SOME LIMIT THEOREMS CONCERNING WITH THE RENEWAL NUMBERS

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Let $\{X_i\}$ be a sequence of independent random variables and let $\{a_i\}$ be a sequence of real numbers. Denoting as $S_n=\sum_1^n X_i$, we shall interest the weighted mean of renewal numbers in the interval $(a, x+h)$, which is defined by

$$A(x, h)=\sum_{n=1}^{\infty} a_n P(x<S_n \leqq x+h). \tag{1}$$

If both $\{E(X_i)\}$ and $\{a_i\}$ are stable sequences[1] with average m and a, respectively, and if some further conditions are satisfied, then it is known that $A(x, h)\to ah/m$ $(x\to\infty)$ by Cox and Smith [1].

But when $\{E(X_i)\}$ is not stable, $A(x, h)$ is not necessarily convergent to a finite limit as $x\to\infty$. In this case, instead of $A(x, h)$, the variable

$$\bar{A}_h(X)=\frac{1}{X}\int_0^X A(x, h)\,dx$$

will converge to ah/m as $X\to\infty$, under suitable conditions. This fact was shown by the analogous argument of [2] by Prof. T. Kawata.

From a practical problem it was necessary to us to find the distribution of $A(x, h)$ or $\bar{A}_h(X)$ when $\{a_i\}$ is a sequence of independent random variables having the mean a. We shall treat in the present paper this problem when a_i are the random variables identically distributed and obeying the exponential distribution.

First of all, we shall prepare the following lemmas.

Lemma 1. *Let X_i $(i=1, 2, \cdots)$ be independent random variables having the distribution function $F_i(x)$ such that $E(X_i)=m_i>0$. Suppose that the following conditions are satisfied:*

$$\int_{-\infty}^0 e^{-sx}dF_i(x)<\infty \qquad \textit{for} \quad 0\leqq s\leqq s_0, \tag{2}$$

$$\lim_{A\to\infty}\int_A^{\infty} x\,dF_i(x)=0, \tag{3}$$

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1) Cox and Smith [1] gave the following

Definition. A sequence $\{\mu_i\}$ such that $\lim_{p\to\infty}\frac{1}{p+1}\sum_{i=n}^{n+p}\mu_i=\mu$, uniformly in n, will be called *stable* with average μ.

$$\lim_{A\to\infty}\int_{-\infty}^{-A} e^{-sx}dF_i(x)=0, \tag{4}$$

where both (3) *and* (4) *hold uniformly with respect to i and $0<s\leqq s_0$. If $(1/n)\sum_{i=1}^{n}m_i\to m>0$ $(n\to\infty)$, then*

$$\lim_{X\to\infty}\frac{1}{X}\int_{-\infty}^{X}\sum_{n=1}^{\infty}P(x<S_n\leqq x+h)\,dx=\frac{h}{m}. \tag{5}$$

LEMMA 2. *The conditions of the Lemma* 1 *are assumed. Furthermore if $\{a_i\}$ is a sequence of real numbers satisfying that $(1/n)\sum_{i=1}^{n}a_i\to a$ $(n\to\infty)$,* then

$$\lim_{X\to\infty}\frac{1}{X}\int_{0}^{X}\sum_{n=1}^{\infty}a_nP(x<S_n\leqq x+h)\,dx=\frac{ah}{m}. \tag{6}$$

These two lemmas are due to Prof. T. Kawata[2]. From Lemma 2, the following theorem will be proved directly.

THEOREM 1. *Let $X_i\,(i=1,2,\cdots)$ be independent random variables satisfying the conditions of Lemma* 1. *Furthermore, if $\{a_i\}$ is a sequence of random variables which obey the strong law of large numbers, i.e.,*

$$P\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}a_i=a\right)=1,$$

then the random variable

$$\bar{A}_h(X)=\frac{1}{X}\int_{0}^{X}\sum_{n=1}^{\infty}a_nP(x<S_n\leqq x+h)\,dx \tag{7}$$

will converge to ah/m as $X\to\infty$ with probability 1.

