ON THE SINGULARITIES OF THE DIFFERENTIAL EQUATION

$$\frac{d^2y}{dx^2} + f(x,y)\frac{dy}{dx} + g(x,y) = P(x)$$

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§ 1

1. In this section we shall consider the differential equation

$$(1) \quad \frac{d^2y}{dx^2} + f(y)\frac{dy}{dx} + g(y) = P(x),$$

where f(y) and g(y) are polynomials of degree n and m respectively, i.e.,

$$f(y) = ay^n + by^{n-1} + \cdots + c,$$
$$a \neq 0$$

and

$$g(y) = \alpha y^m + \beta y^{m-1} + \cdots + \gamma,$$
$$\alpha \neq 0$$

and P(x) is a regular and single-valued function of x in certain neighborhood D of x^* on the x-plane. If we put $dy/dx = z$, we have a simultaneous equation

$$\begin{cases} \dfrac{dy}{dx} = z \\ \dfrac{dz}{dx} = P(x) - f(y)z - g(y). \end{cases}$$

Since the right hand side of it is regular in certain domain containing (x^*, y^*, z^*) in virtue of the hypotheses, there exists the one and only one regular solution through the point (x^*, y^*, z^*). If we continue the solution along a curve C, we may encounter a singular point or tend to the point at infinity. Hence the analytic continuation carries out a problem of singularities. In the sequel we shall exclusively consider a problem of isolated singularities which will appear as essential singularities, poles or branch points. And we always exclude the cases where $n = 0$ and $m = 0$ or 1.

We suppose that we can continue a solution $y = y(x)$ of (1) along any curve C up to a point x_0, but not beyond it. Further we suppose that, if we approach to x_0 along C, $y = y(x)$ tends to ∞. Then, the point $x = x_0$ is an isolated singularity and it may be a branch point. Then, we make a change of variable $x - x_0 = t^k$ if x_0 is finite and $x = t^{-k}$ if $x_0 = \infty$, where k is a positive integer not equal to zero and t is a local parameter which uniformize the solution in a neighborhood of x_0. Then, it follows from the equation (1) that

$$(2) \quad \frac{d^2y}{dt^2} + \left(kt^{k-1}f(y) - \frac{k-1}{t}\right)\frac{dy}{dt} + k^2t^{2(k-1)}g(y) = k^2t^{2(k-1)}P(x_0+t^k)$$

if $x - x_0 = t^k$, and

$$(3) \quad \frac{d^2y}{dt^2} + \left(\frac{k+1}{t} - \frac{kf(y)}{t^{k+1}}\right)\frac{dy}{dt} + \frac{k^2g(y)}{t^{2(k+1)}} = \frac{k^2P(t^{-k})}{t^{2(k+1)}}$$

if $x = t^{-k}$. According to the hypotheses, the solution of (2) is of the form

$$(4) \quad y = \sum_{\nu=-r}^{\infty} a_\nu t^\nu, \qquad a_{-r} \neq 0, \quad r \geqq 1.$$

Substituting (4) into (2), we obtain

$$\frac{r(r+1)a_{-r}}{t^{r+2}} + \cdots$$

$$(5)\quad +\frac{(k-1)r a_r}{t^{r+2}}-\frac{kr a_r^{n+1}}{t^{nr+r+2-k}}+\cdots$$

$$+\frac{k^2 a_r^m}{t^{mr-2k+2}}+\cdots = k^2 t^{2(k-1)} P(x_0+t^k).$$

In order to determine the highest negative power in (5), we put

$$A = r+2, \qquad B = nr+r+2-k,$$

$$C = mr - 2k + 2.$$

Then,

$$B - A = nr - k,$$

$$B - C = (n-m+1)r + k.$$

i) If $n \geqq m-1$, i.e., if $B > C$, we must have $B = A$, i.e., $k = nr$ since the right hand side of (5) is regular at $t = 0$. We shall show that $r = 1$ and $k = n$ if $n \geqq m-1$. In fact, since $B > C$, we have

$$t^C\left(\left(\frac{d^2y}{dt^2}-\frac{k-1}{t}\frac{dy}{dt}\right)+kt^{k-1}f(y)\right)+O(1)=0.$$

