For any $a\in\tilde{\boldsymbol{M}}$ a functional $(x, \varphi)\to\langle a, \omega_r(x, \varphi)\rangle$ is a bounded bilinear one of $E\times E^*$, which determines the bounded operator $\mathfrak{R}_a$ on E^* such as

$$\langle a, \omega_r(x, \varphi)\rangle = \langle x, \mathfrak{R}_a\varphi\rangle$$

for all $x\in E$ and $\varphi\in E^*$. Putting ${}^t\mathfrak{R}_a x = x\circ a$ for all $x\in E^{**}$ and $a\in\tilde{\boldsymbol{M}}$, where ${}^t\mathfrak{R}_a$ is the transpose of $\mathfrak{R}_a$, we have

LEMMA 2. *E^{**} is a normed right $\tilde{\boldsymbol{M}}$-module with respect to the above product.*[1]

Proof. It suffices only to prove

$$x\circ(ab) = (x\circ a)\circ b$$

for all $a, b\in\tilde{\boldsymbol{M}}$ and $x\in E^{**}$. Clearly $x\circ(ab) = (x\circ a)\circ b$ for $a, b\in\boldsymbol{M}$ and $x\in E$. If x belongs to E, then the mapping $a\to x\circ a$ is continuous for $\sigma(\tilde{\boldsymbol{M}}, \boldsymbol{M}^*)$- and $\sigma(E^{**}, E^*)$-topologies. For, if $\{a_\alpha\}$ is a directed sequence of $\tilde{\boldsymbol{M}}$ converging to $a\in\tilde{\boldsymbol{M}}$ for $\sigma(\tilde{\boldsymbol{M}}, \boldsymbol{M}^*)$-topology, then we have

$$\begin{aligned}\lim_\alpha \langle x\circ a_\alpha, \varphi\rangle &= \lim_\alpha \langle x, \mathfrak{R}_{a_\alpha}\varphi\rangle = \lim_\alpha \langle a_\alpha, \omega_r(x, \varphi)\rangle \\ &= \langle a, \omega_r(x, \varphi)\rangle = \langle x, \mathfrak{R}_a\varphi\rangle = \langle x\circ a, \varphi\rangle\end{aligned}$$

for all $\varphi\in E^*$. Hence we get $x\circ(ab) = (x\circ a)\circ b$ for $x\in E$ and $a, b\in\tilde{\boldsymbol{M}}$. Moreover, the mapping $x\to x\circ a$ is $\sigma(E^{**}, E^*)$-continuous for $a\in\tilde{\boldsymbol{M}}$ because, if $\{x_\alpha\}$ is a directed sequence of E^{**} converging to $x\in E^{**}$ for $\sigma(E^{**}, E^*)$-topology, we have

$$\lim_\alpha \langle x_\alpha\circ a, \varphi\rangle = \lim_\alpha \langle x_\alpha, \mathfrak{R}_a\varphi\rangle = \langle x, \mathfrak{R}_a\varphi\rangle = \langle x\circ a, \varphi\rangle$$

for all $\varphi\in E^*$. Therefore we get $x\circ(ab) = (x\circ a)\circ b$ for all $a, b\in\tilde{\boldsymbol{M}}$ and $x\in E^{**}$. This concludes the proof.

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1) In Lemma 1 and Lemma 2, the left and right products are defined non-symmetrically, but this is not avoidable in order that we shall show below $(a\cdot x)\circ b = a\cdot(x\circ b)$.

By these lemmas we see that the second conjugate space E^{**} of a normed $\boldsymbol{M}$-module E is a normed left and right $\tilde{\boldsymbol{M}}$-module. Moreover, we have

THEOREM 1. *If E is a normed $\boldsymbol{M}$-module, then the second conjugate space E^{**} of E is a normed $\tilde{\boldsymbol{M}}$-module with respect to the products in Lemma 1 and Lemma 2.*