In other words, the distribution of $\bar{A}_h(X)$ converges to the unit distribution. Hence we shall introduce the normalized variables by

$$\bar{A}_h'(X)=\frac{\bar{A}_h(X)-ah/m}{B(X)},$$

where $B(X)$ is the normalizing factor. In the following, we shall investigate the distribution of $\bar{A}_h'(X)$ when $a_i\,(i=1,2,\cdots)$ are distributed identically with the probability density given by

$$P(x<a_i\leqq x+dx)=\frac{1}{a}e^{-x/a}dx \qquad (x>0), \tag{8}$$
$$=0 \qquad (x\leqq 0).$$

Then we have following

2) Lemma 1 is the theorem given in the paper [2]. Lemma 2 can be proved by the analogous argument with Lemma 1.

THEOREM 2. *Let X_i $(i=1, 2, \cdots)$ be independent random variables having the positive mean m_i, and let a_i $(i=1, 2, \cdots)$ be independent random variables having the probability density* (8). *Assume that* $(1/n)\sum_{i=1}^{n} m_i \to m$ $(n \to \infty)$, *and put*

$$(9) \qquad B^2(X) = \sum_{i=1}^{\infty} \left\{ \frac{1}{X} \int_0^X P(x < S_n \leqq x+h)\, dx \right\}^2.$$

Then the distribution of $\bar{A}'_h(X)$ *will be approximately* $N(0, a^2)$ as $X \to \infty$.

Proof. Put

$$\alpha_n(X) = \frac{1}{X} \int_0^X P(x < S_n \leqq x+h)\, dx.$$

Obviously, for every n, $0 \leqq \alpha_n(X) \leqq 1$. By Lemma 1,

$$(10) \qquad \sum_{n=1}^{\infty} \alpha_n(X) = O(1) \quad \text{as} \quad X \to \infty.$$

First of all, we shall show that

$$(11) \qquad B^2(X) = o(1).$$

Suppose

$$(12) \qquad B^2(X) = \sum_{n=1}^{\infty} \alpha_n^2(X) \neq o(1) \quad \text{as} \quad X \to \infty.$$

Then we have, for some n,

$$(13) \qquad \alpha_n(X) \neq o(1) \quad \text{as} \quad X \to \infty,$$

For, if for every n,

$$\alpha_n(X) = o(1) \quad \text{as} \quad X \to \infty,$$

then

$$B^2(X) \leqq \sup_n \alpha_n(X) \cdot \sum_{n=1}^{\infty} \alpha_n(X) = o(1) \quad \text{as} \quad X \to \infty,$$

which will be contrary to (12). Now, (13) will be rewritten as

$$(14) \qquad \int_0^X P(x < S_n \leqq x+h)\, dx = C_n X + o(X)$$

where C_n is a positive constant. Thus, for some constant $k > 0$,

$$(15) \qquad \int_X^{X+k} P(x < S_n \leqq x+h)\, dx = C_n \cdot k + o(X).$$

On the other hand,

$$\int_X^{X+k} P(x < S_n \leqq x+h) dx \leqq k \cdot P(X < S_n \leqq X+k+h)$$
$$= k \cdot \int_X^{X+k+h} dG_n(x)$$

where $G_n(x)=P(S_n\leqq x)$, and this will tend to zero as $X\to\infty$. This is contrary to (13). Thus, we can conclude that (11) holds. Next, since

$$\sum_{n=1}^{\infty}\alpha_n^3(X)\leqq\sup_n \alpha_n(X)\sum_{n=1}^{\infty}\alpha_n^2(X)=o(1)\cdot B^2(X)=o(B^2(X));$$

repeating this argument, we have

$$\sum_{n=1}^{\infty}\alpha_n^k(X)=o(B^2(X))\quad\text{as}\quad X\to\infty, \tag{16}$$

for every $k\geqq 3$.

Under the above preparations, we shall now enter into the main discourse in proving Theorem 2. Since the characteristic function of $\bar{A}_h'(X)$ is

$$f(t)=\prod_{n=1}^{\infty}\left(\frac{1}{1-ita\,\alpha^n(X)/B(X)}e^{-itah/mB(X)}\right), \tag{17}$$

where the exchange between $(1/X)\int_0^X\cdot dx$ and $\sum_{n=1}^{\infty}$ in (7) may be allowed with probability 1. Taking the logarithm of (17), we have

$$\log f(t)=-\sum_{n=1}^{\infty}\log\left(1-ita\frac{\alpha_n(X)}{B(X)}\right)-it\frac{ah}{mB(X)}.$$

