STATIONARY PROCESS AND HARMONIC ANALYSIS

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## §1. Introduction.

Let $X(t)$, $-\infty<t<\infty$ be a continuous stationary process in wide sense; that is, $E(|X(t)|^2)<\infty$, the correlation function $\rho(u)=E\{X(t+u)\overline{X(t)}\}$ is independent of $t$ and $\rho(u)$ is continuous at $u=0$. We assume throughout without loss of generality that $E\{X(t)\}=0$. Then $\sigma^2=E\{|X(t)|^2\}$ is independent of $t$ and $\rho(u)$ is continuous everywhere and is represented as

$$(1.1)\quad \rho(u)=\int_{-\infty}^{\infty}e^{iu\alpha}dF(\alpha),$$

where $F(\alpha)$ is bounded non-decreasing function such that

$$(1.2)\quad F(+\infty)-F(-\infty)=\sigma^2.$$

$F(\alpha)$ is the spectral function of $X(t)$.

A large number of papers on a stationary process has been published.(1) The object of the present paper is to develop a Fourier theory of a stationary process.

§2 deals with the filter theory due to Blanc-Lapierre. A slightly general and simpler treatment is given. §3 concerns with the law of large numbers and known results are proved by Fourier analytical method. N.Wiener developped a prediction theory concerning a sample function of a stationary process. In §4 we shall consider the problem with a stationary process itself instead of a sample function. The similar formulation was considered by K. Karhunen (2), and solved in terms of operators in Hilbert space. We follow after N. Wiener and consider the prediction of $X(t)$ in future, by a specified filtered process. The results and methods are essentially identical as N.Wiener.

In Wiener theory of prediction, for a sample function $f(t)$, an average (correlation function)

$$(1.3)\quad \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(t+x)\overline{f(t)}dt=\varphi(x)$$

is considered, while we take the convariance $\rho(x)=E\{X(t+x)\overline{X(t)}\}$ instead of (1.3). If $X(t)$ is strictly stationary, then $\rho(x)$ is idential with $\varphi(x)$. But in general this is not true and to clarify this situation, we consider the harmonic analysis of $X(t+u)\cdot\overline{X(t)}$. These are done in §§5 and 6.

## §2. Filter Theory.

2.1. Let $X(t)$ be a stationary process in wide sense and let its correlation function and spectral function be $\rho(u)$ and $F(\alpha)$ respectively. We consider a function $K(\theta)$, which is of bounded variation in every finite interval.

If the function

$$(2.1)\quad \int_A^B e^{-ix\theta}dK(\theta)$$

converges in $L_2$ with respect to $F(x)$ to $G(x)$ when $A\to-\infty$, $B\to\infty$, then we say that $K(\theta)\in\mathbf{K}(-\infty,\infty)$. That is, if

$$(2.2)\quad \lim_{\substack{A\to-\infty\\ B\to\infty}}\int_{-\infty}^{\infty}\left|\int_A^B e^{-ix\theta}dK(\theta)-G(x)\right|^2dF(x)=0,$$

then $K(\theta)\in\mathbf{K}(-\infty,\infty)$.

If $K(\theta)$ is defined in $[0,\infty)$, and

$$(2.3)\quad \int_0^A e^{-ix\theta}dK(\theta),$$

instead of (2.1), converges to $G(x)$ in $L_2(0,\infty)$ with respect to $F(x)$, then $K(\theta)$ is said to belong to $\mathbf{K}(0,\infty)$.

(2.2) is also represented as

$$(2.4)\quad \underset{\substack{A\to-\infty\\ B\to\infty}}{\overset{L_2(F)}{\mathrm{l.i.m.}}}\int_A^B e^{-ix\theta}dK(\theta)=G(x).$$

This is called the Fourier-Stieltjes transform of $K(\theta)$.

And if $X_A(t)$ is a process depending on a parameter $A$ and

$$\lim_{A\to\infty} E\{|X_A(t)-X(t)|^2\}=0,$$

then we write simply

$$\underset{A\to\infty}{l.i.m.}\, X_A(t) = X(t).$$

Let $(A, B)$ is any interval and consider a division :

$$\Delta \; : \quad A=\theta_0<\theta_1<\cdots<\theta_n=B.$$

If putting

$$\sum_{k=0}^{n-1} X(\theta_k)(K(\theta_{k+1})-K(\theta_k)) = S_\Delta,$$

it holds : $\underset{\Delta}{l.i.m.}\, S_\Delta = S \;(\max(\theta_{k+1}-\theta_k)\to 0)$, then $S$ is denoted as $\int_A^B X(\theta)\,dK(\theta)$.

It is easily seen that the integral $\int_A^B X(\theta)\,dK(\theta)$ ( $K(\theta)$ is of bounded variation in $[A,B]$ ) exists if and only if

$$\int_A^B\int_A^B \rho(\theta-\theta')\,dK(\theta)\,d\overline{K(\theta')}$$

exists.

From this fact it is evident that for any $K(\theta)\in \mathbf{K}(-\infty,\infty)$,

$$\int_A^B X(t-\theta)\,dK(\theta)$$

always exists for any $A$ and $B$.

Theorem 1.

$$(2.5)\qquad \underset{\substack{A\to-\infty\\ B\to\infty}}{l.i.m.}\int_A^B X(t-\theta)\,dK(\theta)$$

exists if and only if $K(\theta)\in\mathbf{K}(-\infty,\infty)$. (2.5) is denoted as $\mathfrak{F}[X(t)]$ and is called the filtered process by $K(\theta)$.

The proof of Theorem 1 is immediate from the following identity.

$$E\left\{\left|\int_A^{A'} X(t-\theta)\,dK(\theta)\right|^2\right\}$$

$$= E\left\{\int_A^{A'}\int_A^{A'} X(t-\theta)\,\overline{X(t-\theta')}\,dK(\theta)\,d\overline{K(\theta')}\right\}$$

$$= \int_A^{A'}\int_A^{A'} \rho(\theta-\theta')\,dK(\theta)\,d\overline{K(\theta')}$$

$$= \int_{-\infty}^{\infty}\int_A^{A'}\int_A^{A'} e^{-i(\theta-\theta')x}\,dF(x)\,dK(\theta)\,d\overline{K(\theta')}$$

$$= \int_{-\infty}^{\infty}\left|\int_A^{A'} e^{-ix\theta}\,dK(\theta)\right|^2 dF(x).$$

Theorem 2. Let $K_1(\theta)$ and $K_2(\theta)$ be functions of $\mathbf{K}$ and their Fourier-Stieltjes transforms be $G_1(x)$ and $G_2(x)$ respectively. And let $Y_1(t)$ and $Y_2(t)$ be filtered processes of $X(t)$ by $K_1(\theta)$ and $K_2(\theta)$ respectively. Then we have

$$(2.6)\quad E\{Y_1(t+u)\overline{Y_2(t)}\} = \int_{-\infty}^{\infty} G_1(x)\overline{G_2(x)}\,e^{iux}\,dF(x).$$

Proof.

$$E\{Y_1(t+u)\overline{Y_2(t)}\}$$

$$= E\Big\{\underset{\substack{A\to-\infty\\ B\to\infty}}{l.i.m.}\int_A^B X(t+u-\theta)\,dK_1(\theta)\cdot \underset{\substack{A'\to-\infty\\ B'\to\infty}}{l.i.m.}\int_{A'}^{B'}\overline{X(t-\theta')}\,d\overline{K_2(\theta')}\Big\}$$

$$=\lim_{\substack{A\to-\infty\\ B\to\infty}}\lim_{\substack{A'\to-\infty\\ B'\to\infty}}\int_A^B\int_{A'}^{B'} E\{X(t+u-\theta)\overline{X(t-\theta')}\}\,dK_1(\theta)\,d\overline{K_2(\theta')}$$

$$=\lim_{\substack{A\to-\infty\\ B\to\infty}}\lim_{\substack{A'\to-\infty\\ B'\to\infty}}\int_A^B\int_{A'}^{B'} \rho(u+\theta'-\theta)\,dK_1(\theta)\,d\overline{K_2(\theta')}$$

$$=\lim_{\substack{A\to-\infty\\ B\to\infty}}\lim_{\substack{A'\to-\infty\\ B'\to\infty}}\int_{-\infty}^{\infty} dF(x)\cdot\int_A^B\int_{A'}^{B'} e^{i(u+\theta'-\theta)x}\,dK_1(\theta)\,d\overline{K_2(\theta')}$$

$$=\int_{-\infty}^{\infty} e^{iux}\,dF(x)\cdot \underset{\substack{A\to-\infty\\ B\to\infty}}{l.i.m.}\;\underset{\substack{A'\to-\infty\\ B'\to\infty}}{l.i.m.}\int_A^B\int_{A'}^{B'} e^{-i\theta x}e^{i\theta' x}\,dK_1(\theta)\,d\overline{K_2(\theta')}$$

$$=\int_{-\infty}^{\infty} e^{iux}\,G_1(x)\,\overline{G_2(x)}\,dF(x).$$

By (2.6), it is seen that $E\{Y_1(t+u)\overline{Y_2(t)}\}$ is independent of $t$. Especially $Y(t)=\mathfrak{F}[X(t)]$, the filtered process is a stationary process, which is stated in the following theorem.