in the neighborhood of $t = 0$. Then, the circumstances of the singularity $t = 0$ are equivalent to the equation

$$t^C\left(\frac{d^2y}{dt^2}-\frac{k-1}{t}\frac{dy}{dt}+kt^{k-1}f(y)\right)=0.$$

Dividing this equation by $k^2t^{2(k-1)}$ and returning to the original variables, we have

$$\frac{d^2y}{dx^2}+f(y)\frac{dy}{dx}=0.$$

Integrating this equation and choosing a suitable branch, we obtain a solution

$$y=\sum_{\nu=-1}^{\infty} a'_\nu (x-x_0)^{\frac{\nu}{n}},$$

$$a'_{-1} \neq 0,$$

from which we obtain $k = n$ and $r = 1$.

If k is not equal to a multiple of n, the following two cases will occur.

ii) $B = C > A$. By multiplying t^A, we obtain

$$t^A\left(kt^{k-1}f(y)\frac{dy}{dt}+k^2t^{2(k-1)}g(y)\right)+O(1)=0.$$

Then, returning to the original variables, the above equation is equivalent to

$$\frac{f(y)}{x}\frac{dy}{dx}+g(y)=0$$

at $x = x_0$. If $m > n+1$, integrating this equation, we have

$$y=\sum_{\nu=-1}^{\infty} a'_{-\nu}(x-x_0)^{\frac{2\nu}{m-n-1}},$$

$$a'_{-1} \neq 0.$$

Hence, we obtain $k = m-n-1$ and $r = 2$ if $m-n-1$ is odd and $k = (m-n-1)/2$ and $r = 1$ if $m-n-1$ is even.

iii) $A = C > B$. By multiplying t^B, we obtain

$$t^B\left(\frac{d^2y}{dt^2}-\frac{k-1}{t}\frac{dy}{dt}+k^2t^{2(k-1)}g(y)\right)+O(1)=0.$$

Then, returning to the original variables, we obtain

$$\frac{d^2y}{dx^2}+g(y)=0$$

at $x = x_0$. This equation corresponds to the case $n = 0$ and $a = 0$. Integrating this equation we have a solution

$$y=\sum_{\nu=-1}^{\infty} a'_\nu (x-x_0)^{\frac{2\nu}{m-1}},$$

$$a'_1 \neq 0.$$

Hence, we have $k = (m-1)/2$ and $r = 1$ if m is odd and $k = m-1$ and $r = 2$ if m is even. Then, we have the following

Theorem 1. We suppose that in the equation (1) $f(y)$ and $g(y)$ are the polynomials of degree n and m respectively and $P(x)$ is a regular and single-valued function in a certain domain D. Further we suppose that we can continue analytically a solution of (1) up to a finite point x_0 along any curve from a point, at which the solution is regular, but not beyond x_0. If the solution tends to ∞ as we approach to x_0, there exists a solution of (1) of such a form that

i) $$y=\sum_{\nu=-1}^{\infty} a_\nu (x-x_0)^{\frac{\nu}{n}}$$

if $n \geq m-1$,

ii) $$y=\sum_{\nu=-1}^{\infty} a_\nu (x-x_0)^{\frac{2\nu}{m-n-1}}$$

if $n < m-1$,

Remark: If $P(x)$ is accidentally uniformized by a local parameter t, it is unnecessary that $P(x)$ is regular and single-valued. That is, $P(x)$ may be multiple-valued and may have $x=x_0$ or $x=\infty$ as a pole or branch point. In the following theorems, this remark will remain valid.

