ABEL SUMMABILITY OF DERIVED CONJUGATE FOURIER SERIES

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The Abel summability of the derived conjugate series has been discussed by Plessner [4], Moursund [3] and Misra [2]. Moursund's result is very complicated and Misra proved a simpler theorem, but it is not general. The object of this note is to prove a simpler and more general theorem. In §1, we shall prove a summability theorem of the conjugate series. This is another result of Misra [1], and our method of the proof is simpler than Misra's. In §2, we shall reduce the summability theorem of the derived conjugate series to the case of §1.

1. Let f(x) be an integrable and periodic function and

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

$$\psi(x,t) \equiv \psi^{(0)}(x,t) = \frac{1}{2}\{f(x+t) - f(x-t)\}$$

$$\sim \sum_{n=1}^{\infty}(b_n \cos nx - a_n \sin nx)\sin nt$$

$$\equiv \sum_{n=1}^{\infty} B_n(x)\sin nt.$$

Since $t/(1+t^2)$ is of bounded variation in $(0,\infty)$ and tends to zero as $t\to\infty$, we have for any fixed $\varepsilon > 0$

$$\int_0^{\infty}\psi(x,t)\,\frac{t/\varepsilon}{1+(t/\varepsilon)^2}\,dt$$

$$= \sum_{n=1}^{\infty} B_n(x)\int_0^{\infty}\frac{t/\varepsilon}{1+(t/\varepsilon)^2}\sin nt\,dt$$

$$= \frac{\pi\varepsilon}{2}\sum_{n=1}^{\infty} B_n(x)e^{-\varepsilon n}.$$

The Abel mean of $\sum B_n(x)$ is

$$V(x,\varepsilon) \equiv \sum_{n=1}^{\infty} B_n(x)e^{-\varepsilon n}$$

$$= \frac{2}{\pi}\int_0^{\infty}\frac{t}{\varepsilon^2+t^2}\psi(x,t)\,dt$$

$$= \frac{2}{\pi}\int_0^{\infty}\psi(x,t)\bar{P}(\varepsilon,t)\,dt,$$

say. We denote by

$\psi_n(x,t)$ the n-th integral of $\psi(x,t)$, then

$$|\psi_n(x,t)| \leqq Mt^{n-1}, \quad \text{as} \quad t\to\infty$$

Since for $n = 0, 1, 2, \cdots$

(a) $$\frac{\partial^n \bar{P}(\varepsilon,t)}{\partial t^n} = O(\varepsilon^{-(n+1)}) \qquad (t \leqq \varepsilon),$$

(b) $$\frac{\partial^n \bar{P}(\varepsilon,t)}{\partial t^n} = O(t^{-(n+1)}) \qquad (t\to\infty),$$

we have, by successive partial integration,

(1) $V(x,\varepsilon)$

$$= \frac{2}{\pi}\Big[\psi_1(x,t)\bar{P}(\varepsilon,t)\Big]_0^{\infty} - \frac{2}{\pi}\int_0^{\infty}\psi_1(x,t)\frac{\partial\bar{P}(\varepsilon,t)}{\partial t}dt$$

$$= -\frac{2}{\pi}\int_0^{\infty}\psi_1(x,t)\frac{\partial\bar{P}(\varepsilon,t)}{\partial t}dt$$

$$= (-1)^n\frac{2}{\pi}\int_0^{\infty}\psi_n(x,t)\frac{\partial^n\bar{P}(\varepsilon,t)}{\partial t^n}dt.$$

Let us put

$$\bar{P}(\varepsilon,t) = \frac{1}{t} - \frac{\varepsilon^2}{t(\varepsilon^2+t^2)} = \frac{1}{t} - Q(\varepsilon,t),$$

then

(c) $$\frac{\partial^n Q(\varepsilon,t)}{\partial t^n} = O(\varepsilon^2 t^{-(n+3)}) \quad \text{as} \quad t\to\infty.$$