Theorem 3. The filtered process by $K(\theta)\in\mathbf{K}$, $Y(t)$ is a stationary process and its correlation function $\rho_Y(u)$ is given by

$$(2.7)\qquad \rho_Y(u)=\int_{-\infty}^{\infty}|G(x)|^2 e^{iux}\,dF(x)$$

and in particular

$$(2.8)\qquad E\{|Y(t)|^2\}=\int_{-\infty}^{\infty}|G(x)|^2\,dF(x).$$

2.2 If, in the definition of $\mathbf{K}$ $A$ and $B$ in (2.1) are replaced by $-A$, $A$ ; that is if

$$\underset{A\to\infty}{\overset{L_2(F)}{l.i.m}} \int_{-A}^{A} e^{-ix\theta} dK(\theta)$$

exists, then $K(\theta)$ is said to belong to $\mathbf{K}_1$. Theorems 1-3 are also valid if the definition of filtering is replaced by

$$\underset{A\to\infty}{l.i.m.} \int_{-A}^{A} X(t-\theta)\, dK(\theta) = \mathcal{F}\{X(t)\}.$$

Now we put

$$(2.9)\qquad l(\theta; a, b) = \frac{1}{2\pi i\theta}\left(e^{+ib\theta} - e^{+ia\theta}\right) = \frac{1}{2\pi}\int_a^b e^{ix\theta}\, dx.$$

Then

$$\lim_{A\to\infty}\int_{-A}^{A} l(\theta; a, b)\, e^{-iu\theta} d\theta = \begin{cases} 1, & a < u < b, \\ \frac{1}{2}, & u = a, b, \\ 0, & u < a,\ u > b. \end{cases}$$

Putting

$$(2.10)\qquad L(\theta; a, b) = L(\theta) = \int_0^\theta l(\theta; a, b)\, d\theta,$$

we define

$$(2.11)\qquad \mathcal{F}_{ab}\{X(t)\} = \underset{A\to\infty}{l.i.m.}\int_{-A}^{A} X(t-\theta)\, dL(\theta)$$

and consider

$$(2.12)\qquad Z(a,b) = \mathcal{F}_{ab}\{X(0)\}$$

which is a random variable depending on an interval ( $a$ , $b$ ). Thus for any interval I whose end points are continuity points of $F(x)$, we define a random variable $Z(I)$. $Z(I)$ has following properties which are easily verified by Theorem 2.

(1°) if $I_1 \cup I_2 = I$, then $Z(I) = Z(I_1) + Z(I_2)$, where $I_1$ and $I_2$ have no common interval, and the interval I with $I_1$ and $I_2$, has the continuity points of $F(x)$ as its end points.

(2°) $E\{|Z(I)|^2\} = \int_I dF(x)$,

(3°) $E\{Z(I_1)\overline{Z(I_2)}\} = \int_{I_1\cap I_2} dF(x)$.

Now we appeal to the following lemma which is due to H.Cramér.(3)

Lemma 1. Let $Z(S)$ be a random variable defined on every continuity intervals of a spectral function $F(x)$ of a stationary process $X(t)$. If $Z(S)$ satisfies the conditions (1°), (ii) and (iii) above, then we can uniquely define the random variable $Z(S)$ defined on every Borel set $S$ on the real axis, such that

(I) $Z(S_1 \cup S_2) = Z(S_1) + Z(S_2)$, when $S_1 \cap S_2 = 0$,

(II) $E\{Z(S_1)\cdot\overline{Z(S_2)}\} = \int_{S_1\cap S_2} dF(x)$, $(S_1\cap S_2 \neq 0)$,
$= 0$, $(S_1 \cap S_2 = 0)$,

(III) $E\{|Z(S)|^2\} = \int_S dF(x)$.

By this lemma, starting from (2.12) we can define $Z(S)$ depending on any Borel set $S$. Let $A > 0$ and consider a division

$$-A = \nu_0 < \nu_1 < \cdots < \nu_{n-1} < \nu_n = A$$

and denote $Z(S)$, $S$ being an interval, $a < x \leq b$ as $Z(a, b)$. Put

$$(2.13)\qquad S_n(t) = \sum_{k=0}^{n-1} Z(\nu_k, \nu_{k+1})\, e^{i\nu_k t},$$

$$(2.14)\qquad X_A(t) = \mathcal{F}_{(-A,A)}\{X(t)\}.$$

Then we have

$$(2.15)\qquad E\{|S_n(t)|^2\} = \int_{-A+0}^{A+0} dF(x),$$

$$(2.16)\qquad E\{|X_A(t)|^2\} = \int_{-A+0}^{A+0} dF(x),$$

$$(2.17)\qquad E\{S_n(t)\overline{X_A(t)}\} = \sum \int_{\nu_k+0}^{\nu_{k+1}+0} e^{-itx} dF(x)\cdot e^{i\nu_k t}.$$

These are easily seen, for example

$$E\{S_n(t)\overline{S_n(t)}\} = E\Big\{\sum_k\sum_j Z(\nu_k, \nu_{k+1})\overline{Z(\nu_j, \nu_{j+1})}\, e^{i(\nu_k - \nu_j)t}\Big\}$$
$$= \sum_k\sum_j E\{Z(\nu_k, \nu_{k+1})\overline{Z(\nu_j, \nu_{j+1})}\}\cdot e^{i(\nu_k-\nu_j)t}.$$

By Theorem 2 and (II)

$$E\{Z(\nu_k,\nu_{k+1})\overline{Z(\nu_j,\nu_{j+1})}\}=0,\ (k\neq j)$$
$$=\int_{\nu_k+0}^{\nu_{k+1}+0} dF(x),\ (k=j),$$

which yields (2.15). (2.16) is obvious. (2.17) is also proved easily. (2.15), (2.16) and (2.17) show that

$$E\{|S_n(t)-X_A(t)|^2\}$$
$$=E\{|S_n(t)|^2\}+E\{|X_A(t)|^2\}-2\mathcal{R}E\{S_n(t)\cdot\overline{X_A(t)}\}$$
$$=2\int_{-A+0}^{A+0} dF(x)-2\mathcal{R}\sum_k\int_{\nu_k+0}^{\nu_{k+1}+0} e^{i(\nu_k-x)t}dF(x),$$

which tends to zero as $n\to\infty$, $\max(\nu_{k+1}-\nu_k)\to 0$.

That is, it holds:

$$(2.17)\quad \underset{}{\mathrm{l.i.m.}}\,S_n(t)=X_A(t)$$

Furthermore

$$E\{|X_A(t)-X(t)|^2\}$$
$$=E\{|X_A(t)|^2\}+E\{|X(t)|^2\}-2\mathcal{R}E\{X_A(t)\overline{X(t)}\},$$

and

$$E\{X_A(t)\overline{X(t)}\}$$
$$=E\{\underset{B\to\infty}{\mathrm{l.i.m.}}\int_{-B}^{B}X(t-\theta)dL(\theta;-A,A)\cdot\overline{X(t)}\}$$
$$=\lim_{B\to\infty}E\{\int_{-B}^{B}X(t-\theta)dL(\theta;-A,A)\overline{X(t)}\}$$
$$=\lim_{B\to\infty}\int_{-B}^{B}E\{X(t-\theta)\overline{X(t)}\}dL(\theta;-A,A)$$
$$=\lim_{B\to\infty}\int_{-B}^{B}\rho(-\theta)\,dL(\theta;-A,A)$$
$$=\lim_{B\to\infty}\int_{-B}^{B}\int_{-\infty}^{\infty}e^{-i\theta x}dF(x)\,dL(\theta;-A,A)$$
$$=\int_{-\infty}^{\infty}dF(x)\lim_{B\to\infty}\int_{-B}^{B}e^{-i\theta x}dL(\theta;-A,A)$$
$$=\int_{-A+0}^{A+0}dF(x).$$

Hence using (2.16), we have

$$E\{|X_A(t)-X(t)|^2\}$$
$$=\int_{-\infty}^{\infty}dF(x)-\int_{-A+0}^{A+0}dF(x)$$

which tends to $0$ as $A\to\infty$

Thus we have proved the

**Theorem 4.** <u>The stationary process</u> $X(t)$ <u>can be represented as</u>

$$(2.18)\quad X(t)=\int_{-\infty}^{\infty}e^{itx}dZ(x)$$

<u>where</u> $Z(x)=Z(-\infty,x)$ <u>and</u> (2.18) <u>means</u>

$$(2.19)\quad X(t)=\underset{A\to\infty}{\mathrm{l.i.m}}\ \underset{n\to\infty}{\mathrm{l.i.m}}\ S_n(t),$$

$S_n(t)$ <u>being given by</u> (2.13).

Here it seems worthwhile to give some remarks on the integral.

