NOTE ON UNIPOTENT INVERSIBLE SEMIGROUPS[1]

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A semigroup with only one idempotent is called unipotent [2]. In this note we shall investigate the construction of unipotent inversible semigroup (defined as below). After all the study of such a semigroup will be reduced to that of a zero-semigroup [3].

Lemma 1. A semigroup is unipotent if and only if it contains the greatest group [4].

Proof. Suppose that a semigroup S has its greatest group G, and S contains idempotents e and f. Then, since $\{e\}$ and $\{f\}$ are groups in S, we see that $\{e\} \subset G$ and $\{f\} \subset G$; e and f are idempotents contained in G. Hence $e = f$; S is unipotent. Conversely, if S is unipotent, S has at least one group as a subsemigroup. Let $\{G_\alpha\}$ $(\alpha \in \Gamma)$ be the set of all groups in S. Since every G_α has the idempotent e of S in common, the semigroup G generated by all G_α $(\alpha \in \Gamma)$ is proved to be a group. It is easy to see that G is greatest.

When a unipotent semigroup S, for example, is finite, the greatest group G is represented as $G = Se$ where e is an idempotent. What is the necessary and sufficient condition in order that Se is the greatest group of S ?

Let S be a unipotent semigroup with an idempotent e. If, for any $a \in S$, there exists $b \in S$ such that $ab = e$ $(ba = e)$, S is called right (left) inversible, and b is a right (left) inverse of a. Of course b depends on a. Then since e is a right (left) zeroid [5] of S, a unipotent right (left) inversible semigroup is equivalent to a unipotent semigroup with zeroids [5]. The following lemmas follow immediately from the general theories of a semigroup with zeroids.

Lemma 2. Let S be a unipotent semigroup. The following conditions are equivalent.

(1) S is right inversible.
(2) S is left inversible.
(3) Se is a group.
(4) eS is a group.

We need no distinction between right inversibleness and left inversibleness. If S is right or left inversible, it is said to be inversible.

Lemma 3. Let S be a unipotent inversible semigroup, and G be its greatest group.

(1) $G = Se = eS$
(2) G is a two-sided ideal of S as well as the least one-sided ideal of S.
(3) e commutes with every $x \in S$.
(4) S is homomorphic on G by the mapping $\varphi(x) = xe = ex$.

We denote by Z the difference semigroup of S modulo G [6]. Z is a zero-semigroup.

Now we shall discuss the structure of a semigroup with zeroids in preparation for the theory of a unipotent inversible semigroup.

Let S be a semigroup having zeroids, and U be its group of zeroids. Since U is a two-sided ideal, we can consider the difference semigroup M of S modulo U; and M is a semigroup with a zero. Conversely, if we are given arbitrarily a semigroup M with a zero and a group U disjoint from M, there exists always at least one ramified homomorphism[7] ψ of M into U, e.g., the mapping of all non-zero elements of M into the unit of U. Consequently we have the following lemma[7].

Lemma 4. Given a semigroup M with a zero 0, and a non-trivial group U which is disjoint from M, and given a ramified homomorphism ψ of M into U, we can construct uniquely a semigroup S with zeroids such that

(1) S is the union of U and $\overline{M}$ where $\overline{M}$ is the set of all non-zero elements of M.

(2) U is the group of zeroids of S and is an ideal of S.

(3) M is the differenc semigroup of S modulo U.

(4) ψ is the ramified homomorphism of M into U.

In the case that a group is trivial, i.e., a group formed by only one element e, S is isomorphic with M; the lemma is trivial.

Thus the semigroup S with zeroids is determined in this fashion by G, M and ψ. We denote by $x \cdot y$ the product of x and y in G, by $x * y$ in M. Then the product xy in S is defined as:

$$xy = \begin{cases} x \cdot y & \text{if } x \in G,\ y \in G, \\ x \cdot \psi(y) & \text{if } x \in G,\ y \in \overline{M}, \\ \psi(x) \cdot y & \text{if } x \in \overline{M},\ y \in G, \\ \psi(x) \cdot \psi(y) & \text{if } x, y \in \overline{M} \text{ and } x * y = 0, \\ x * y & \text{if } x, y \in \overline{M} \text{ and } x * y \neq 0. \end{cases}$$

The mapping f of S onto G is defined as follows.

(1) $f(x) = x$ if $x \in G$,

(2) $f(x) = \psi(x)$ if $x \in \overline{M}$.

It is easy to see that f is a homomorphism of S onto G and ψ is a contraction of f to $\overline{M}$. We may say that a semigroup S with zeroids is determined by G, M and f; and S is written as $S = (G, M, f)$ where the product is given as

$$xy = \begin{cases} f(x) \cdot f(y) & \text{if at least one of } x \text{ and } y \text{ belongs to } G, \text{ or if } x, y \in \overline{M} \text{ and } x * y = 0, \\ x * y & \text{if } x, y \in \overline{M} \text{ and } x * y \neq 0. \end{cases}$$

Now S is unipotent if and only if M is a zero-semigroup. Then M is called the characteristic zero-semigroup of the unipotent semigroup S. By applying Lemma 4 to this case, we get immediately the following theorem.

Theorem 1. A non-trivial group G, a zero-semigroup Z disjoint from G, and a homomorphism f above mentioned determine uniquely a unipotent inversible semigroup S such that $S = (G, Z, f)$, that is to say,

(1) $S = G \cup \overline{Z}$,

(2) G is the greatest group of S and is an ideal of S,

(3) Z is the characteristic zero-semigroup of S,

(4) f is a homomorphism of S onto G.