If we assume

$$\psi_n(x,t) = o(t^n),$$

$$\bar{V}(x,\varepsilon)$$

$$= (-1)^n \frac{2}{\pi}\int_0^\infty \psi_n(x,t)\frac{\partial^n \bar{P}(\varepsilon,t)}{\partial t^n}dt$$

$$= (-1)^n \frac{2}{\pi}\left\{\int_0^\varepsilon + \int_\varepsilon^\infty\right\}\psi_n(x,t)\frac{\partial^n \bar{P}(\varepsilon,t)}{\partial t^n}dt$$

$$= (-1)^n \frac{2}{\pi}\left\{\int_0^\varepsilon \psi_n(x,t)\frac{\partial^n \bar{P}}{\partial t^n}dt\right.$$

$$\left.+(-1)^n\cdot n!\int_\varepsilon^\infty \frac{\psi_n(x,t)}{t^{n+1}}dt - \int_\varepsilon \psi_n(x,t)\frac{\partial^n Q}{\partial t^n}dt\right\}$$

$$= I + J + K,$$

say. From (a),

$$I = \int_0^\varepsilon o(t^n)O(\varepsilon^{-(n+1)})dt = o(1)$$

and

$$K = \int_\varepsilon^\infty \psi_n(x,t)\frac{\partial^n Q}{\partial t^n}dt$$

$$= \int_\varepsilon^\delta + \int_\delta^\infty = K_1 + K_2,$$

say. From (c),

$$K_1 = \int_\varepsilon^\delta \psi_n \frac{\partial^n Q}{\partial t^n}dt$$

$$= \int_\varepsilon^\delta o(t^n)O(\varepsilon^2 t^{-(n+3)})dt$$

$$= o(\varepsilon^2)\int_\varepsilon^\delta \frac{dt}{t^3} = o(1),$$

and

$$K_2 = \int_\delta^\infty O(t^{n-1})O(\varepsilon^2 t^{-(n+3)})dt$$

$$= O(\varepsilon^2)\int_\delta^\infty \frac{dt}{t^4} = \varepsilon^2 \to 0.$$

Thus if $\psi_n(x,t) = o(t^n)$, we get

$$\bar{V}(x,\varepsilon) - \frac{2}{\pi}n!\int_\varepsilon^\infty \frac{\psi_n(x,t)}{t^{n+1}}dt$$

$$\to 0 \quad \text{as} \quad \varepsilon \to 0.$$

On the other hand, since

$$\int_\varepsilon^\infty \frac{\psi_n(x,t)}{t^n}dt$$

$$= \left[\frac{\psi_n(x,t)}{t^n}\right]_\varepsilon^\infty + n\int_\varepsilon^\infty \frac{\psi_n(x,t)}{t^{n+1}}dt,$$

we have

$$\int_\varepsilon^\infty \frac{\psi_{n-1}(x,t)}{t^n}dt - n\int_\varepsilon^\infty \frac{\psi_n(x,t)}{t^{n+1}}dt$$

$$\to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Thus we get the following theorem:

Theorem 1. If

$$\bar{V}(x,\varepsilon) = \sum_{n=1}^\infty (b_n \cos nx - a_n \sin nx)e^{-\varepsilon n},$$

then

$$\lim_{\varepsilon\to 0}\left[\bar{V}(x,\varepsilon) - \frac{2}{\pi}(n-1)!\int_\varepsilon^\infty \frac{\psi_{n-1}(t)}{t^n}dt\right]$$

$$= 0,$$

provided that

$$\psi_n(x,t) = o(t^n),$$

where $\psi_n(x,t)$ is the n-th integral of $\psi(x,t) = \frac{1}{2}\{f(x+t) - f(x-t)\}$ and n is a positive integer.