Let $g(x)$ be a function of $L_2(F)$; that is

$$\int_{-\infty}^{\infty}|g(x)|^2dF(x)$$

exists (in Lebesgue sense). Then we can define

$$(2.20)\quad \int_{-\infty}^{\infty}g(x)\,dZ(x)$$

by approximating $g(x)$ in $L_2(F)$ by simple functions.(4) Then besides ordinary fundamental properties of integral, (2.20) has following properties, among others,

(i) $$(2.21)\quad E\{|\int_{-\infty}^{\infty}g(x)dZ(x)|^2\}=\int_{-\infty}^{\infty}|g(x)|^2dF(x),$$

(ii) if $S_1$ and $S_2$ are Borel sets, and $f(x)$ and $g(x)$ are of $L_2(F)$, then

$$E\{\int_{S_1}f(x)dZ(x)\int_{S_2}\overline{g(x)}d\overline{Z(x)}\}=\int_{S_1\cap S_2}f(x)\overline{g(x)}\,dF(x).$$

(iii) if $f_\alpha(x)$ converges in mean $L_2(F)$ to $f(x)$, then

$$\underset{\alpha\to\infty}{l.i.m.}\int_{-\infty}^{\infty} f_\alpha(x)\,dZ(x) = \int_{-\infty}^{\infty} f(x)\,dZ(x),$$

(iv) if $f(x)$ is bounded continuous, then $\int_{-\infty}^{\infty} f(x)\,dZ(x)$ exists.

It is immediate that the integral in (2.18) can also be considered as the one defined in (2.20), $g(x)$ being $e^{itx}$.

We add a following theorem.

Theorem 5. The filtered process $\mathfrak{F}_k[X(t)]$ of the stationary process $X(t)$ by a function $K(\theta)\in \mathbf{K}$ is represented as

$$(2.22)\quad \mathfrak{F}_k\{X(t)\} = \int_{-\infty}^{\infty} e^{itx} G(x)\,dZ(x),$$

where $G(x)$ is the Fourier-Stieltjes transform of $K(\theta)$

We have

$$\mathfrak{F}_k\{X(t)\} = \int_{-\infty}^{\infty} X(t-\theta)\,dK(\theta)$$

$$= \underset{\substack{A\to-\infty\\ B\to\infty}}{l.i.m.} \int_A^B X(t-\theta)\,dK(\theta)$$

$$= \underset{\substack{A\to-\infty\\ B\to\infty}}{l.i.m.} \int_A^B dK(\theta)\, l.i.m.\, l.i.m.\, S_n(t-\theta)$$

$$= \underset{\substack{A\to-\infty\\ B\to\infty}}{l.i.m.} \lim \lim \int_A^B \sum_0^n Z(\nu_k,\nu_{k+1}) e^{i\nu_k(t-\theta)}\,dK(\theta)$$

$$= \underset{\substack{A\to-\infty\\ B\to\infty}}{l.i.m.} \lim \lim \sum_0^n Z(\nu_k,\nu_{k+1}) e^{i\nu_k t}\int_A^B e^{-i\nu_k\theta}\,dK(\theta),$$

which is by (iv)

$$\underset{A,B}{l.i.m.}\int_{-\infty}^{\infty} e^{ixt}\left(\int_A^B e^{-ix\theta}\,dK(\theta)\right) dZ(x)$$

$$= \int_{-\infty}^{\infty} e^{ixt} G(x)\,dZ(x)$$

by (iii).

§3. The Law of Large Numbers.

3.1. We shall prove the following known theorem.

Theorem 6. Let $X(t)$ be a stationary process with random spectral function $Z(x)$. Then

$$(3.1)\quad \underset{T\to\infty}{l.i.m.} \frac{1}{T}\int_0^T X(t) e^{-i\xi t}\,dt = Z(\xi+0) - Z(\xi-0).$$

We note that $Z(x\pm 0)$ exists, for example

$$E\{\,|Z(x+\varepsilon) - Z(x+\varepsilon')|^2\} = E\left\{\left|\int_{x+\varepsilon'+0}^{x+\varepsilon+0} dZ(x)\right|^2\right\} = \int_{x+\varepsilon'+0}^{x+\varepsilon+0} dF(x)$$

and converge to $0$ as $\varepsilon', \varepsilon \to 0$.

Before proving the theorem it is convenient to state a lemma.

Lemma 2. Let $\varphi(t)$ be continuous on $a\le t\le b$, and $X_A(t)$ and $X(t)$ be stochartic processes continuous in mean. If

$$\underset{A\to\infty}{l.i.m.}\, X_A(t) = X(t),\quad a\le t\le b,$$

$$E\{\,|X_A(t) - X(t)|^2\} \le K < \infty,\ a\le t\le b,$$

then

$$\underset{A\to\infty}{l.i.m.}\int_a^b \varphi(t) X_A(t)\,dt = \int_a^b \varphi(t) X(t)\,dt,$$

the integral being taken in Riemann sense.

The proof is immediate by the definition of integral. The similar facts holds in Lebesgue sense which will be stated as Lemma 5 in §6 in the sequel.

We shall prove the theorem. Let $u(x,\xi) = 0\ (x\neq\xi)$, $u(x,\xi) = 1\ (x=\xi)$. We have

$$E\left\{\left|\frac{1}{T}\int_0^T X(t)e^{-i\xi t}\,dt - \int_{-\infty}^{\infty} u(x,\xi)\,dZ(x)\right|^2\right\}$$

$$= E\left\{\left|\frac{1}{T}\int_0^T e^{-i\xi t}\,dt\int_{-\infty}^{\infty} e^{itx}\,dZ(x) - \int_{-\infty}^{\infty} u(x,\xi)\,dZ(x)\right|^2\right\}$$

(Lemma 2 is used)

$$= E\left\{\left|\int_{-\infty}^{\infty}\left(\frac{1}{T}\int_0^T e^{it(x-\xi)}\,dt\right) dZ(x) - \int_{-\infty}^{\infty} u(x,\xi)\,dZ(x)\right|^2\right\}$$

Here the inversion of the order of integral is legitimate as easily verified. The above is

$$E\left\{ \left| \int_{-\infty}^{\infty} \left\{ \frac{e^{iT(x-\xi)}-1}{iT(x-\xi)} - u(x,\xi) \right\} dZ(x) \right|^2 \right\}$$

$$(3.2) \quad = \int_{-\infty}^{\infty} \left| \frac{e^{iT(x-\xi)}-1}{iT(x-\xi)} - u(x,\xi) \right|^2 dF(x),$$

by (i) in §2, $F(x)$ being the spectral function of $x(t)$. Since the integrand of (3.2) converges to zero boundedly, (3.2) tends to zero which is to be proved.

Next we shall discuss the convergence of

$$\int^{\infty} \frac{x(t)}{t} e^{-i\xi t} dt.$$

<u>Theorem 7.</u> <u>If for some</u> $\varepsilon > 0$,

$$(3.3) \quad \int_0^{\varepsilon} \frac{F(\xi+x)-F(\xi-x)}{x} \log \frac{1}{x}\, dx < \infty,$$

<u>then</u>

$$(3.4) \quad \underset{T\to\infty}{l.i.m.} \int_1^T \frac{x(t)}{t} e^{-i\xi t} dt$$

<u>exists.</u> <u>Especially</u> <u>if</u> <u>for</u> <u>some</u> $\alpha > 0$,

$$(3.5) \quad F(\xi+x) - F(\xi-x) = O(x^{\alpha}), \quad \text{as } x \to +0,$$

<u>then</u> (3.4) <u>exists.</u>

Proof. We shall prove in the case $\xi = 0$. We have

$$\int_1^T \frac{x(t)}{t} dt = \int_1^T \frac{dt}{t} \int_{-\infty}^{\infty} e^{itx} dZ(x)$$

$$= \int_{-\infty}^{\infty} \left( \int_1^T \frac{e^{itx}}{t} dt \right) dZ(x)$$

$$= \int_{-\infty}^{\infty} dZ(x) \int_1^T \frac{\cos tx}{t} dt + i \int_{-\infty}^{\infty} dZ(x) \int_1^T \frac{\sin tx}{t} dt$$

$$(3.6) \quad = J_1 + iJ_2,$$

say. Since $\int_1^T \frac{\sin tx}{t} dt$ converges boundedly as $T \to \infty$, it also converges in $L_2(F)$, and hence by (iii) in §2, $\underset{T\to\infty}{l.i.m.}\, J_2$ exists.