Proof. It suffices only to prove

$$(a\cdot x)\circ b = a\cdot(x\circ b)$$

for $a, b \in \tilde{\boldsymbol{M}}$ and $x \in E^{**}$. Suppose a belonging to $\boldsymbol{M}$, then the mapping $x \to a\cdot x$ is $\sigma(E^{**}, E^*)$-continuous for, if $\{x_\alpha\}$ is a directed sequence of E^{**} converging to x for $\sigma(E^{**}, E^*)$-topology, we have

$$\begin{aligned}\lim_\alpha \langle a\cdot x_\alpha, \varphi\rangle &= \lim_\alpha \langle a, \omega_l(x_\alpha, \varphi)\rangle = \lim_\alpha \langle x_\alpha, \mathfrak{L}_a\varphi\rangle \\ &= \langle x, \mathfrak{L}_a\varphi\rangle = \langle a, \omega_l(x, \varphi)\rangle = \langle a\cdot x, \varphi\rangle\end{aligned}$$

for all $\varphi \in E^*$. Therefore we get $(a\cdot x)\circ b = a\cdot(x\circ b)$ for all $a, b \in \boldsymbol{M}$ and $x \in E^{**}$.

On the other hand, the mapping $a \to a\cdot x$ is continuous for $\sigma(\tilde{\boldsymbol{M}}, \boldsymbol{M}^*)$- and $\sigma(E^{**}, E^*)$-topologies and the mapping $x \to x\circ b$ is $\sigma(E^{**}, E^*)$-continuous by the arguments in Lemma 1 and Lemma 2. Hence we have

$$(a\cdot x)\circ b = a\cdot(x\circ b)$$

for all $a, b \in \tilde{\boldsymbol{M}}$ and $x \in E^{**}$. This concludes the proof.

In the following, we denote the second conjugate space E^{**} of a normed $\boldsymbol{M}$-module by $\tilde{E}$ as a normed $\tilde{\boldsymbol{M}}$-module.

If we consider $\boldsymbol{M}$, itself, as a normed $\boldsymbol{M}$-module, then we have

$$xy = x\cdot y = x\circ y$$

for all $x, y \in \tilde{\boldsymbol{M}}$. In fact, the mappings $y \to xy$ and $x \to xy$ are $\sigma(\tilde{\boldsymbol{M}}, \boldsymbol{M}^*)$-continuous and coincide with the mappings $y \to x\cdot y$ and $x \to x\circ y$ for $x, y \in \boldsymbol{M}$ respectively.

Furthermore, if we consider a Banach algebra B instead of a normed $\boldsymbol{M}$-module, then the slight modification of the above arguments points out that the second conjugate space B^{**} of B becomes a Banach algebra in two different manners. But we shall omit the detail.

Next, we consider a certain linear mapping from a $\boldsymbol{M}$-module E into $\boldsymbol{M}$. A linear mapping θ from E into a normed $\boldsymbol{M}$-module F called a left (resp. right) $\boldsymbol{M}$-linear mapping if

$$\theta(ax) = a\theta(x) \qquad (\text{resp. } \theta(xa) = \theta(x)a)$$

for every $a \in \boldsymbol{M}$ and $x \in E$. If θ is two-sided $\boldsymbol{M}$-linear, it is called $\boldsymbol{M}$-linear simply. Combining this definition and Theorem 1, we have

LEMMA 3. *If θ is a bounded M-linear mapping from E into F, then the bitranspose ${}^{tt}\theta = \tilde{\theta}$ of θ is $\tilde{M}$-linear.*

Proof. From the proof of Lemma 2, the mapping $b \to x \circ b$ is $\sigma(\tilde{M}, M^*)$- and $\sigma(\tilde{E}, E^*)$-continuous for $x \in E$. Hence we have $\theta(x \circ b) = \theta(x) \circ b$ for $x \in E$ and $b \in \tilde{M}$. Using the $\sigma(\tilde{E}, E^*)$-continuity of the mapping $x \to x \circ b$, we get

$$\theta(x \circ b) = \theta(x) \circ b$$

for all $x \in \tilde{E}$ and $b \in \tilde{M}$. Moreover, the continuity of the mapping $a \to a \cdot x$ implies

$$\theta(a \cdot x) = a \cdot \theta(x)$$

for all $a \in \tilde{M}$ and $x \in \tilde{E}$. This concludes the proof.