2. Now, we consider the case, $k=1$, that is, $x=x_0$ is a pole, but not a branch point. Then, we have

$$B-A=nr-1,$$

$$B-C=(n-m+1)r+1.$$

By the same reason as above, we have the following

Theorem 2. In order that x_0 is not a branch point, but a pole, it is necessary and sufficient that

i) $\begin{cases} n \geq 2 \\ m=n+2 \end{cases}$

ii) $\begin{cases} n=1 \\ m=0,\ 1,\ 2,\ 3,\ 4 \end{cases}$

iii) $\begin{cases} n=0 \\ m=2,\ 3 \end{cases}$

§ 2

In this section we consider the equation

(6) $$\frac{d^2y}{dx^2}+\frac{f(y)}{x}\frac{dy}{dx}+g(y)=P(x)$$

and

(7) $$\frac{d^2y}{dx^2}+\frac{f(y)}{x}\frac{dy}{dx}+\frac{g(y)}{x^2}=P(x).$$

We suppose that the hypotheses in § 1 concerning $f(y)$, $g(y)$ and $P(x)$ are satisfied. Then, there exists a solution regular and single-valued in certain neighborhood of $x=0$ with an exception of $x=0$.

1. At the outset, we suppose that $x=0$ is not a branch point, but a pole of the solution. Then, the solution is of the form

(8) $$y=\sum_{\nu=-r}^{\infty} a_\nu x^\nu,$$

$$a_{-r} \neq 0, \quad r \geq 1.$$

Substituting (8) into (6), we obtain

$$\frac{r(r+1)a_{-r}}{x^{r+2}}+\cdots$$
$$-\frac{r\,a\,a_{-r}^{n+1}}{x^{nr+r+2}}+\cdots$$
$$+\frac{\alpha a_{-r}^{m}}{x^{mr}}+\cdots \qquad =P(x).$$

As in § 1, we put

$$A=r+2, \qquad B=nr+r+2,$$

$$C=mr.$$

Then, we have

$$B-A=nr,$$

$$B-C=2+(n+1-m)r.$$

We distinguish two cases:

i) $B-A=0$. Then, we have $n=0$ since $r \geq 1$. It follows from $B-C \geq 0$ that $2+(1-m)r \geq 0$, i.e., $(m-1)r \leqq 2$.
Hence we have

$$\begin{cases} m=2 \\ r=1 \end{cases}, \quad \begin{cases} m=2 \\ r=2 \end{cases}, \quad \begin{cases} m=3 \\ r=1 \end{cases}.$$

ii) $B-A>0$. Then, $n \geq 1$, $r \geq 1$ and $B-C=0$, i.e., $(m-n-1)r=2$. Then, it is necessary that $m-n-1>0$ in order that $x=0$ is a pole. Since $r \geq 1$, we obtain

$$\begin{cases} m = n+2 \\ r = 2 \end{cases}, \quad \begin{cases} m = n+3 \\ r = 1 \end{cases}.$$

iii) If $n > 0$, $n = m - 1$ and r is finite, we have $n = 0$, which is a contradiction. Hence, $r = \infty$ if $n > 0$ and $n = m - 1$, that is, $x = 0$ is an essential singularity.

Theorem 3. In order that in the equation (6) $x = 0$ is not a branch point, but a pole, it is necessary and sufficient that

i) $\begin{cases} n = 0 \\ m = 2 \end{cases}, \quad \begin{cases} n = 0 \\ m = 2 \end{cases}, \quad \begin{cases} n = 0 \\ m = 3 \end{cases}.$

ii) $\begin{cases} n > 0 \\ m = n+2 \end{cases}, \quad \begin{cases} n > 0 \\ m = n+3 \end{cases}.$

Corollary 1. If $n > 0$ and $n = m - 1$, $x = 0$ is an essential singularity.

Corollary 2. If $n > 0$ and $n > m - 1$, $x = 0$ is a regular point.

2. We suppose that $x = 0$ is a branch point. If we make a change of variable $x = t^k$, where k is a positive integer and t is a local parameter which uniformize the solution of (6), the equation (6) leads to

$$\frac{d^2y}{dt^2} + \frac{kf(y) - k + 1}{t}\frac{dy}{dt} + k^2 t^{2(k-1)} g(y) = k^2 t^{2(k-1)} P(t^k). \tag{9}$$

The solution are supposed to be of the form
$$y = \sum_{\nu=-r}^{\infty} a_\nu t^\nu,$$
$$a_{-r} \neq 0, \quad r \geq 1.$$