Next we have

$$J_1 = J_1(T) = \int_{-\infty}^{\infty} dZ(x) \int_x^{xT} \frac{\cos t}{t} dt$$

and

$$J_1(T) - J_1(T') = \int_{-\infty}^{\infty} dZ(x) \int_{xT'}^{xT} \frac{\cos t}{t} dt.$$

Hence by (2.2.1)

$$E\{|J_1(T) - J_1(T')|^2\}$$

$$= \int_{-\infty}^{\infty} \left| \int_{xT'}^{xT} \frac{\cos t}{t} dt \right|^2 dF(x)$$

$$= \int_{|x| \le \varepsilon} + \int_{|x| > \varepsilon} = J_{11} + J_{12},$$

say, $\varepsilon$ being any positive number. Since, for $|x| > \varepsilon$, $\int_{xT'}^{xT} \frac{\cos t}{t} dt$ converges boundedly to zero, we have

$$(3.7) \quad \underset{T,T'\to\infty}{l.i.m.}\, J_{12} = 0.$$

We have

$$J_{11} = \int_0^{\varepsilon} dF(x) \left| \int_{xT'}^{xT} \frac{\cos t}{t} dt \right|^2 + \int_{-\varepsilon}^{0} dF(x) \left| \int_{xT'}^{xT} \frac{\cos t}{t} dt \right|^2$$

$$= \int_0^{\varepsilon} d\{F(x) - F(-x)\} \cdot \left| \int_{xT'}^{xT} \frac{\cos t}{t} dt \right|^2$$

which is, by integration by parts

$$\{F(\varepsilon) - F(-\varepsilon)\} \cdot \left( \int_{\varepsilon T'}^{\varepsilon T} \frac{\cos t}{t} dt \right)^2$$

$$- \lim_{x \to 0} \{F(x) - F(-x)\} \left( \int_{xT'}^{xT} \frac{\cos t}{t} dt \right)^2$$

$$(3.8) \quad -2 \int_0^{\varepsilon} \{F(x) - F(-x)\} dx \cdot \int_{xT'}^{xT} \frac{\cos t}{t} dt \left( \frac{\cos xT}{x} - \frac{\cos xT'}{x} \right).$$

The first term converges to zero as $T \to \infty$, and the second term is zero, since $F(x)$ is continuous at $x = 0$, which is a consequence of the condition (3.3), for

$$\int_{xT'}^{xT} \frac{\cos t}{t}\, dt$$
$$= O\left(\int_{xT'}^{xT} \frac{dt}{t}\right)$$
$$= O\left(\log \frac{T}{T'}\right).$$

Further if we consider

$$K = \int_0^{\varepsilon} \{F(x) - F(-x)\} \int_{xT}^{\infty} \frac{\cos t}{t}\, dt \cdot \frac{\cos xT'}{x}$$
$$= \int_0^{\frac{1}{T}} + \int_{\frac{1}{T}}^{\varepsilon}$$
$$= K_1 + K_2,$$

say, then

$$K_1 = \int_0^{\frac{1}{T}} \{F(x) - F(-x)\} \int_{xT}^{1} \frac{\cos t - 1}{t}\, dt \cdot \frac{\cos xT'}{x}\, dx$$
$$+ \int_0^{\frac{1}{T}} \{F(x) - F(-x)\} \log \frac{1}{xT}\, \frac{\cos xT'}{x}\, dx$$
$$+ \int_0^{\frac{1}{T}} \{F(x) - F(-x)\} \int_1^{\infty} \frac{\cos t}{t}\, dt\ \frac{\cos xT'}{x}\, dx,$$

and noticing that $\int_{\xi}^{1} (\cos t - 1)/t \cdot dt = O(1)$, uniformly in $0 < \xi < 1$, the first term of $K_1$ is

$$\int_0^{\frac{1}{T}} \{F(x) - F(-x)\} \cdot O\left(\tfrac{1}{x}\right) dx = o(1),$$

as $T \to \infty$ by (3.3) and the second term is

$$o\left\{\int_0^{\frac{1}{T}} \{F(x) - F(-x)\} \tfrac{1}{x} \log \tfrac{1}{x}\, dx\right\}$$
$$+ O\left(\log T \cdot \int_0^{\frac{1}{T}} \frac{F(x) - F(-x)}{x}\, dx\right)$$

$$= O\left(\int_0^{\frac{1}{T}} \{F(x) - F(-x)\} \tfrac{1}{x} \log \tfrac{1}{x}\, dx\right)$$

also by (3.3), and furthermore the last term of $K_1$ is also

$$o\left(\int_0^{\frac{1}{T}} \{F(x) - F(-x)\} \tfrac{1}{x}\, dx\right)$$
$$= o(1).$$

Hence we have

(3.9) $\quad K_1 = o(1) \quad , \quad$ as $T \to \infty$.

Next

$$K_2 = \int_{\frac{1}{T}}^{\varepsilon} \{F(x) - F(-x)\} \int_{xT}^{\infty} \frac{\cos t}{t}\, dt \cdot \frac{\cos xT'}{x}\, dx$$
$$= O\left\{\int_{\frac{1}{T}}^{\varepsilon} \{F(x) - F(-x)\} \frac{dx}{x}\right\},$$

Since $xT > 1$, which is arbitrarily small by taking $\varepsilon$ small. Combining this with (3.9) we have

(3.10) $\quad K = o(1),$

by letting $T \to \infty$ and then $\varepsilon \to 0$.

The similar integrals arising in the last term of (3.8) are also treated quite similarly and we can prove that (3.8) converges to zero as $T, T' \to \infty$ which results with (3.7),

$$E\{|J_1(T) - J_1(T')|^2\} \to 0.$$

3.2. We shall now prove the

Theorem 8. If $X(t)$ is a stationary process, then

(3.11) $$\underset{T \to \infty}{\mathrm{l.i.m.}}\ \frac{1}{\pi} \int_{-T}^{T} \frac{\sin A(t-x)}{t-x} X(t)\, dt$$
$$= \int_{-A}^{A} e^{ix\lambda}\, dZ(\lambda),$$

$Z(\lambda)$ being the random spectral function of $X(t)$ and it is assumed that at the discontinuities of the spectral function $F(\lambda)$, $Z(\lambda)$ is defined as

$$Z(\lambda) = \tfrac{1}{2}\{Z(\lambda+0) + Z(\lambda-0)\}$$
$$= \tfrac{1}{2} \underset{\varepsilon \to 0}{\mathrm{l.i.m.}} \{Z(\lambda+\varepsilon) + Z(\lambda-\varepsilon)\}.$$
$$I = \frac{1}{\pi} \int_{-T}^{T} \frac{\sin A(t-x)}{t-x} X(t)\, dt$$

which is by Lemma 1

$$= \frac{1}{\pi}\int_{-T}^{T} \frac{\sin A(t-x)}{t-x} dt \int_{-\infty}^{\infty} e^{it\lambda} d Z(\lambda)$$

$$= \int_{-\infty}^{\infty} d Z(\lambda) \frac{1}{\pi}\int_{-T}^{T} \frac{\sin A(t-x)}{t-x} e^{it\lambda} dt.$$

Since

$$\frac{1}{\pi}\int_{-T}^{T} \frac{\sin Au}{u} e^{i\lambda u} du \to \begin{cases} 0, & |\lambda| > A, \\ \frac{1}{2}, & |\lambda| = A, \\ 1, & |\lambda| < A, \end{cases}$$

and

$$\left| \frac{1}{\pi}\int_{T}^{T+x} \frac{\sin Au}{u} e^{iu\lambda} du \cdot e^{ix\lambda} \right|$$

$$\leqq \frac{1}{\pi}\int_{T}^{T+x} \frac{du}{u}$$

$$= \frac{1}{\pi} \log \left(1 + \frac{x}{T}\right),$$

we have, by (iii) of 2.2,

$$\underset{T\to\infty}{l.i.m.}\ I = \int_{-A}^{A} e^{ix\lambda} d Z(\lambda).$$

From Theorem 8, following Theorems 9 and 10 are easily obtained.

Theorem 9.

$$(3.12) \quad Z(\lambda) - Z(0) = \frac{1}{2\pi} \underset{T\to\infty}{l.i.m.} \int_{-T}^{T} \frac{1-e^{-it\lambda}}{it} X(t)\, dt.$$

Theorem 10. Putting

$$D_A X(t) = \underset{T\to\infty}{l.i.m.} \frac{1}{\pi}\int_{-T}^{T} \frac{\sin A(t-x)}{t-x} X(t)\, dt,$$

we have

$$(3.13) \quad D_A D_B X(t) = D_C X(t), \quad C = \min.(A,B).$$

Further we shall state

Theorem 11. The necessary and sufficient condition for that the spectrum of the spectral function $F(x)$ of a stationary process $X(t)$, is bounded, is that

$$(3.14) \quad D_A X(t) = X(t)$$

for some $A > 0$.

If (3.14) holds, then using (3.11)

$$E\{|X(t)|^2\} = E\{|D_A X(t)|^2\}$$

$$= E\left\{\left|\int_{-A}^{A} e^{it\lambda} d Z(\lambda)\right|^2\right\} = \int_{-A}^{A} d F(x).$$

Since $E\{|X(t)|^2\} = F(+\infty) - F(-\infty)$, we have

$$F(+\infty) - F(-\infty) = F(A) - F(-A)$$

so that the spectrum of $F(x)$ is contained in $(-A, A)$.