Now we can state one of our main results in the following

THEOREM 2. (*Generalized Hahn-Banach Theorem*) *Let A be a commutative AW^*-algebra, E a normed A-module and V an invariant subspace of E, i. e. $aVb \subset V$ for a and $b \in A$. If θ is a bounded A-linear A-valued mapping on V, then θ can be extended to an A-linear A-valued mapping θ_0 on E preserving its norm.*[2)]

Proof. At first, we recall that the second conjugate space E of E is a normed $\tilde{A}$-module by Theorem 1. Since the $\sigma(\tilde{E}, E^*)$-closure $\tilde{V}$ of V is the second conjugate space of V, θ is uniquely extended to an $\tilde{A}$-linear $\tilde{A}$-valued mapping $\tilde{\theta}$ on $\tilde{V}$ by Lemma 3 as the bitranspose of θ. Let Ω be the spectrum space of A and A_0 the space of all bounded complex valued functions on Ω, that is, $A_0 = l^\infty(\Omega)$, then A_0 becomes a subalgebra of $\tilde{A}$. Hence $\tilde{E}$ is considered as a normed A_0-module. For any fixed point $t \in \Omega$, put $\varphi_t = {}^t\theta(\sigma_t)$ where σ_t is the pure state of A corresponding to t, then we have that $\varphi_t \in V^*$ and

$$\begin{aligned}\langle a \cdot x \circ b, \varphi_t \rangle &= \langle a \cdot x \circ b, {}^t\theta(\sigma_t) \rangle = \langle \tilde{\theta}(a \cdot x \circ b), \sigma_t \rangle = \langle a\tilde{\theta}(x)b, \sigma_t \rangle \\ &= a(t)b(t) \langle \tilde{\theta}(x), \sigma_t \rangle = a(t)b(t) \langle x, \varphi_t \rangle\end{aligned}$$

for all $a, b \in A_0$ and $x \in \tilde{V}$. Next, let e_t be the carrier projection of σ_t in $\tilde{A}$, then e_t belongs to A_0 and we have

$$\langle e_t \cdot x, \varphi_t \rangle = \langle x, \varphi_t \rangle, \qquad \langle x \circ e_t, \varphi_t \rangle = \langle x, \varphi_t \rangle$$

2) We call a commutative C^*-algebra A AW^*-algebra if the self-adjoint part of A becomes a conditionally complete vector lattice with respect to the usual ordering of operators. Then the characterization of a commutative C^*-algebra A to be AW^*-algebra is given as follows: the closure of any open set in the spectrum space Ω of A becomes open again. And such a compact space is called stonean space (cf. [2] and [5]).

for all $x \in \tilde{V}$ and $e_t a = a(t)e_t$ for $a \in A_0$. Put $\tilde{V}_t = e_t \cdot \tilde{V} \circ e_t$ and $\tilde{E}_t = e_t \cdot \tilde{E} \circ e_t$, then $\tilde{V}_t$ is an A_0-invariant subspace of $\tilde{E}_t$ and one can consider φ_t as an element of V_t^*. Let $\tilde{\varphi}_t$ be an extension of φ_t to $\tilde{E}_t$ by the usual Hahn-Banach Theorem and $\bar{\varphi}_t$ the element of $\tilde{E}^*$ which is defined by the equation

$$\langle x, \bar{\varphi}_t \rangle = \langle e_t \cdot x \circ e_t, \tilde{\varphi}_t \rangle$$

for $x \in \tilde{E}$, we have

$$\langle a \cdot x \circ b, \bar{\varphi}_t \rangle = \langle e_t \cdot a \cdot x \circ b \circ e_t, \tilde{\varphi}_t \rangle = \langle a(t)b(t)e_t \cdot x \circ e_t, \tilde{\varphi}_t \rangle = a(t)b(t) \langle x, \bar{\varphi}_t \rangle$$

for all $a, b \in A_0$ and $x \in \tilde{E}$. Consider the mapping $\bar{\theta}$ from $\tilde{E}$ to A_0 such as $\bar{\theta}(x)(t) = \langle x, \bar{\varphi}_t \rangle$ for all $x \in \tilde{E}$ and $t \in \Omega$, then we have

$$\bar{\theta}(a \cdot x \circ b) = a\bar{\theta}(x)b$$

for all $a, b \in A_0$ and $x \in \tilde{E}$, for

$$\begin{aligned} \bar{\theta}(a \cdot x \circ b)(t) &= \langle a \cdot x \circ b, \bar{\varphi}_t \rangle = a(t)b(t) \langle x, \bar{\varphi}_t \rangle \\ &= a(t)b(t)\bar{\theta}(x)(t) = [a\bar{\theta}(x)b](t). \end{aligned}$$