Substituting it into (9), we obtain

$$\frac{r(r+1)a_{-r}}{t^{r+2}} + \cdots - \frac{(k-1)a_{-r}}{t^{r+2}} + \frac{k a a_{-r}^{n+1}}{t^{nr+r+2}} + \cdots + \frac{k^2 \alpha a_{-r}^m}{t^{mr-2k+2}} + \cdots = k^2 t^{2(k-1)} P(t^k).$$

As in § 1, we put

$$A = r + 2, \qquad B = nr + r + 2,$$
$$C = mr - 2k + 2.$$

Then,

$$B - A = nr,$$
$$B - C = (n - m + 1)r + 2k.$$

i) $B - A > 0$, i.e., $n \geq 1$. Then, we have $B - C = 0$,

$$2k = (m - n - 1)r.$$

Since $k \geq 1$, we obtain $m > n + 1$. We shall show that $r = 1$ or 2. In fact, multiplying t^A, we obtain

$$t^A\left(\frac{kf(y)}{t}\frac{dy}{dt} + k^2 t^{2(k-1)} g(y)\right) + O(1) = 0.$$

Then, the circumstances of the singularity $t = 0$ are equivalent to the equation

$$t^A\left(\frac{kf(y)}{t}\frac{dy}{dt} + k^2 t^{2(k-1)} g(y)\right) = 0.$$

Returning to the original variables, we obtain

$$\frac{f(y)}{x}\frac{dy}{dx} + g(y) = 0.$$

Integrating this equation, there exists a solution such that

$$y = \sum_{\nu=-1}^{\infty} a'_\nu x^{\frac{2\nu}{m-n-1}},$$
$$a'_{-1} \neq 0.$$

Hence, we obtain $k = (m - n - 1)/2$ and $r = 1$ if $m - n - 1$ is even and $k = m - n - 1$ and $r = 2$ if $m - n - 1$ is odd.

If $n \geq 1$ and $m \leq n + 1$, then there exists no solutions having the point $x = 0$ as a pole or branch point.

ii) $B - A = 0$, i.e., $n = 0$. Then, $B \geq C$. If $f(y) \equiv a \neq 0$, we consider the equation

$$\frac{d^2y}{dx^2} + \frac{a}{x}\frac{dy}{dx} + g(y) = P(x).$$

This equation may have a logarithmic branch point at $x = 0$ if we choose a suitable a.

If $a = 0$, i.e., if $f(y) \equiv 0$, we consider the equation

$$\frac{d^2y}{dx^2} + g(y) = P(x).$$

Then, there exists a solution of such a form that

$$y = \sum_{\nu=-1}^{\infty} a'_\nu\, x^{\frac{2\nu}{m-1}},$$

$$a'_{-1} \neq 0.$$

Hence, we have $k = (m - 1)/2$ and $r = 1$ if m is odd and $k = m - 1$ and $r = 2$ if m is even.

Theorem 4. In order that in the equation (6) $x = 0$ is a branch point, it is necessary and sufficient that $m > n + 1$. Then, there exists a solution of such a form that

$$y = \sum_{\nu=-1}^{\infty} a_\nu\, x^{\frac{2\nu}{m-n-1}},$$

$$a_{-1} \neq 0.$$

if $n > 0$.

If $n = 0$ and $f(y) \not\equiv 0$, $x = 0$ may be a logarithmic branch point.

If $n = 0$ and $f(y) \equiv 0$, there exists a solution of such a form that

$$y = \sum_{\nu=-1}^{\infty} a_\nu\, x^{\frac{2\nu}{m-1}},$$

$$a_{-1} \neq 0.$$

Corollary 1. If $n > 0$ and $n = m - 1$, $x = 0$ is an essential singularity.

Corollary 2. If $n > 0$ and $n > m - 1$, $t = 0$ is a regular point.

3. We consider the equation (7). If we make a change of variable $x = \exp t$, we have

$$(10)\quad \frac{d^2y}{dt^2} + (f(y)-1)\frac{dy}{dt} + g(y) = e^{2t}P(e^t).$$

Hence, we can apply the same method in §1 to the equation (10) and obtain the analogous results. If we return to the original equation (6), there exists a logarithmic branch point.

§ 3

In this section we shall consider the properties at the point at infinity.