If the spectrum of $F(x)$ is bounded, then there exists $A$ such that $F(+\infty) = F(A)$, $F(-\infty) = F(-A)$. And since

$$E\left\{\left|\int_{A}^{\infty} e^{it\lambda} d Z(\lambda)\right|^2\right\} = \int_{A}^{\infty} d F(\lambda) = 0,$$

$$E\left\{\left|\int_{-\infty}^{-A} e^{it\lambda} d Z(\lambda)\right|^2\right\} = \int_{-\infty}^{-A} d F(\lambda) = 0,$$

we have

$$X(t) = \int_{-\infty}^{\infty} e^{it\lambda} d Z(\lambda) = \int_{-A}^{A} e^{it\lambda} d Z(\lambda).$$

We shall, lastly, add a remark that $D_A X(t) = X(t)$ is equivalent to

$$D_A \rho(t) = \rho(t),$$

$\rho(t)$ being the correlation function of $X(t)$.

## §4. Wiener's prediction theory and the Fourier Stieltjes transform.

4.1. Let $K(\theta)$ be a function of $\mathbf{K}(0,\infty)$ defined in 2.1. Suppose throughout that $X(t)$ is a stationary process and $Z(x)$, $F(x)$ are the random spectral function and the spectral function of $X(t)$ as before. We consider the problem to predict $X(t+\alpha)$, $(\alpha > 0)$ by the values before $t$ of the filtered process by $K(\theta)$. The following arguments are essentially due to N.Wiener [5], but the formulation is different in some points. The class of $K(\theta)$ is slightly general than Wiener's. He considered $K(\theta)$ of bounded variation in

$[0,\infty)$. This generalization is more natural for his theory, and our procedure is more simple in some points.

We begin with the following fact.

Theorem 12. Let $\alpha > 0$ and the Fourier-Stieltjes transform of $K(\theta)$ be $G(x)$ (in the sense in 2.1). Then we have

$$(4.1)\quad E\left\{\left|X(t+\alpha) - \int_0^\infty X(t-\theta)\,dK(\theta)\right|^2\right\}$$

$$= \int_{-\infty}^\infty |e^{i\alpha x} - G(x)|^2\,dF(x).$$

Proof. $\int_0^A e^{-i\theta x}\,dK(\theta)$ converges in mean $L_2(F)$ to $G(x)$. This we denote as

$$\underset{A\to\infty}{\overset{L_2(F)}{l.i.m.}} \int_0^A e^{-i\theta x}\,dK(\theta).$$

We have

$$E\left\{\overline{X(t+\alpha)}\int_0^\infty X(t-\theta)\,dK(\theta)\right\}$$

$$= E\left\{\overline{X(t+\alpha)}\ \underset{A\to\infty}{l.i.m.}\int_0^A X(t-\theta)\,dK(\theta)\right\}$$

$$= \lim_{A\to\infty} E\left\{\overline{X(t+\alpha)}\int_0^A X(t-\theta)\,dK(\theta)\right\}$$

$$= \lim_{A\to\infty}\int_0^A E\left\{\overline{X(t+\alpha)}\,X(t-\theta)\right\}dK(\theta)$$

$$= \lim_{A\to\infty}\int_0^A\int_{-\infty}^\infty e^{-i(\theta+\alpha)u}\,dF(u)\,dK(\theta)$$

$$= \lim_{A\to\infty}\int_{-\infty}^\infty e^{-i\alpha u}\,dF(u)\int_0^A e^{-i\theta u}\,dK(\theta)$$

$$= \int_{-\infty}^\infty e^{-i\alpha u}\,dF(u)\ \underset{A\to\infty}{\overset{L_2(F)}{l.i.m.}}\int_0^A e^{-i\theta u}\,dK(\theta)$$

$$(4.2)\quad = \int_{-\infty}^\infty e^{-i\alpha u}G(u)\,dF(u).$$

Now the left hand side of (4.1) is

$$E\{|X(t+\alpha)|^2\}$$
$$-2\mathcal{R}\,E\left\{\overline{X(t+\alpha)}\int_0^\infty X(t-\theta)\,dK(\theta)\right\}$$
$$+E\left\{\left|\int_0^\infty X(t-\theta)\,dK(\theta)\right|^2\right\}$$

which is, by Theorem 3, and (4.2)

$$(4.3)\quad \int_{-\infty}^\infty dF(x) - 2\mathcal{R}\int_{-\infty}^\infty e^{-i\alpha x}G(x)\,dF(x)$$
$$+\int_{-\infty}^\infty |G(x)|^2\,dF(x)$$
$$= \int_{-\infty}^\infty |e^{i\alpha x} - G(x)|^2\,dF(x).$$

Lemma 3. Suppose that there exists a function $H(x)$ of $L_2(F)$ which is the Fourier-Stieltjes transform of a function of $\boldsymbol{K}(0,\infty)$ such that for a positive number $\alpha$, it holds

$$(4.4)\quad \int_{-\infty}^\infty e^{i\tau x}H(x)\,dF(x) = \int_{-\infty}^\infty e^{i\tau x}e^{i\alpha x}\,dF(x),$$

for all $\tau > 0$.

Then, for the Fourier-Stieltjes transform $G(x)$ of any function of $\boldsymbol{K}$, we have

$$(4.5)\quad \int_{-\infty}^\infty H(x)\,\overline{G(x)}\,dF(x)$$
$$= \int_{-\infty}^\infty e^{i\alpha x}\,\overline{G(x)}\,dF(x)$$

and consequently

$$(4.6)\quad \int_{-\infty}^\infty |H(x)|^2\,dF(x)$$
$$= \int_{-\infty}^\infty e^{-i\alpha x}H(x)\,dF(x)$$
$$= \int_{-\infty}^\infty e^{i\alpha x}\,\overline{H(x)}\,dF(x)$$
$$= \mathcal{R}\int_{-\infty}^\infty e^{-i\alpha x}H(x)\,dF(x).$$

Proof. Let

$$H(x) = \underset{A\to\infty}{\overset{L_2(F)}{l.i.m.}}\int_0^A e^{-i\theta x}\,dL(\theta),$$

$$G(x)=\underset{A\to\infty}{\overset{L_2(F)}{l.i.m.}}\int_0^A e^{-i\theta x}dK(\theta),$$

$$L(\theta),\ K(\theta)\in \mathbf{K}(0,\infty).$$

The left hand side of (4.5) is

$$\int_{-\infty}^{\infty}H(x)dF(x)\underset{A\to\infty}{\overset{L_2(F)}{l.i.m.}}\int_0^A e^{i\theta x}d\overline{K(\theta)}$$

$$=\lim_{A\to\infty}\int_0^A d\overline{K(\theta)}\int_{-\infty}^{\infty}H(x)e^{i\theta x}dF(x)$$

which is, by (4.4)

$$\lim_{A\to\infty}\int_0^A d\overline{K(\theta)}\int_{-\infty}^{\infty}e^{i\theta x}e^{i\alpha x}dF(x)$$

$$=\int_{-\infty}^{\infty}e^{i\alpha x}dF(x)\underset{A\to\infty}{\overset{L_2(F)}{l.i.m.}}\int_0^A e^{i\theta x}d\overline{K(\theta)}$$

$$=\int_{-\infty}^{\infty}e^{i\alpha x}\overline{G(x)}\,dF(x).$$

This is (4.5). (4.6) are immediate since the left hand side of (4.6) is real.

Now we put

$$(4.7)\qquad J(G)=\int_{-\infty}^{\infty}|e^{i\alpha x}-G(x)|^2dF(x).$$

And we shall prove the following theorem.

<u>Theorem 13. If $G(x)$ is the Fourier-Stieltjes transform of a function of $\mathbf{K}(0,\infty)$ and $H(x)$ is the function in (4.4), then</u>

$$(4.8)\qquad J(G)\geq J(H).$$

<u>The equality holds if and only if $G(x)=H(x)$ almost everywhere with respect to measure function $F(x)$</u> .

Proof. By (4.3), we have

$$J(G)=\int_{-\infty}^{\infty}dF(x)-2\mathcal{R}\int_{-\infty}^{\infty}e^{-i\alpha x}G(x)dF(x)$$
$$+\int_{-\infty}^{\infty}|G(x)|^2dF(x),$$

and

$$J(H)=\int_{-\infty}^{\infty}dF(x)-2\mathcal{R}\int_{-\infty}^{\infty}e^{-i\alpha x}H(x)dF(x)$$
$$+\int_{-\infty}^{\infty}|H(x)|^2dF(x)$$

which is, by Lemma 2,

$$\int_{-\infty}^{\infty}dF(x)-\int_{-\infty}^{\infty}|H(x)|^2dF(x).$$

Hence we have

$$(4.9)\qquad \begin{aligned}J(G)-J(H)&=\int_{-\infty}^{\infty}|H(x)|^2dF(x)\\&-2\mathcal{R}\int_{-\infty}^{\infty}e^{-i\alpha x}G(x)dF(x)+\int_{-\infty}^{\infty}|G(x)|^2dF(x).\end{aligned}$$

Using (4.5), we get

$$J(G)-J(H)=\int_{-\infty}^{\infty}|H(x)|^2dF(x)$$
$$-2\mathcal{R}\int_{-\infty}^{\infty}G(x)\overline{H(x)}dF(x)+\int_{-\infty}^{\infty}|G(x)|^2dF(x)$$
$$=\int_{-\infty}^{\infty}|H(x)-G(x)|^2dF(x),$$

from which the conclusions of our theorem are obvious.