Finally we shall take in question the condition for two semigroups, which are thus obtained, to be isomorphic.

Theorem 2. There are two unipotent inversible semigroups S_1 and S_2. $S_1 = (G_1, Z_1, f)$ is isomorphic with $S_2 = (G_2, Z_2, g)$ if and only if there exists a one-to-one mapping σ of S_1 onto S_2 such that

(1) G_1 is isomorphic with G_2 by σ,

(2) Z_1 is isomorphic with Z_2 by the modified mapping σ' defined as below,

(3) $f = \sigma^{-1} g \sigma$

Here the modified mapping σ' is a mapping of Z_1 on Z_2 such that

$\sigma'(0_1) = 0_2$ where 0_1 and 0_2 are zeros of Z_1 and Z_2 respectively,

$\sigma'(x_1) = \sigma(x_1)$ if $0_1 \neq x_1 \in Z_1$.

Proof. Suppose that S_1 is isomorphic with S_2. Let σ be the isomorphism of S_1 onto S_2: $S_1 \ni x_1 \longrightarrow \sigma(x_1) \in S_2$. Since σ maps the idempotent $e_1 \in S_1$ to the idempotent $e_2 \in S_2$, it is easily seen that $G_1 = S_1 e_1$ is isomorphic with $G_2 = S_2 e_2$ by σ. Also (2) is clear, for σ makes an element of $S_1 - G_1$[10] correspond to one of $S_2 - G_2$. We shall show (3). By the definition of the product, for every $x_1 \in S_1$,

$$\sigma(x_1 e_1) = \sigma(f(x_1) \cdot f(e_1)) = \sigma(f(x_1) \cdot e_1) = \sigma(f(x_1)),$$

on the other hand,

$$\sigma(x_1)\sigma(e_1) = g(\sigma(x_1)) \cdot g(\sigma(e_1)) = g(\sigma(x_1)) \cdot g(e_2) = g(\sigma(x_1)) \cdot e_2 = g(\sigma(x_1)).$$

From the assumption that $\sigma(x_1 e_1) = \sigma(x_1)\cdot\sigma(e_1)$,

$$\sigma(f(x_1)) = g(\sigma(x_1)).$$

Hence we have $\sigma f = g\sigma$, i.e., $f = \sigma^{-1} g \sigma$.

Consequently, suppose that a mapping σ exists, then we shall prove that $\sigma(x_1 y_1) = \sigma(x_1)\,\sigma(y_1)$ for $x_1, y_1 \in S_1$.

At first, if $x_1, y_1 \in G_1$,

$\sigma(x_1 y_1) = \sigma(f(x_1)\cdot f(y_1))$ by the definition of the product,

while $\sigma(x_1)\,\sigma(y_1) = g(\sigma(x_1))\cdot g(\sigma(y_1)) = (\sigma f(x_1))(\sigma f(y_1))$ by the definition and (3). Since $f(x)$ and $f(y)$ lie in G_1, it follows from (1) that

$$\sigma(f(x_1)\cdot f(y_1)) = (\sigma f(x_1))(\sigma f(y_1)).$$

Therefore we have $\sigma(x_1 y_1) = \sigma(x_1)\,\sigma(y_1)$.

Secondly, if $x_1 y_1 \notin G_1$ i.e.,
and $x_1 \times y_1 \neq 0_1$, $\sigma(x_1 y_1) = \sigma(x_1 \times y_1)$
and $\sigma(x_1)\,\sigma(y_1) = \sigma(x_1) \times \sigma(y_1)$
because $\sigma(x_1) \times \sigma(y_1) \neq 0_2$. Since $\sigma(x_1 \times y_1) = \sigma(x_1) \times \sigma(y_1)$ by (2), we have $\sigma(x_1 y_1) = \sigma(x_1)\,\sigma(y_1)$. Thus we have proved that σ is an isomorphism of S_1 onto S_2.

Remark. Theorem 2 is also valid for a semigroup S with zeroids.

In order to complete the study of unipotent inversible semigroups, we require the determination of the structure of zero-semigroups, which we shall call in question in another article.

References.

[1] I found a part of the theory of the previous paper [8], [9] contained in the paper [5] by A. H. Clifford. We argue them here synthetically by using Clifford's theory.

[2] We once called it one-idempotent.

[3] By a zero-semigroup we mean a unipotent semigroup whose idempotent is a two-sided zero.

[4] The greatest group G of S is the group G contained in S such that $G_1 \subset G$ for every group $G_1 \subset S$. Of course, the subset $\{e\}$ formed by only an idempotent element is considered as a group.

[5] A. H. Clifford & D. D. Miller, Semigroups having zeroid elements, Amer. Jour. of Math. Vol.LXX, No. 1, 1948, pp.117-125.

[6] D. Rees, On semigroups, Proc. Cambridge Philo. Soc., Vol. 36, 1940, pp.387-400.

[7] A. H. Clifford, Extensions of semigroups, Tran. of Amer. Math. Soc., Vol.68, No.2, 1950, pp. 165-173.

[8] T. Tamura, On finite one-idempotent semigroups, Jour. of Gakugei, Tokushima Univ., Vol.IV, 1954, pp.11-20.

[9] T. Tamura, On compact one-idempotent semigroups. Kodai Math. Semi. Rep. No.1, 1954, pp.17-21.
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[10] $S_1 - G_1$ means the complementary set of G_1 to S.

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