For $x \in \tilde{V}$, we have

$$\bar{\theta}(x)(t) = \langle x, \varphi_t \rangle = \langle e_t \cdot x \circ e_t, \rangle = \langle e_t \cdot x \circ e_t, \varphi_t \rangle = \langle x, \varphi_t \rangle = \tilde{\theta}(x)(t),$$

so that $\bar{\theta}$ coincides with $\tilde{\theta}$ on $\tilde{V}$. Moreover, we have

$$\begin{aligned} \|\bar{\theta}\| &= \sup [\|\bar{\theta}(x)\| : \|x\| \leqq 1] = \sup [|\bar{\theta}(x)(t)| : \|x\| \leqq 1, t \in \Omega] \\ &= \sup [|\langle x, \bar{\varphi}_t \rangle| : \|x\| \leqq 1, t \in \Omega] = \sup [\|\bar{\varphi}_t\| : t \in \Omega] \\ &= \sup [\|\varphi_t\| : t \in \Omega] = \sup [|\langle x, \varphi_t \rangle| : x \in V \|x\| \leqq 1, t \in \Omega] \\ &= \sup [|\theta(x)(t)| : x \in V \|x\| \leqq 1, t \in \Omega] \\ &= \sup [\|\theta(x)\| : x \in V \|x\| \leqq 1] = \|\theta\| . \end{aligned}$$

Hence we get $\|\bar{\theta}\| = \|\theta\|$.

Now, there exists a projection π of norm one from A_0 to A by Nachbin-Hasumi Theorem [4] and [6]. Put $\theta_0(x) = \pi[\bar{\theta}(x)]$ for $x \in E$, then θ_0 is required one. In fact, we have

$$\begin{aligned} \theta_0(axb) &= \pi[\bar{\theta}_0(axb)] = \pi[a\bar{\theta}(x)b] = a\pi[\bar{\theta}(x)]b \qquad \text{(cf. [11])} \\ &= a\theta_0(x)b \qquad \text{for all } a, b \in A \text{ and } x \in E, \\ \theta_0(x) &= \pi[\bar{\theta}(x)] = \pi[\theta(x)] = \theta(x) \qquad \text{for } x \in V \end{aligned}$$

and $\|\theta\| \leqq \|\theta_0\| = \|\pi \cdot \bar{\theta}\| \leqq \|\bar{\theta}\| = \|\theta\|$. This concludes the proof.

Connecting with this theorem, we consider a W^*-algebra $\boldsymbol{M}$ with its commutative W^*-subalgebra $\boldsymbol{A}$ as a normed $\boldsymbol{A}$-module, then the σ-weak continuities of θ and θ_0 come into our consideration. However, it can be shown that there exists no σ-weakly continuous projection of norm one from the full operator algebra $\boldsymbol{M}$ on an infinite dimensional Hilbert space to its commutative W^*-subalgebra which contains no non-zero minimal projection (cf. [12]).

2. Let $\boldsymbol{M}$ be a C^*-algebra and $\boldsymbol{N}$ its C^*-subalgebra, then $\boldsymbol{M}$ becomes a normed $\boldsymbol{N}$-module. If θ is a bounded positive $\boldsymbol{N}$-linear $\boldsymbol{N}$-valued mapping on $\boldsymbol{M}$ such that $\theta(a)=a$ for every $a\in\boldsymbol{N}$, then θ is called an expectation from $\boldsymbol{M}$ to $\boldsymbol{N}$. If $\boldsymbol{M}$ and $\boldsymbol{N}$ contain units respectively, then the characterization of a mapping from $\boldsymbol{M}$ to $\boldsymbol{N}$ to be an expectation is given as $\theta(I_{\boldsymbol{M}})=I_{\boldsymbol{N}}$, $\|\theta\|\leqq 1$ and $\theta(a)=a$ for all $a\in\boldsymbol{N}$ in [11], where $I_{\boldsymbol{M}}$ and $I_{\boldsymbol{N}}$ are units of $\boldsymbol{M}$ and $\boldsymbol{N}$, respectively. In other words, the expectation is a generalized state, i. e. it is an operator valued state (cf. [8]). In this section, we shall prove the generalization of Takeda-Grothendieck's Theorem.