The above discussions are settled into the following theorem.

<u>Theorem 14. If there exists $H(x)$ which satisfies (4.4) in Lemma 2</u>

$$H(x)=\underset{A\to\infty}{\overset{L_2(F)}{l.i.m.}}\int_0^A e^{-i\theta x}dL(\theta),$$

$$L(\theta)\in\mathbf{K}(0,\infty),$$

<u>then the error $J(G)$ when we predict $X(t+\alpha)$, $(\alpha>0)$ by $\int_0^\infty X(t-\theta)\,dK(\theta)$, is minimum when $K(\theta)=L(\theta)$</u> .

4.2. Throughout this section, we set a further assumption after N.Wiener [6] that the spectral function $F(x)$ of $X(t)$ is absolutely continuous and such that

$$(4.10)\qquad \int_{-\infty}^{\infty}\frac{|\log F'(x)|}{1+x^2}dx<\infty.$$

If we put $F'(x)=\Phi(x)$, then by the well-known theorem of Paley and Wiener [7], there exists a function $\Psi(x)$ such that

$$(4.11)\qquad \Phi(x)=|\Psi(x)|^2$$

and the Fourier transform in $L_2$ of $\Psi(x)$

$$(4.12)\quad \psi(t) = \underset{\substack{A\to-\infty\\ B\to\infty}}{\overset{L_2}{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_A^B \Psi(x) e^{ixt} dx$$

is such that

$$\psi(t) = 0 \quad , \quad (t<0).$$

Moreover we assume that, there exists a function $L(\theta) \in \mathbf{K}(0,\infty)$ such that

$$(4.13)\quad \underset{\substack{A\to-\infty\\ B\to\infty}}{\overset{L_2}{l.i.m.}} \int_0^\infty \frac{1}{\sqrt{2\pi}} \left(\int_A^B \Psi(x) e^{ix(t-\theta)} dx\right) dL(\theta)$$

$$= \psi(t+\alpha), \quad (t>0)$$

$$= 0, \qquad (t<0).$$

The left hand side of (4.12) actually exists. For if we put

$$(4.14)\quad H(x) = \underset{C\to\infty}{\overset{L_2(F)}{l.i.m.}} \int_0^C e^{-ix\theta} dL(\theta),$$

then, since $H(x) \in L_2(F)$ and hence $H(x)\Psi(x) \in L_2(-\infty,\infty)$, the Fourier transform

$$\underset{\substack{A\to-\infty\\ B\to\infty}}{\overset{L_2}{l.i.m}} \int_A^B H(x)\Psi(x) e^{ixt} dx$$

exists and this is

$$\underset{\substack{A\to-\infty\\ B\to\infty}}{\overset{L_2}{l.i.m.}} \int_A^B \Psi(x) e^{ixt} dx \underset{C\to\infty}{\overset{L_2(F)}{l.i.m.}} \int_0^C e^{-ix\theta} dL(\theta)$$

$$= \underset{\substack{A\to-\infty\\ B\to\infty}}{\overset{L_2}{l.i.m.}} \lim_{C\to\infty} \int_0^C dL(\theta) \int_A^B \Psi(x) e^{ix(t-\theta)} dx$$

$$(4.15)\quad = \underset{\substack{A\to-\infty\\ B\to\infty}}{\overset{L_2}{l.i.m.}} \int_0^\infty dL(\theta) \int_A^B \Psi(x) e^{ix(t-\theta)} dx.$$

<u>Lemma 3.</u> <u>If there exists a function $H(x)$ such that (4.13) and (4.14) hold, then $H(x)$ satisfies the condition (4.4).</u>

Proof. By (4.15) and (4.13)

$$\underset{\substack{A\to-\infty\\ B\to\infty}}{\overset{L_2}{l.i.m.}} \int_A^B H(x)\Psi(x) e^{ixt} dx$$

$$= \sqrt{2\pi}\, \psi(t+\alpha), \quad t>0, \alpha>0.$$

Therefore, for any positive number $\tau$,

$$\int_{-\infty}^\infty e^{i\tau x} H(x)\, dF(x)$$

$$= \int_{-\infty}^\infty e^{i\tau x} H(x)\Psi(x)\overline{\Psi(x)}\, dx$$

which equals, by Parseval relation, to

$$\int_{-\infty}^\infty \psi(t+\tau+\alpha)\overline{\psi(t)}\, dt$$

which is, again by Parseval relation

$$\int_{-\infty}^\infty e^{i(\tau+\alpha)x} \Psi(x)\overline{\Psi(x)}\, dx$$

$$= \int_{-\infty}^\infty e^{i(\tau+\alpha)x} dF(x).$$

<u>Theorem 15.</u> <u>If (4.10) holds and there exists a function $L(\theta) \in \mathbf{K}(0,\infty)$ which satisfies (4.13), then $J(G)$ attains its minimum value when and only when $G(x)$ is the Fourier-Stieltjes transform $H(x)$ of $L(\theta)$ (except possibly in a set of $F(x)$-measure zero). Moreover in this case</u>

$$(4.16)\quad J(H) = \int_0^\alpha |\psi(t)|^2 dt.$$

We have only to prove the latter part. By (4.7),

$$J(H) = \int_{-\infty}^\infty |e^{i\alpha x} - H(x)|^2 \Phi(x)\, dx$$

$$= \int_{-\infty}^\infty |\Psi(x) e^{i\alpha x} - H(x)\Psi(x)|^2 dx.$$

The Fourier transforms of $\Psi(x)e^{i\alpha x}$ and $H(x)\Psi(x)$ are $\psi(t+\alpha)$ and $\xi(t) = \psi(t+\alpha)$ $(t>0)$, $= 0$ $(t<0)$ respectively. Hence by Parseval relation, we have

$$J(H) = \int_{-\infty}^\infty |\psi(t+\alpha) - \xi(t)|^2 dt$$

$$= \int_{-\infty}^0 |\psi(t+\alpha)|^2 dt = \int_{-\alpha}^0 |\psi(t+\alpha)|^2 dt$$

$$= \int_0^\alpha |\psi(t)|^2 dt$$

which is to be proved.

Lastly we add a remark that, under the assumptions of Theorem 15, $H(x)$ can be written as

$$(4.17)\quad H(x) = \frac{1}{\Phi(\alpha)} \underset{A\to\infty}{\overset{L_2}{\mathrm{l.i.m.}}} \frac{1}{\sqrt{\pi}} \int_0^A \psi(t+\alpha) e^{-itx}\, dt .$$

This is obvious from the proof of Lemma 3.

## §5. A class of stationary processes.

5.1. Let $X(t)$ be a continuous stationary process and let its spectral and random spectral functions be $F(x)$ and $Z(x)$ respectively, so that

$$(5.1)\quad X(t) = {}^{*}\!\!\int_{-\infty}^{\infty} e^{it\alpha}\, d\, Z(\alpha) .$$

Consider divisions $D_i$ of $(-\infty, \infty)$ :

$$D_i : \quad -\infty \leqq \alpha_{i,1} < \alpha_{i,2} < \cdots < \alpha_{i,n-1} < \alpha_{i,n} \leqq \infty$$

and put

$$Z(\alpha_{i,k+1}) - Z(\alpha_{i,k}) = Z(\alpha_{i,k}, \alpha_{i,k+1}) = \Delta Z(\alpha_{i,k}),$$

$$Z(-\infty) = 0 , \quad Z(\infty) = F(\infty) - F(-\infty)$$

Suppose that $E\{|X(t)|^8\} < C$, $C$ being independent of $t$, and that there exists a constant $M$ such that for any divisions $D_i$ $(i = 1, 2, \cdots, 8)$

$$(5.2)\quad \sum_{\substack{p,q,r,s,\\ i,j,k,l}} \{E\{|\Delta Z(\alpha_{1p})\Delta Z(\alpha_{2q})\Delta Z(\alpha_{3r})\Delta Z(\alpha_{4s}) \cdot \overline{\Delta Z(\alpha_{5i})}\cdot\overline{\Delta Z(\alpha_{6j})}\,\overline{\Delta Z(\alpha_{7k})}\,\overline{\Delta Z(\alpha_{8l})}|\}\} \leq M < \infty .$$

We denote the class of such $X(t)$ as $\mathbf{S}$ . Similar class has been introduced by Blanc-Lapierre [7] .

Let $E\{|X(t)|^4\} < \infty$ and

$$E\{X(t+h_1)X(t+h_2)\overline{X(t+h_3)}\,\overline{X(t+h_4)}\}$$

$$(5.3)\qquad = \varphi(h_1, h_2, h_3, h_4)$$

is independent of $t$ for every $h_1$, $h_2$, $h_3$ and $h_4$ . In this case $X(t)$ is said the stationary process of the fourth order. Following theorem is essentially due to Blanc-Lapierre [8].