1. Since the point at infinity $x = \infty$ corresponds to the point $t = 0$ by $x = t^{-k}$, the properties of $x = \infty$ are reduced to those of $t = 0$. We consider the equation

$$(11)\quad \frac{d^2y}{dx^2} + f(y)\frac{dy}{dx} + g(y) = 0.$$

By making use of a change of variable $x = t^{-k}$, we have

$$(12)\quad \frac{d^2y}{dt^2} + \left(\frac{k+1}{t} - \frac{k f(y)}{t^{k+1}}\right)\frac{dy}{dt} + \frac{k^2 g(y)}{t^{2(k+1)}} = 0.$$

We suppose that we can analytically continue a solution from a regular point along any curve C up to $t = 0$, but not beyond it, and if we approach to $t = 0$ along C the solution tends to ∞. Then, $t = 0$ will be a pole. We suppose the solution is of such a form that

$$(13)\quad y = \sum_{\nu=-r}^{\infty} a_\nu t^\nu, \qquad a_{-r} \neq 0, \quad r \geq 1.$$

Substituting (13) into (12), we obtain

$$\frac{r(r+2)a_{-r}}{t^{r+2}} + \cdots - \frac{r(k+1)a_{-r}}{t^{r+2}} + \frac{rka\,a_{-r}^{n+1}}{t^{nr+r+2+k}} + \cdots + \frac{\alpha k^2 a_{-r}^m}{t^{mr+2k+2}} + \cdots = 0.$$

As in § 1, we put

$A = r + 2$, $\qquad B = nr + r + 2 + k$,

$C = mr + 2k + 2$.

Then,

$$B - A = nr + k,$$

$$B - C = (n - m + 1)r - k.$$

Since $k \geqq 1$, we have $B - C = 0$, i.e., $k = (n - m + 1)r$, which show that $n > m - 1$ is necessary in order that $t = 0$ is to be a pole. By multiplying t^A, we obtain

$$t^A\left(-\frac{kf(y)}{t^{k+1}}\frac{dy}{dt} + \frac{k^2 g(y)}{t^{2(k+1)}}\right) + O(1) = 0.$$

Hence, the circumstances at $t = 0$ is equivalent to the equation

$$t^A\left(-\frac{kf(y)}{t^{k+1}}\frac{dy}{dt} + \frac{k^2 g(y)}{t^{2(k+1)}}\right) = 0.$$

Returning to the original variables, we obtain

$$f(y)\frac{dy}{dt} + g(y) = 0.$$

Integrating this equation, we have the solution of the form

$$y = \sum_{\nu=-\infty}^{1} a'_\nu x^{\frac{\nu}{n-m+1}},$$

$$a'_1 \neq 0.$$

which shows $k = n - m + 1$ and $r = 1$. Hence, if $t = 0$ is not a branch point, but a pole, it is necessary and sufficient that $n = m$.

Theorem 5. We suppose that in the equation (10) the hypotheses in § 1 concerning $f(y)$ and $g(y)$ are satisfied. Further we suppose that we can analytically continue a solution along any curve C up to $x = \infty$ and the solution tends to ∞ if we approach to $x = \infty$ along C. Then, it is necessary and sufficient $n > m - 1$ in order that $x = \infty$ is not an essential singularity, but a pole or branch point.

If $n > m - 1$, there exists a solution of such a form that

$$y = \sum_{\nu=-\infty}^{1} a_\nu x^{\frac{\nu}{n-m+1}},$$

$$a_1 \neq 0.$$

Corollary 1. If $x = \infty$ is not a branch point, but a pole, it is necessary and sufficient that $n = m$.

Corollary 2. If $n = m - 1$, there exists a solution having $x = \infty$ as an essential singularity.

Corollary 3. If $n < m - 1$, $t = 0$ is a regular point.