LEMMA 4. *Let $\boldsymbol{M}$ be a W^*-algebra, $\boldsymbol{N}$ a finite W^*-subalgebra of $\boldsymbol{M}$ and θ a bounded *-preserving σ-weakly continuous $\boldsymbol{N}$-linear $\boldsymbol{N}$-valued mapping on $\boldsymbol{M}$, then there exist two positive σ-weakly continuous $\boldsymbol{N}$-linear $\boldsymbol{N}$-valued mappings θ^+ and θ^- on $\boldsymbol{M}$ such as $\theta=\theta^+-\theta^-$.*

Proof. At first, we recall that we have

$$\|\varphi_1-\varphi_2\|=\|\varphi_1\|+\|\varphi_2\|$$

for σ-weakly continuous positive linear functionals φ_1 and φ_2 on $\boldsymbol{M}$ if and only if the carrier projections of φ_1 and φ_2 are orthogonal each other by [3]. Hence putting

$$\langle u^{-1}xu, \varphi\rangle=\langle x, \varphi_u\rangle$$

for self-adjoint $\varphi\in\boldsymbol{M}^*$ and unitary $u\in\boldsymbol{M}$, we have

$$\varphi_u=(\varphi_u)^+-(\varphi_u)^-=(\varphi^+)_u-(\varphi^-)_u$$

and

$$\|\varphi_u\|=\|\varphi\|=\|(\varphi_u)^+\|+\|(\varphi_u)^-\|=\|(\varphi^+)_u\|+\|(\varphi^-)_u\|.$$

That is, $(\varphi_u)^+=(\varphi^+)_u$ and $(\varphi_u)^-=(\varphi^-)_u$.

(1) Case of $\boldsymbol{N}$ to be countably decomposable: From our assumption for $\boldsymbol{N}$ it has a faithful trace τ. Putting $\varphi={}^t\theta(\tau)$, we have

$$\langle x, \varphi_u\rangle=\langle u^{-1}xu, \varphi\rangle=\langle\theta(u^{-1}xu), \tau\rangle=\langle u^{-1}\theta(x)u, \tau\rangle=\langle\theta(x), \tau\rangle=\langle x, \tau\rangle$$

for all $x\in\boldsymbol{M}$ and unitary $u\in\boldsymbol{N}$, so that $\varphi_u=\varphi$. Hence $(\varphi^+)_u=\varphi^+$ and $(\varphi^-)_u=\varphi^-$ for unitary $u\in\boldsymbol{N}$ from our above remark. Let e and f be the carrier projections of φ^+ and φ^- respectively, we have

$$[x\in\boldsymbol{M}\colon \langle x^*x, \varphi^+\rangle=0]=\boldsymbol{M}(I-e),$$
$$[x\in\boldsymbol{M}\colon \langle x^*x, \varphi^-\rangle=0]=\boldsymbol{M}(I-f)$$

and

$$L_e\varphi = \varphi^+, \qquad L_f\varphi = -\varphi^-.$$

The invariancy of φ^+ and φ^- with respect to unitary of $\boldsymbol{N}$ implies that

$$u^{-1}\boldsymbol{M}(I-e)u = \boldsymbol{M}(I-e) \quad \text{and} \quad u^{-1}\boldsymbol{M}(I-f)u = \boldsymbol{M}(I-f)$$

for unitary u of $\boldsymbol{N}$. Hence we have

$$u^{-1}eu = e \quad \text{and} \quad u^{-1}fu = f,$$

so that e and f belong to $\boldsymbol{N}'$ which is the commutator of $\boldsymbol{N}$.