Theorem 16. Let $X(t) \in \mathbf{S}$ . Then in order that $X(t)$ is a stationary process of the 4-th order, it is necessary and sufficient that if the hyper-rectangle with sides $\Delta\alpha$ , $\Delta\beta$ , $\Delta\gamma$ and $\Delta\delta$ in Euclidean space $R_4$ has no common point with any of hyper-planes

$$(5.4)\quad x + y - z - w = 0$$

( $x$ , $y$ , $z$ , $w$ are current coordinates), then

$$(5.5)\quad E\{(Z(\alpha+\Delta\alpha) - Z(\alpha))\cdot(Z(\beta+\Delta\beta) - Z(\beta)) \cdot (\overline{Z(\gamma+\Delta\gamma)} - \overline{Z(\gamma)})\cdot(\overline{Z(\delta+\Delta\delta)} - \overline{Z(\delta)})\}$$

$$= 0 .$$

Before proving the theorem, we shall show that

$$(5.6)\quad X(t+h_1)X(t+h_2)\overline{X(t+h_3)}\,\overline{X(t+h_4)}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{it(\alpha+\beta-\gamma-\delta)} \cdot e^{i(h_1\alpha + h_2\beta - h_3\gamma - h_4\delta)} \cdot dZ(\alpha)\, dZ(\beta)\, d\overline{Z(\gamma)}\, d\overline{Z(\delta)} .$$

holds with probability 1. The right side is defined as

$$(5.7)\quad \mathrm{l.i.m.} \sum_{i,j,k,l} e^{it(\alpha_i+\beta_j-\gamma_k-\delta_l)} \cdot e^{i(h_1\alpha_i + h_2\beta_j - h_3\gamma_k - h_4\delta_l)} \cdot \Delta Z(\alpha_i)\Delta Z(\beta_j)\overline{\Delta Z(\gamma_k)}\,\overline{\Delta Z(\delta_l)},$$

which exists by the condition (5.2) by the similar arguments as in the proof of existence of Riemann-Stieltjes integral. The equality of (5.6) can be shown as follows.

$$(5.8)\quad E\{|X(t+h_1)X(t+h_2)\overline{X(t+h_3)}\,\overline{X(t+h_4)} - \sum_i \sum_j \sum_k \sum_l |\}$$

(the summand is the one in (5.7))

$$\leq E\{|X(t+h_1)\sum_j\sum_k\sum_l - \sum_i\sum_j\sum_k\sum_l|\}$$

$$+ E\{|X(t+h_1)X(t+h_2)\sum_k\sum_l - X(t+h_1)\sum_j\sum_k\sum_l|\}$$

$$+ E\{ | X(t+h_1) X(t+h_2) \overline{X(t+h_3)} \cdot \sum_{\ell}$$

$$- X(t+h_1) X(t+h_2) \sum_k \sum_\ell | \}$$

$$+ E\{ | X(t+h_1) X(t+h_2) \overline{X(t+h_3)}\, \overline{X(t+h_4)}$$

$$- X(t+h_1) X(t+h_2) \overline{X(t+h_3)} \cdot \sum_{\ell} | \}$$

in which, for example $\sum_k \sum_\ell$ means

$$\sum_k \sum_\ell e^{it(-\gamma_k - \delta_\ell)} \cdot e^{i(-h_3\gamma_k - h_4\delta_\ell)} \cdot \Delta \overline{Z(\gamma_k)}\, \Delta \overline{Z(\delta_\ell)}$$

other summations are easily analogized. For example, the second term of the right side of (5.8) is not greater than, by Schwarz inequality,

$$(5.9) \quad \Big[ E\{ | X(t+h_1) \sum_k \sum_\ell |^2 \} \Big]^{\frac{1}{2}} \cdot \Big[ E\{ | X(t+h_2) - \sum_j |^2 \} \Big]^{\frac{1}{2}}.$$

The former factor is

$$\leq E\{ | X(t+h_1)|^4 \}^{\frac{1}{4}} \cdot \{ E | \sum_k \sum_\ell |^4 \}^{\frac{1}{4}}$$

which is bounded by (5.2). The second factor tends to zero as the division is made indefinitely minute. Other terms of the right side of (5.8) are similarly shown to vanish in the limit. Hence we obtained that

$\sum_i \sum_j \sum_k \sum_\ell$ tends in $L_1$-mean to $X(t+h_1) X(t+h_2) \overline{X(t+h_3)}\, \overline{X(t+h_4)}$.

But since $\mathrm{l.i.m.} \sum_i \sum_j \sum_k \sum_\ell$ (in $L_2$-mean $E\{| \cdot |^2\}$) exist, this is equal to $X(t+h_1) X(t+h_2) \overline{X(t+h_3)}\, \overline{X(t+h_4)}$ with probability 1, which proves (5.6).

Now we shall prove Theorem 16. Let (5.5) hold for hyper rectangles with no common points with every hyper-plane (5.4).

$$(5.10) \quad E\{ X(t+h_1) X(t+h_2) \overline{X(t+h_3)}\, \overline{X(t+h_4)} \}$$

$$= E\{ \mathrm{l.i.m.} \sum_{i,j,k,\ell} \},$$

$\sum_{i,j,k,\ell}$ being the sum in (5.7),

$$= \lim E\{ \sum_{i,j,k,\ell} \}$$

$$(5.11) \quad = \lim \sum_{i,j,k,\ell} e^{it(\alpha_i + \beta_j - \gamma_k - \delta_\ell)} \cdot e^{i(h_1\alpha_i + h_2\beta_j - h_3\gamma_k - h_4\delta_\ell)} \cdot E\{ \Delta Z(\alpha_i) \Delta Z(\beta_j) \Delta \overline{Z(\gamma_k)} \Delta \overline{Z(\delta_\ell)} \}$$

in which $E\{\cdot\}$ is zero if the hyper rectangles with sides $\Delta\alpha_i$, $\Delta\beta_j$, $\Delta\gamma_k$, $\Delta\delta_\ell$ has no common point with (5.4). And the difference between (5.11) and

$$(5.12) \quad \lim \sum_{i,j,k,\ell} e^{i(h_1\alpha_i + h_2\beta_j - h_3\gamma_k - h_4\delta_\ell)} \cdot E\{ \Delta Z(\alpha_i) \Delta Z(\beta_j) \Delta \overline{Z(\gamma_k)} \Delta \overline{Z(\delta_\ell)} \}$$

is easily seen to be zero. Hence (5.10) is (5.12) which is independent of $t$.

Conversely let (5.10) be independent of $t$ for all real numbers $h_1$, $h_2$, $h_3$ and $h_4$. Then (5.11) and (5.12) are equal for all $t$. For sufficiently large $A$, $B$, $C$, and $D$, the sum (5.11) over $|\alpha_i| > A$, $|\beta_j| > B$, $|\gamma_k| > C$ and $|\delta_\ell| > D$ are arbitrarily small, which is the consequence of (5.2). Using this fact and approximating the trapesoidal functions

$$g(\alpha) = 1, \quad (\alpha^{(1)} < \alpha < \alpha^{(2)}),$$
$$= 0 \quad (x > \alpha^{(2)} + \varepsilon, \ x < \alpha^{(1)} - \varepsilon)$$

by $\sum_\nu c_\nu e^{ih_i^{(\nu)}\alpha}$, we can prove that

$$(5.13) \quad \lim \sum_{i,j,k,\ell}' (e^{it(\alpha_i + \beta_j - \delta_k - \delta_\ell)} - 1) \cdot E\{ \Delta Z(\alpha_i) \Delta Z(\beta_j) \Delta \overline{Z(\delta_k)} \Delta \overline{Z(\delta_\ell)} \}$$

$$= 0,$$

where $\sum'$ denotes to take the sum over $\alpha^{(1)} < \alpha_i < \alpha^{(2)}$, $\beta^{(1)} < \beta_j < \beta^{(2)}$, $\gamma^{(1)} < \gamma_k < \gamma^{(2)}$ and $\delta^{(1)} < \delta_\ell < \delta^{(2)}$.