2. Now, we consider the equation

$$(14) \qquad \frac{d^2y}{dx^2} + \frac{f(y)}{x}\frac{dy}{dx} + g(y) = 0.$$

By making use of a change of variable $x = t^{-k}$, we have

$$(15) \qquad \frac{d^2y}{dt^2} + \frac{k+1-kf(y)}{t}\frac{dy}{dt} + \frac{k^2 g(y)}{t^{2(k+1)}} = 0.$$

If a solution is of such a form that

$$y = \sum_{\nu=-r}^{\infty} a_\nu t^\nu,$$

$$a_{-r} \neq 0, \quad r \geqq 1,$$

we substitute it into (15) and obtain, as in § 1,

$$\frac{r(r+1)a_{-r}}{t^{r+2}} + \cdots$$

$$- \frac{r(k+1)a_{-r}}{t^{r+2}} - \frac{a r k a_{-r}^{n+1}}{t^{nr+r+2}} + \cdots$$

$$+ \frac{\alpha k^2 a_{-r}^m}{t^{mr+2k+2}} + \cdots = 0.$$

As in § 1, we put

$$A = r + 2, \qquad B = nr + r + 2,$$

$$C = mr + 2k + 2.$$

Then,

$$B - A = nr,$$

$$B - C = (n - m + 1)r - 2k.$$

i) $B - A > 0$, i.e., $n \geqq 1$, $r \geqq 1$. Then, $(n - m + 1)r = 2k$. Hence it necessary that $n > m - 1$ since $k > 0$. By multiplying t^A, we have

$$t^A\left(\frac{-kf(y)}{t}\frac{dy}{dt} + \frac{k^2 g(y)}{t^{2(k+1)}}\right) + O(1) = 0.$$

Returning to the original variables, the circumstances at $x = \infty$ is equivalent to

$$\frac{f(y)}{x}\frac{dy}{dx} + g(y) = 0.$$

Integrating this equation, we obtain

$$y = \sum_{\nu=-\infty}^{1} a'_\nu \, x^{\frac{2\nu}{n-m+1}},$$

$$a'_1 \neq 0.$$

Hence, we have $k = (n - m + 1)/2$ and $r = 1$ if $n - m + 1$ is even, and $k = n - m + 1$ and $r = 2$ if $n - m + 1$ is odd.

ii) $B - A = 0$, i.e., $n = 0$ and $r \geqq 1$. Then, $B \geqq C$, i.e., $2k \leqq (1 - m) \times r$. Since $k \geqq 1$ and $r \geqq 1$, we obtain $m = 0$, the case which we exclude.

Theorem 6. We suppose that the hypotheses in Theorem 5 are satisfied. Then, it is necessary and sufficient that $n > m - 1$. Then, there exists a solution of such a form that

$$y = \sum_{\nu=-\infty}^{1} a_\nu \, x^{\frac{2\nu}{n-m+1}},$$

$$a_1 \neq 0.$$

Corollary 1. If $n = m - 1$, $x = \infty$ is an essential singularity.

Corollary 2. If $n < m - 1$, $t = 0$ is a regular point.

3. We consider the case $k = 1$, that is, $x = \infty$ is not a branch point, but a pole. Then,

i) $n > 0$, $(n - m + 1)r = 2$.

Since r is a positive integer larger than 0, we have

$$\begin{cases} n = m \\ r = 2, \end{cases} \quad \begin{cases} n = m + 1 \\ r = 1 \end{cases}$$

ii) $n = 0$, i.e., $(m - 1)r \leqq 2$.

Then, we obtain

$$\begin{cases} m = 2 \\ r = 1, \end{cases} \quad \begin{cases} m = 2 \\ r = 2, \end{cases} \quad \begin{cases} m = 3 \\ r = 1. \end{cases}$$

Theorem 7. In order that $x = \infty$ is not a branch point, but a pole, it is necessary and sufficient that the following relations hold goods:

i) $\begin{cases} n > 0 \\ n = m, \end{cases} \quad \begin{cases} n > 0 \\ n = m + 1 \end{cases}$

ii) $\begin{cases} n = 0 \\ m = 2, \end{cases} \quad \begin{cases} n = 0 \\ m = 2, \end{cases} \quad \begin{cases} n = 0 \\ m = 3. \end{cases}$

Corollary 1. If $n = m - 1$, $x = \infty$ is an essential singularity.

Corollary 2. If $n < m - 1$, $t = 0$ is a regular point.

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