Expanding this and noting (16), we have for large X,

$$\begin{aligned}\log f(t)&\sim\sum_{n=1}^{\infty}\left\{ita\frac{\alpha_n(X)}{B(X)}-\frac{a^2t^2}{2}\frac{\alpha_n^2(X)}{B^2(X)}\right\}-it\frac{ah}{mB(X)}\\&=it\frac{a}{B(X)}\sum_{n=1}^{\infty}\alpha_n(X)-\frac{a^2t^2}{2}\frac{1}{B^2(X)}\sum_{n=1}^{\infty}\alpha_n^2(X)-it\frac{ah}{mB(X)}\\&=\frac{ita}{B(X)}\sum_{n=1}^{\infty}\frac{1}{X}\int_0^X P(x<S_n\leqq x+a)\,dx\\&\qquad-\frac{a^2t^2}{2}\frac{1}{B^2(X)}\sum_{n=1}^{\infty}\alpha_n^2(X)-it\frac{ah}{mB(X)}\\&\sim\frac{a^2t^2}{2}\end{aligned}$$

which is the characteristic function of the normal distribution. Thus the theorem has been proved.

An analogous argument gives the following theorem in the case where $\{X_i\}$ is a sequence of identically distributed random variables. In the case where $\{E(X_i)\}$ is a stable sequence, a similar theorem will be proved under the conditions given by [1].

THEOREM 2′. *Let X_i $(i=1,2,\cdots)$ be independent random variables identically distributed with the mean $m>0$. If we replace $A_h(X)$ and $B(X)$ by*

$$A'(x,h)=\frac{1}{B'(x)}\left\{\sum_{n=1}^{\infty}a_nP(x<S_n\leqq x+h)-\frac{ah}{m}\right\}, \tag{18}$$

(19) $$B'^2(x)=\sum_{n=1}^{\infty}\{P(x<S_n\leqq x+h)\}^2,$$

respectively, then the conclusion of Theorem 2 remains valid.

In the remaing part of the paper, we shall calculate $B'^2(x)$ in the case where tne central limit theorem will hold on $\{X_i\}$.

THEOREM 3.[3] *If X_i $(i=1, 2, \cdots)$ are independent random variables identically distributed and having the mean $m>0$ and the variance v, and if a_i $(i=1, 2, \cdots)$ are independent random variables having the probability density* (8), *then the distribution of*

$$\frac{\sqrt{2}(\pi mvx)^{1/4}}{h}\left\{\sum_{n=1}^{\infty}a_nP(x<S_n\leqq x+h)-\frac{ah}{m}\right\}$$

tends to $N(0, a^2)$ as $x\to\infty$.

Proof. Based on Theorem 2′, it is sufficient to show the relation

(20) $$B'^2(x)=\frac{h^2}{2\sqrt{\pi mv}}x^{-1/2}+o(x^{-1/2}).$$

By the central limit theorem, for large n, the distributions of S_n are approximately $N(nm, nv)$; more precisely, there is a positive constant $N(\varepsilon)$ such as

(21) $$P(x<S_n\leqq x+dx)=\frac{1}{\sqrt{2\pi nv}}e^{-(x-nm)^2/2nv}dx+E(n, x)\,dx$$

and

(22) $$\sum_{n=N}^{\infty}E(n, x)<\frac{\varepsilon}{h^2},\quad \text{uniformly respect to } x,$$

for all $n>N(\varepsilon)$, where ε is a negligibly small positive constant. Now, by the mean value theorem on integral,

(23)
$$\begin{aligned}&\sum_{n=1}^{\infty}\{P(x<S_n\leqq x+h)\}^2\\ &=\sum_{n=1}^{N}\{P(x<S_n\leqq x+h\}^2+\sum_{n=N+1}^{\infty}\frac{h^2}{2\pi nv}e^{-(x+\theta_nh-nm)^2/nv}\\ &\quad+2h^2\sum_{n=N+1}^{\infty}\frac{1}{2\pi nv}e^{-(x+\theta_nh-nm)^2/2nv}E(n, x+\theta'_nh)+h^2\sum_{n=N+1}^{\infty}E^2(n, x+\theta'_nh),\end{aligned}$$

where $0<\theta_n, \theta'_n<1$. Choosing so large $x\gg Nm$ that $P(x<S_n\leqq x+h)<\sqrt{\varepsilon/N}$ $(n=1, 2, \cdots, N)$, we have

3) This theorem is one referred to in [3].