Now if we put $\theta^+(x) = \theta(ex)$ and $\theta^-(x) = -\theta(fx)$, this is a desired decomposition. In fact, we have clearly

$$\theta = \theta^+ - \theta^-.$$

For any $a, b \in \boldsymbol{N}$ and $x \in \boldsymbol{M}$, we have

$$\theta^+(axb) = \theta(eaxb) = \theta(aexb) = a\theta(ex)b = a\theta^+(x)b$$

and

$$\theta^-(axb) = -\theta(faxb) = -\theta^-(afxb) = -a\theta(fx)b = a\theta^-(x)b.$$

For any fixed positive element x of $\boldsymbol{M}$ and every positive a of $\boldsymbol{N}$, we have

$$\begin{aligned}\langle a\theta^+(x), \tau\rangle &= \langle a\theta(ex), \tau\rangle = \langle a^{1/2}\theta(ex)a^{1/2}, \tau\rangle = \langle \theta(ea^{1/2}xa^{1/2}), \tau\rangle \\ &= \langle ea^{1/2}xa^{1/2}, \varphi\rangle = \langle a^{1/2}xa^{1/2}, \varphi^+\rangle \geqq 0\end{aligned}$$

and similarly $\langle a\theta^-(x), \tau\rangle \geqq 0$, so that $\theta^+(x)$ and $\theta^-(x)$ are positive in $\boldsymbol{N}$. Therefore θ^+ and θ^- are positive.

(2) General case: There exists a family $\{z_\alpha\}$ of orthogonal central projections of $\boldsymbol{N}$ such that $\sum_\alpha z_\alpha = I$ and each $\boldsymbol{N}z_\alpha$ is countably decomposable. Suppose θ_α to be the restriction of θ on $z_\alpha\boldsymbol{M}z_\alpha$, there exist projections e_α and f_α in $(\boldsymbol{N}z_\alpha)' \cap z_\alpha\boldsymbol{M}z_\alpha$ such that $\theta_\alpha(e_\alpha x)$ and $-\theta_\alpha(f_\alpha x)$ are positive mappings and

$$\theta_\alpha(x) = \theta_\alpha(e_\alpha x) - \theta_\alpha(f_\alpha x)$$

by the arguments in case (1). Putting $\sum_\alpha e_\alpha = e$ and $\sum_\alpha f_\alpha = f$, $\theta^+(x) = \theta(ex)$ and $\theta^-(x) = -\theta(fx)$ are desired ones. In fact, we have

$$\begin{aligned}\theta(x) &= \theta((\textstyle\sum_\alpha z_\alpha)x(\sum_\alpha z_\alpha)) = \sum_{\alpha,\alpha'} \theta(z_\alpha x z_{\alpha'}) = \sum_{\alpha,\alpha'} z_\alpha\theta(x)z_{\alpha'} \\ &= \textstyle\sum_\alpha z_\alpha\theta(x) = \sum_\alpha z_\alpha\theta(x)z_\alpha = \sum_\alpha \theta_\alpha(z_\alpha x z_\alpha) \\ &= \textstyle\sum_\alpha [\theta_\alpha(e_\alpha x) + \theta_\alpha(f_\alpha x)] = \sum_\alpha [\theta(e_\alpha x) + \theta(f_\alpha x)] = \theta(ex) + \theta(fx).\end{aligned}$$

For any positive $x \in \boldsymbol{M}$, $z_\alpha x z_\alpha$ is positive so that $\theta_\alpha(e_\alpha x)$ and $-\theta_\alpha(f_\alpha x)$ are positive. Hence $\theta(ex)$ and $-\theta(fx)$ are positive. Finally, we have

$$\theta^+(axb) = \theta(eaxb) = \theta(aexb) = a\theta(ex)b = a\theta^+(x)b$$

and similarly

$$\theta^-(axb) = a\theta^-(x)b$$

for all $a, b \in \boldsymbol{N}$ and $x \in \boldsymbol{M}$. This concludes the proof.