2. Concerning with the derived conjugate series, it is legitimate to differentiate under the integral. So

$$\frac{\partial}{\partial x}\int_0^\infty f(x+t)\frac{t}{\varepsilon^2+t^2}dt$$

$$= \frac{\partial}{\partial x}\int_x^\infty f(t)\frac{t-x}{\varepsilon^2+(t-x)^2}dt$$

$$= \int_x^\infty f(t)\frac{\partial}{\partial x}\left(\frac{t-x}{\varepsilon^2+(t-x)^2}\right)dt + f(t)\bar{P}(\varepsilon,0)$$

$$= -\int_0^\infty f(x+t)\frac{\partial}{\partial t}\left(\frac{t}{\varepsilon^2+t^2}\right)dt$$

$$= -\int_0^\infty f(x+t)\frac{\partial \bar{P}(\varepsilon,t)}{\partial t}dt$$

and

$$\frac{\partial}{\partial x}\int_0^\infty f(x-t)\frac{t}{\varepsilon^2+t^2}dt$$

$$= \int_0^\infty f(x-t) \frac{\partial \bar{P}(\varepsilon,t)}{\partial t} dt$$

Since

$$\int_0^\infty \frac{\partial \bar{P}(x,t)}{\partial t} dt = 0,$$

if we put

$$\psi^{(1)}(x,t) = \frac{1}{2}\left\{f(x+t)+f(x-t)-2\alpha_0\right\},$$

where α_0 is a constant, then

$$\frac{\partial V(x,\varepsilon)}{\partial x}$$

$$= -\frac{2}{\pi}\int_0^\infty \psi^{(1)}(x,t)\frac{\partial \bar{P}(\varepsilon,t)}{\partial t}dt.$$

More generally, if we put

$$\psi^{(r)}(x,t)$$

$$= \frac{1}{2}\Big\{f(x+t)+(-1)^{r+1}f(x-t)$$

$$- 2\sum_{k=0}^{[r-\frac{1}{2}]} \alpha_{r-1-2k}\, t^{r-1-2k}\Big\},$$

where α_i is a constant, then

$$\frac{\partial^r V(x,\varepsilon)}{\partial x^r}$$

$$= (-1)^r \frac{2}{\pi}\int_0^\infty \psi^{(r)}(x,t)\, \frac{\partial^r \bar{P}(\varepsilon,t)}{\partial t^r} dt.$$

It is easy to see

$$\psi^{(r)}(x,t) \leqq M t^{r-1}$$

and, by the partial integration, we get

$$(2) \qquad \frac{\partial^r V(x,\varepsilon)}{\partial x^r}$$

$$= (-1)^{r+n}\frac{2}{\pi}\int_0^\infty \psi_n^{(r)}(x,t)\, \frac{\partial^{r+n}\bar{P}(\varepsilon,t)}{\partial t^{r+n}}dt.$$

Since this formula is the type of formula (1), we have the following theorem which is proved by the analogous method.

Theorem 2. If

$$V(x,\varepsilon) = \sum_{n=1}^\infty (b_n \cos nx - a_n \sin nx)\, e^{-\varepsilon n},$$

then

$$\lim_{\varepsilon\to 0}\left[\frac{\partial^r V(x,\varepsilon)}{\partial x^r} - \frac{2}{\pi}(n+r-1)!\int_\varepsilon^\infty \frac{\psi_{n-1}^{(r)}(x,t)}{\partial t^{n+r}}dt\right]$$

$$= 0$$

provided that

$$\psi_n^{(r)}(x,t) = o(t^{r+n}),$$

where $\psi_n^{(r)}(x,t)$ is the n-th integral of

$$\psi^{(r)}(x,t) = \frac{1}{2}\Big\{f(x+t)+(-1)^{r+1}f(x-t)$$

$$- 2\sum_{k=0}^{[r-\frac{1}{2}]} \alpha_{r-1-2k}\, t^{r-1-2k}\Big\}$$

and α_{r-1-2k}'s are constants.

