ON CONTINUABILITY OF BILINEAR DIFFERENTIALS

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Schiffer and Spencer [3] have derived a condition under which bilinear differentials are continuable. In this paper, applying the results due to Aronszajn [1], we shall give a condition in terms of positive definite kernels.

Let D be a domain in the z-plane. A function $\psi(z, \bar{\zeta})$ of $z, \zeta \in D$ is called a Hermitian kernel on D, if it satisfies $\psi(z, \bar{\zeta}) = \psi(\zeta, \bar{z})$. If for any points $y_1, y_2, \cdots, y_n \in D$ and any complex numbers $\xi_1, \xi_2, \cdots, \xi_n$ the inequality

$$\sum_{i,j=1}^{n} \psi(y_i, \bar{y}_j)\xi_i\bar{\xi}_j \geqq 0 \qquad (n = 1, 2, \cdots)$$

is satisfied, then $\psi(z, \bar{\zeta})$ is called a positive definite kernel on D. Further, we denote by P_D the aggregate of all positive definite kernels $\psi(z, \bar{\zeta})$, which are analytic in z, $\bar{\zeta}$ respectively. Let $\psi, \varphi \in P_D$. We denote $\varphi \ll \psi$ if for any points $y_1, y_2, \cdots, y_n \in D$ and any complex numbers $\xi_1, \xi_2, \cdots, \xi_n$

$$\sum_{i,j=1}^{n} \psi(y_i, \bar{y}_j)\xi_i\xi_j - \sum_{i,j=1}^{n} \varphi(y_i, \bar{y}_j)\xi_i\xi_j \geqq 0 \qquad (n = 1, 2, \cdots).$$

Now, generally, the following lemma is well known (cf. [4]).

Lemma 1. *Let E be an abstract set. If a function $k(x, y)$ of $x, y \in E$ satisfies*

$$\sum_{i,j=1}^{n} k(y_i, y_j)\bar{\xi}_i\xi_j \geqq 0 \qquad (n = 1, 2, \cdots)$$

for any points $y_1, y_2, \cdots, y_n \in E$ and any complex numbers $\xi_1, \xi_2, \cdots, \xi_n$, we can construct a Hilbert space which has $k(x, y)$ as its reproducing kernel.

Proof. Let F_1 be the family of functions f_1 which are of the form

$$f_1(x) = \sum_{j=1}^{n} \alpha_j k(x, y_j)$$

where $y_1, \cdots, y_n$ are any points of E, $a_1, \cdots, \alpha_n$ any complex numbers and n any natural number. Let the inner product be defined by

$$(f_1, g_1) = \sum_{j,i=1}^{\max(m,n)} \alpha_j\bar{\gamma}_i k(u_i, y_j), \qquad (f_1, f_1) = \|f_1\|^2,$$

where

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$$g_1(x)=\sum_{i=1}^{m}\gamma_i k(x,\ u_i)\in F_1.$$

Then we have a normed space and $k(x, y)$ is a reproducing kernel of F_1, that is, if $f\in F_1$,

$$f(y)=(f(x),\ k(x,\ y))$$

for any $y\in E$. Therefore we have

$$|(f_1,\ k(x,\ y))|\leqq\|f_1\|\ \|k(x,\ y)\|$$

and we can easily see that $\|f_1\|=0$ is equivalent to $f_1\equiv 0$. Completing F_1, we get a Hilbert space F and $k(x, y)$ remains to possess the reproducing property for F.

LEMMA 2 (Moore [2]). *To every positive matrix $k(x, y)$ there corresponds one and only one class of functions with a uniquely determined quadratic form in it, which forms a Hilbert space admitting $k(x, y)$ as a reproducing kernel.*

LEMMA 3 (Aronszajn [1]). *If k is the reproducing kernel of the class F of functions defined in the set E with the norm $\|\ \ \|$, then k restricted to a subset $E_1\subset E$ is the reproducing kernel of the class F_1 of all restrictions of F to the subset E_1. For any such restriction, $f_1\in F$, the norm $\|f_1\|$ is equal to the minimum of $\|f\|$ for all $f\in F$ whose restriction to E_1 is f_1.*

LEMMA 4 (Aronszajn [1]). *If k and k_1 are the reproducing kernels of the classes F and F_1 with the norms $\|\ \ \|$ and $\|\ \ \|_1$, respectively, and if $k_1\ll k$, then $F_1\subset F$, and $\|f_1\|_1\geqq\|f_1\|$ for every $f_1\in F_1$.*

LEMMA 5 (Aronszajn [1]). *If k is the reproducing kernel of the class F with the norm $\|\ \ \|$, and if the linear class $F_1\subset F$ forms a Hilbert space with the norm $\|\ \ \|_1$ such that $\|f_1\|_1\geqq\|f_1\|$ for every $f_1\in F_1$, then the class F_1 possesses a reproducing kernel k_1 which satisfies $k_1\ll k$.*

Applying these lemmas, we have following results.

THEOREM 1. *Let $\psi(s, t)\in P_V$, where V denote an arbitrary open set in D. If*

$$\psi(s,\ \bar{t})\ll k(s,\ \bar{t})\qquad in\quad V,$$

then $\psi(s, \bar{t})$ is continuable to the whole D and

$$\psi(s,\ \bar{t})\ll k(s,\ \bar{t})\qquad in\quad D.$$

Here $k(s, \bar{t})$ denotes the Bergman's kernel corresponding to D.

