A PROOF OF A THEOREM OF TAKESAKI

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Since theory of von Neumann algebra in the sense of Dixmier [1] is recognized as a non-commutative extension of the integration theory, various theorems on additive set functions are translated into that of linear functionals of von Neumann algebras: For example, the Jordan decomposition is generalized by [3, 6], the Radon-Nikodym theorem by [2], the Vitali-Hahn-Saks theorem by [5, 8], etc.

Very recently, Takesaki [7] pointed out, among others, a generalization of the decomposition theorem of Yosida-Hewitt [9]: *A positive linear functional of a von Neumann algebra is decomposed into the sum of its normal and singular components*; where a positive linear functional ρ is called *normal* if $\rho(x_\alpha)\to\rho(x)$ for $x_\alpha\uparrow x$, and *singular* if ρ has no normal minorant. Clearly our normality (singularity) corresponds to the complete additivity (purely finite additivity) of additive set functions.

Instead of the total decomposition of the conjugate space being employed by Takesaki, here a direct verion of Yosida-Hewitt's proof will show that the theorem of Takesaki is a consequense of a known convergence theorem due to Ogasawara [4], which states that *the weak* limit ρ of a monotone increasing net of normal positive linear functionals $\{\rho_\delta\}$ is also normal*: Indeed, $0\leqq a_\alpha\uparrow a$ implies $\rho(a_\alpha)\geqq\rho_\delta(a_\alpha)$ for any α, which implies $\lim_\alpha \rho(a_\alpha)\geqq\rho_\delta(a)$ or $\lim_\alpha\rho(a_\alpha)\geqq\rho(a)$, and the converse inequality is obvious.

Ogasawara's theorem implies at once, that the set F of all normal linear positive functionals ρ_α such as $0\leqq\rho_\alpha\leqq\rho$ is inductively ordered in its usual natural ordering, whence F contains a maximal ρ' by Zorn's Lemma. Put $\rho''=\rho-\rho'$. Then ρ'' is positive by the definition. Moreover, ρ'' is singular, for otherwise there exists a normal member ρ''' such as $0<\rho'''\leqq\rho''\leqq\rho$ or $\rho\geqq\rho'+\rho'''>\rho'$ which contradicts the maximality of ρ'. Thus $\rho=\rho'+\rho''$ where ρ' is normal and ρ'' singular as desired.

At the end, it is to be noticed that the method of the present proof is also applicable for the decompositions of linear applications of some kinds. However, it will be remained in another occasion.

References

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