3. Under the hypothesis of theorem 2, if we assume that $\psi^{(r)}(x,t)/t^r$ is integrable in the sence of Cauchy-Lebesgue, we get

Theorem 3. Under the hypothesis of theorem 2, if $\psi^{(r)}(x,t)/t^r$ is integrable in Cauchy-Lebesgue sence, then

$$\lim_{\varepsilon\to 0}\Big[\frac{\partial^r V(x,\varepsilon)}{\partial x^r}$$

$$- (C, n-1)\frac{2}{\pi}r!\int_\varepsilon^\infty \frac{\psi^{(r)}(x,t)}{t^{r+1}}dt\Big]$$

$$= 0,$$

where

$$(C, n-1)\int_\varepsilon^\infty \frac{\psi^{(r)}(x,t)}{t^{r+1}}dt$$

means the n-1-th Cesàro mean of conjugate integral

$$\int_\varepsilon^\infty \frac{\psi^{(n)}(x,t)}{t^{r+1}}dt.$$

Let us put

$$L_n(t) = \frac{2}{\pi}(n+r-1)!\,\frac{\psi_n^{(r)}(x,t)}{t^{n+r}},$$

then, by the Cauchy integrability,

$$\int_0^t \frac{\psi_{n-1}^{(r)}(u)}{u^{n+r-1}}du$$

$$= \left[\frac{\psi_n^{(r)}(u)}{u^{n+r-1}} \right]_0^t + (n+r-1) \int_0^t \frac{\psi_n^{(r)}(u)}{u^{n+r}} du$$

$$= \frac{\psi_n^{(r)}(x,t)}{t^{n+r-1}} + (n+r-1) \int_0^t \frac{\psi_n^{(r)}(u)}{u^{n+r}} du$$

and

$$\frac{1}{t} \int_0^t \frac{\psi_{n-1}^{(r)}(u)}{u^{n+r-1}}$$

$$= \frac{\psi_n^{(r)}(x,t)}{t^{n+r}} + (n+r-1) \frac{1}{t} \int_0^t \frac{\psi_n^{(r)}(u)}{u^{n+r}} du.$$

If $L_n(t) \to 0$, as $t \to 0$, then successively, we get

$$(3) \quad L_{n-1}(t) \to 0 \quad (C,1),$$

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$$L_0(t) \to 0 \quad (C,n),$$

that is

$$\psi^{(r)}(x,t)/t^r \to 0 \quad (C,n)$$

as $t \to 0$.

The method of reduction is well-known; see Misra [1]. Put

$$\underline{K}_n(\varepsilon) = \frac{2}{\pi}(n+r)! \int_\varepsilon^\infty \frac{\psi_n^{(r)}(x,t)}{t^{n+r+1}} dt,$$

then, by the integration by part,

$$(4) \quad \underline{K}_n(\varepsilon) = L_n(\varepsilon) + \underline{K}_{n-1}(\varepsilon).$$

On the other hand, since we can see

$$\varepsilon \underline{K}_{n-1}(\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

$$(5) \quad \int_0^\varepsilon \underline{K}_{n-1}(t) dt$$

$$= \varepsilon \underline{K}_{n-1}(\varepsilon) + (n+r-1) \int_0^\varepsilon L_{n-1}(t) dt.$$

From (4) and (5),

$$\underline{K}_n(\varepsilon)$$

$$= L_n(\varepsilon) + \frac{1}{\varepsilon} \int_0^\varepsilon \underline{K}_{n-1}(t) dt$$

$$- (n+r-1) \frac{1}{\varepsilon} \int_0^\varepsilon L_{n-1}(t) dt$$

$$= (C,1) \underline{K}_{n-1}(\varepsilon) + L_n(\varepsilon) - \alpha (C,1) L_{n-1}(\varepsilon),$$

where α is a fixed constant. Continuing this reduction formula and by (3), we get $\underline{K}_n(\varepsilon)$ and $(C,n)\underline{K}_0(\varepsilon)$ is equi-convergent, and it is easy to see that this is equi-convergent with

$$(C,n-1) \frac{2}{\pi} r! \int_\varepsilon^\infty \frac{\psi^{(r)}(x,t)}{t^r} dt,$$

under the condition $\psi_n^{(r)}(x,t) = o(t^{r+n})$.

Literatures

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