Proof. We apply Lemma 4 to k and $k_1=\psi$ in V, k being also restricted to V. Let F_k be the space corresponding to k. In view of the analyticity of $\mathcal{L}^2(D)$ we have

$$F_k = \mathcal{L}^2(D), \qquad \|f_k\|_k = \|f_k\|.$$

Let F_ψ be the space corresponding to ψ. Now, by Lemma 4, we have

$$F_\psi \subset F_k = \mathcal{L}^2(D)$$

and

$$\|f_\psi\|_\psi \geqq \|f_\psi\| \qquad \text{for every} \quad f_\psi \in F_k,$$

where $\|\ \|_\psi$ and $\|\ \|$ denote the norms corresponding to F_ψ and $\mathcal{L}^2(D)$, respectively. Hence $\psi(z, \bar{\zeta})$ belongs to $F_\psi \subset \mathcal{L}^2(D)$ for any fixed $\zeta \in D$, i.e. it is continuable to the whole D and $\psi(z, \bar{\zeta})$ is analytic in D. As $\psi(z, \bar{\zeta}) = \overline{\psi(\zeta, \bar{z})}$, it is also analytic in $\bar{\zeta}$. Therefore we can apply Lemma 5. Namely, the class F_ψ possesses a reproducing kernel k_1 satisfying $k_1 \ll k$. But by Lemma 2 we obtain $k_1 = \psi$. Thus we have $\psi \ll k$ in D.

We can obtain the inverse of this theorem as follows.

THEOREM 2. *Let $\psi(s, \bar{t})$ belong to P_D and also to $\mathcal{L}^2(D)$ for fixed $t \in D$. Then there exists a positive number λ such that*

$$\lambda \psi(s, \bar{t}) \ll k(s, \bar{t}).$$

Proof. Let D_1 be a subdomain of D such that $\bar{D}_1 \subset D$, and its boundary be obtained from that of D by a suitable analytic deformation depending on a parameter ε. Let $k_1(s, \bar{t})$ be the Bergman's kernel function of D_1. Let further $\bar{S}$ be any compact subdomain of D_1. It is known [3] that under these circumstances

$$l_F(s, \bar{t}) = -k_1(s, \bar{t}) + k(s, \bar{t}) = O(\varepsilon)$$

uniformly with respect to $s, t \in \bar{S}$. The kernel $k_1(s, \bar{t})$ may be expressed in terms of a complete orthonormal system $\{\varphi_j\}$:

$$k_1(s, \bar{t}) = \sum_{j=1}^{\infty} \varphi_j(s)\overline{\varphi_j(t)}.$$

Now $\psi(s, \bar{t})$ is regular in $\bar{D}_1$, and we can apply Mercer's theorem which implies

$$\psi(s, \bar{t}) = \sum_{j=1}^{\infty} \lambda_j^{-1}\varphi_j(s)\overline{\varphi_j(t)}.$$

Here, $\{\lambda_j\}\,(\lambda_j > 0,\ j = 1, 2, \cdots)$ is the corresponding sequence of characteristic numbers of the equation

$$\lambda \psi \varphi = \varphi.$$

We may suppose that λ_1 is the least among the characteristic numbers. Thus we have

$$\sum_{i,j=1}^{n} k_1(z_i, \bar{z}_j)\xi_i\bar{\xi}_j = \sum_{l=1}^{\infty} \sum_{i,j=1}^{n} \varphi_l(z_i)\overline{\varphi_l(z_j)}\xi_i\bar{\xi}_j,$$

$$\sum_{i,j=1}^{n} \psi(z_i, \bar{z}_j) \xi_i \bar{\xi}_j = \sum_{l=1}^{\infty} \sum_{i,j=1}^{n} \varphi_l(z_i) \overline{\varphi_l(z_j)} \frac{\xi_i \bar{\xi}_j}{\lambda_l}$$

and hence

$$\begin{aligned} &\sum_{i,j=1}^{n} k_1(z_i, \bar{z}_j)\xi_i\bar{\xi}_j - \lambda_1 \sum_{i,j=1}^{n} \psi(z_i, \bar{z}_j)\xi_i\bar{\xi}_j \\ &= \sum_{l=1}^{\infty} \sum_{i,j=1}^{n} \varphi_l(z_i)\overline{\varphi_l(z_j)}\xi_i\bar{\xi}_j\left(1 - \frac{\lambda_1}{\lambda_l}\right) \\ &= \sum_{l=1}^{\infty} \left(1 - \frac{\lambda_1}{\lambda_l}\right) \left\| \sum_{i=1}^{n} \varphi_l(z_i)\xi_i \right\|^2 \geqq 0 \end{aligned}$$

i.e.

$$\lambda_1 \psi(s, \bar{t}) \ll k_1(s, \bar{t}).$$

Since $k_1(s, \bar{t}) - k(s, \bar{t}) = O(\varepsilon)$ holds uniformly for $s, t \in \bar{S}$, we have

$$\lambda_1 \psi(s, \bar{t}) \ll k(s, \bar{t}) + O(\varepsilon) \qquad \text{in} \quad \bar{S}.$$

Consequently, letting ε tend to zero, we have

$$\lambda_1 \psi(s, \bar{t}) \ll k(s, \bar{t}) \qquad \text{in} \quad D.$$

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