LEMMA 5. *Let $\boldsymbol{A}$ be a commutative AW^*-algebra and $\boldsymbol{M}$ a C^*-algebra, with a unit, containning $\boldsymbol{A}$. If θ is a positive $\boldsymbol{A}$-linear $\boldsymbol{A}$-valued mapping on $\boldsymbol{M}$, then there exist a positive element a of $\boldsymbol{A}$ and an expectation θ_0 such that $\theta(x) = a\theta(x)$.*

Proof. Suppose Ω to be the spectrum space of $\boldsymbol{A}$, Ω is a stonean space. Putting $\theta(I) = a$ and $G = [t \in \Omega\colon a(t) > 0]$, G is an open subset of Ω. Let e be the characteristic function of the closure of G, then e is a projection of $\boldsymbol{A}$. Putting $\theta_0'(x)(t) = \theta(x)(t)/a(t)$ for $t \in G$ and $x \in e\boldsymbol{M}e$, the function $\theta_0'(x)(t)$ is bounded and continuous on G because of the mapping $x \to \theta_0'(x)(t)$ to be a state on $e\boldsymbol{M}e$, which implies

$$|\theta_0'(x)(t)| \leqq \|x\|.$$

Hence it is uniquely extended to a continuous function on the closure of G by [2], i. e. $\theta_0'(x)$ is considered as an element of $\boldsymbol{A}e$. This θ_0' is an expectation from $e\boldsymbol{M}e$ to $\boldsymbol{A}e$, and $\theta(x) = a\theta_0'(x)$ for $x \in e\boldsymbol{M}e$. Now there exists an expectation θ_0'' from $(I-e)\boldsymbol{M}(I-e)$ to $\boldsymbol{A}(I-e)$ by Nachbin-Hasumi's Theorem. Putting $\theta_0(x) = \theta_0'(exe) + \theta_0''((I-e)x(I-e))$ for $x \in \boldsymbol{M}$, θ_0 is the expectation which is $\theta(x) = a\theta_0(x)$ for $x \in \boldsymbol{M}$. This concludes the proof.

Combining these lemmas, we get

THEOREM 3. *Let $\boldsymbol{A}$ be a commutative AW^*-algebra and $\boldsymbol{M}$ a C^*-algebra, with unit, containning $\boldsymbol{A}$. If θ is a bounded $*$-preserving $\boldsymbol{A}$-linear $\boldsymbol{A}$-valued mapping on $\boldsymbol{M}$, then there exist two positive elements a_1 and a_2 and two expectations θ_1 and θ_2 such that*

$$\theta = a_1\theta_1 - a_2\theta_2.$$

Proof. Considering their universal enveloping algebras $\widetilde{\boldsymbol{M}}$, $\widetilde{\boldsymbol{A}}$ and the bitranspose $\tilde{\theta} = {}^{tt}\theta$ of θ, there exist σ-weakly continuous positive $\widetilde{\boldsymbol{A}}$-linear $\widetilde{\boldsymbol{A}}$-valued mapping $\tilde{\theta}^+$ and $\tilde{\theta}^-$ such that

$$\tilde{\theta} = \tilde{\theta}^+ - \tilde{\theta}^-$$

by Lemma 4. There exists a projection π of norm one from $\widetilde{\boldsymbol{A}}$ to $\boldsymbol{A}$ by Nachbin-Hasumi's Theorem. Putting $\theta^+(x) = \pi[\tilde{\theta}^+(x)]$ and $\theta^-(x) = \pi[\tilde{\theta}^-(x)]$ for $x \in \boldsymbol{M}$, θ^+ and θ^- become positive $\boldsymbol{A}$-linear $\boldsymbol{A}$-valued mapping on $\boldsymbol{M}$ such that

$$\theta = \theta^+ - \theta^-.$$

Applying Lemma 5 to θ^+ and θ^- respectively, we obtain

$$\theta^+ = a_1\theta_1 \quad \text{and} \quad \theta^- = a_2\theta_2$$

where a_1 and a_2 are positive element of A and θ_1 and θ_2 are expectations from $\boldsymbol{M}$ to A. Thus we have

$$\theta = a_1\theta_1 - a_2\theta_2.$$

This concludes the proof.

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