$$\sum_{n=1}^{\infty}\{P(x<S_n\leqq x+h)\}^2=\left\{\int_0^{x/m-\alpha\sqrt{x}/m}+\int_{x/m-\alpha\sqrt{x}/m}^{x/m+\alpha\sqrt{x}/m}+\int_{x/m+\alpha\sqrt{x}/m}^{\infty}\right\}\frac{h^2}{2\pi yv}e^{-(x-my)^2/yv}dy+R$$
$$=I_1+I_2+I_3+R,\quad \text{say,}$$

where $|R|$ is negligibly small relatively to other terms. Putting $y=x/m+t\sqrt{x}/m$,

$$I_2=\frac{2h^2}{2\pi v}\int_0^{\alpha}\frac{1}{\sqrt{x}+t}e^{-t^2/\{(v/m)(1+t/\sqrt{x})\}}dt$$
$$=\frac{h^2}{\pi v}\int_0^{\alpha}\frac{1}{\sqrt{x}(1+t/\sqrt{x})}e^{-mt^2/v}e^{(t^3/v)\cdot O(x^{-1/2})}dt$$
$$(25)\qquad =\frac{h^2}{\pi v}x^{-1/2}\int_0^{\alpha}e^{-mt^2/v}dt(1+t^3O(x^{-1/2}))$$
$$=\frac{h^2}{\pi v}x^{-1/2}\left(\frac{1}{2}\sqrt{2\pi}\sqrt{\frac{v}{2m}}-\delta(\alpha)\right)+o(x^{-1/2})$$
$$=\frac{h^2}{2\sqrt{\pi mv}}x^{-1/2}-\frac{h^2}{\pi v}x^{-1/2}\delta(\alpha)+o(x^{-1/2}).$$

with $\delta(\alpha)\to 0$ as $\alpha\to\infty$. Next

$$(26)\qquad I_3=\frac{h^2}{2\pi v}e^{2xm/v}\int_{x/m+\alpha\sqrt{x}/m}^{\infty}\frac{1}{y}e^{-(1/v)(x^2/y+m^2y)}dy.$$

Now, we shall put the integrand as

$$(27)\qquad \frac{1}{y}e^{-z/v},\quad \text{where}\quad z=\frac{x^2}{y}+m^2y.$$

Since z is a convex function of y for $y>0$, the tangent of the curve at $x/m+\alpha\sqrt{x}/m$ lies below the curve. The equation of the tangent is

$$(28)\qquad z=m^2\frac{2\alpha\sqrt{x}+\alpha^2}{(\sqrt{x}+\alpha)^2}y+n\frac{2x^2}{x+\alpha\sqrt{x}},$$

and we have

$$I_3<\frac{h^2}{2\pi v}e^{(2mx-2mx^2/(x+\alpha\sqrt{x}))/v}\cdot\frac{1}{x/m+\alpha\sqrt{x}/m}\int_{x/m+\alpha\sqrt{x}/m}^{\infty}e^{-(m^2y/v)(2\alpha\sqrt{x}+\alpha^2)/(\sqrt{x}+\alpha)^2}dy$$
$$(29)\qquad =\frac{mh^2}{2\pi v}e^{2\alpha mx\sqrt{x}/(x+\alpha\sqrt{x})\cdot(1/v)}\cdot\frac{v(\sqrt{x}+\alpha)}{m^2\cdot 2\alpha x(1+\alpha/2\sqrt{x})}e^{-(m/v)(2\alpha\sqrt{x}+\alpha^2)/(\sqrt{x}+\alpha)\cdot\sqrt{x}}$$
$$=\frac{mh^2}{2\pi v}\frac{v}{2m^2}\frac{1+\alpha/\sqrt{x}}{1+\alpha/2\sqrt{x}}\alpha^{-1/2}x^{-1/2}e^{(m/v)\cdot(-\alpha^2\sqrt{x})/(\sqrt{x}+\alpha)}$$
$$\sim \text{const}\cdot\alpha^{-1/2}x^{-1/2}+o(x^{-1/2})\qquad \text{as}\quad x\to\infty.$$

Taking α so large that $\text{const}\cdot\alpha^{-1/2}$ is negligibly small compared with $h^2/4\sqrt{\pi mv}$, we have

$$(30)\qquad I_3=o(x^{-1/2}).$$

The same argument shows

$$I_1 = o(x^{-1/2}), \tag{31}$$

hence (24), (25), (30) and (31) imply (20).

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