$\alpha^{(1)}, \alpha^{(2)}$ ; $\beta^{(1)}, \beta^{(2)}$ ; $\gamma^{(1)}, \gamma^{(2)}$; $\delta^{(1)}, \delta^{(2)}$ are arbitrary real numbers. (5.13) can be represented as

$$\int_{\alpha^{(1)}}^{\alpha^{(2)}} \int_{\beta^{(1)}}^{\beta^{(2)}} \int_{\gamma^{(1)}}^{\gamma^{(2)}} \int_{\delta^{(1)}}^{\delta^{(2)}} (e^{it(\alpha+\beta-\gamma-\delta)} - 1) \cdot$$
$$\cdot E\{ dZ(\alpha) dZ(\beta) d\overline{Z(\gamma)} d\overline{Z(\delta)} \}$$
$$= 0$$

from which it easily results that if the hyper-rectangle with sides $(\alpha^{(1)}, \alpha^{(2)})$ $(\beta^{(1)}, \beta^{(2)})$, $(\gamma^{(1)}, \gamma^{(2)})$, and $(\delta^{(1)}, \delta^{(2)})$ has no common point with $\alpha + \beta - \gamma - \delta = 0$, then

$$E\{ \Delta Z(\alpha^{(1)}, \alpha^{(2)}) \cdot \Delta Z(\beta^{(1)}, \beta^{(2)}) \overline{\Delta Z(\gamma^{(1)}, \gamma^{(2)})}$$
$$\cdot \Delta \overline{Z(\delta^{(1)}, \delta^{(2)})} \} = 0,$$

which is to be proved.

5.2. We shall now consider the harmonic analysis of $X(t+u)\overline{X(t)}$ ($u$ being fixed). Let us suppose throughout that a continuous stationary process $X(t) \in \mathbf{S}$ is further a stationary process of the 4th order.

We first remark:

Lemma 3. $X(t)$ is continuous in mean of the 4th order, that is

$$\lim_{h\to 0} E\{ |X(t+h) - X(t)|^4 \} = 0.$$

We have

$$= E\{ |X(t+h) - X(t)|^4 \}.$$
$$= E\{ |X(t+h) - X(t)| \cdot |X(t+h) - X(t)|^3 \}$$
$$\leqq [E\{ |X(t+h) - X(t)|^2 \}]^{\frac{1}{2}} \cdot$$
$$\cdot [E\{ |X(t+h) - X(t)|^6 \}]^{\frac{1}{2}}$$
$$\leqq C_1 [E\{ |X(t+h) - X(t)|^2 \}]^{\frac{1}{2}} \cdot$$
$$\cdot ([E\{ |X(t+h)|^6 \}]^{\frac{1}{2}} + [E\{ |X(t)|^6 \}]^{\frac{1}{2}})$$
$$\leqq C_1 [E\{ |X(t+h) - X(t)|^2 \}]^{\frac{1}{2}}$$
$$\cdot ([E\{ |X(t+h)|^8 \}]^{\frac{3}{8}} + [E\{ |X(t)|^8 \}]^{\frac{3}{8}})$$
$$\leqq C_2 [E\{ |X(t+h) - X(t)|^2 \}]^{\frac{1}{2}}$$

($C_1$, $C_2$ being constants) which tends to zero as $h \to 0$.

Now if we consider $X(t+u)\overline{X(t)}$, then this is a continuous stationary process of the 2nd order, $t$ being a parameter, and so we have

$$(5.14) \quad X(t+u)\overline{X(t)} = \int_{-\infty}^{\infty} e^{itv} dW(v),$$

where $W(v) = W(v, u)$ is the random spectral function.

Theorem 17.

$$(5.15) \quad W(v) = \iint_{\beta-\alpha \leqq v} e^{i\beta u} d\overline{Z(\alpha)} dZ(\beta),$$

and if $u = 0$, then

$$(5.16) \quad W(v) = \int_{-\infty}^{\infty} Z(v+\alpha) d\overline{Z(\alpha)},$$

and furthermore

$$(5.17) \quad E\{W(v)\} = \int_{-\infty}^{\infty} e^{i\alpha u} dF(\alpha), \quad v > 0$$
$$= 0, \qquad v < 0,$$

$F(x)$ being the spectral function of $X(t)$.

Proof. Using the spectral representation of $X(t)$, we have

$$\overline{X(t)} X(t+u)$$
$$= \int_{-\infty}^{\infty} e^{-i\alpha t} d\overline{Z(\alpha)} \int_{-\infty}^{\infty} e^{i(t+u)\beta} dZ(\beta)$$
$$= \mathrm{l.i.m} \sum_i \sum_j e^{it(\beta_j - \alpha_i)} e^{i\beta_j u}$$
$$\cdot \Delta \overline{Z(\alpha_i)} \Delta Z(\beta_j)$$

which is seen by the fact $E\{ |\sum e^{-i\alpha_i t} \Delta Z(\alpha_i)|^2 \} < M$ and (5.2), $\{\alpha_i\}$, $\{\beta_j\}$ are arbitrary divisions of $(-\infty, \infty)$, and further is verified to be

$$\int_{-\infty}^{\infty} e^{itv} d\left( \iint_{\beta-\alpha<v} e^{i\beta u} d\overline{Z(\alpha)} dZ(\beta) \right).$$

Hence we have

$$W(v) = \iint_{\beta-\alpha<v} e^{i\beta u} d\overline{Z(\alpha)} dZ(\beta),$$

which is (5.15). If $u = 0$, then

$$(5.18) \quad W(v)$$
$$= \mathrm{l.i.m} \sum_{i,j}{}' \Delta\overline{Z(\alpha_i)} \Delta Z(\beta_j)$$

where $\sum_{i,j}'$ means to sum up $\Delta\overline{Z(\alpha_i)}, \Delta Z(\beta_j))$ with respect to $i, j$

such that the hyper-rectangle with sides $(\alpha_i, \alpha_{i+1})$ , $(\beta_j, \beta_{j+1})$ is contained in the half-space $\beta - \alpha < \nu$ . Then (5.17) is

$$\text{l.i.m.} \sum_{\alpha_i} \sum_{\xi_j < \nu}{}' \Delta \overline{Z(\alpha_i)} \Delta_j Z(\xi_j + \alpha_i)$$

(in which the meanings of $\sum'_{\xi_j < \nu}$ and $\Delta_j Z(\xi_j + \alpha_i)$ are easily understood)

$$= \text{l.i.m.} \sum_{\xi_j < \nu}{}' \Delta_j \Big( \sum_{\alpha_i} \Delta Z(\alpha_i) \cdot Z(\xi_j + \alpha_i) \Big)$$

which is seen to be

$$\int_{-\infty}^{\nu} d_\xi \Big( \int_{-\infty}^{\infty} Z(\xi + \alpha) d\overline{Z(\alpha)} \Big)$$

$$= \int_{-\infty}^{\infty} Z(\nu + \alpha) d \overline{Z(\alpha)}.$$

Next by an analogous expression as (5.18)

$$E\{W(\nu)\}$$

$$= E\Big\{ \text{l.i.m.} \sum_{i,j}{}' e^{i\beta_j u} \Delta \overline{Z(\alpha_i)} \Delta Z(\beta_j) \Big\}$$

(5.19)

$$= \lim \sum_{i,j}{}' e^{i\beta_j u} \cdot E\{\Delta \overline{Z(\alpha_i)} \cdot \Delta Z(\beta_j)\}.$$

By II in 2.1., assuming $\nu > 0$ and taking $\alpha_i = \beta_i$ ,

$$E\{W(\nu)\} = \lim \sum_i e^{i\alpha_i u} (F(\alpha_{i+1}) - F(\alpha_i))$$

$$= \int_{-\infty}^{\infty} e^{i\alpha u} dF(\alpha).$$

If $\nu < 0$ , then in $\sum'_{i,j}$ in (5.19), $\beta_j$ can not be $\alpha_i$ , and hence

$$E\{W(\nu)\} = 0.$$

Thus we have proved the theorem.

Now applying the law of large number, Theorem 6, to the stationary process $\overline{X(t)} X(t+u) - \rho(u)$ ( $u$ being taken as a constant), we have

Theorem 18.

$$(5.20) \quad \underset{T\to\infty}{\text{l.i.m.}} \frac{1}{T} \int_0^T \overline{X(t)} X(t+u) dt$$

$$= W(+0) - W(-0),$$

where $W(\nu)$ is the one in Theorem 17.

The existence of the right hand side of (5.20) is obvious since $W(\nu)$ is a random spectral function.

Theorem 19. We have

$$(5.21) \quad \underset{T\to\infty}{\text{l.i.m.}} \frac{1}{T} \int_0^T |X(t)|^2 dt$$

$$= \underset{\nu\to+0}{\text{l.i.m.}} \int_{-\infty}^{\infty} \{Z(\alpha+\nu) - Z(\alpha-\nu)\} d\overline{Z(\alpha)}$$

$$(5.22) \quad = \underset{h\to 0}{\text{l.i.m.}} \frac{1}{2h} \int_{-\infty}^{\infty} |Z(\alpha+h) - Z(\alpha-h)|^2 d\alpha.$$

(5.21) is a special case of Theorem 18. The proof of (5.22) shall be postponed to the latter section 6.2.

Theorem 20.

$$(5.23) \quad \underset{T\to\infty}{\text{l.i.m.}} \frac{1}{T} \int_0^T \overline{X(t)} X(t+u) e^{-i\xi t} dt$$

$$= W(\xi + 0) - W(\xi - 0).$$

This is easily seen as in Theorem 18.

We shall further add a remark that

$$(5.24) \quad E\{W(+0) - W(-0)\} = \rho(u),$$

$$(5.25) \quad E\{W(\xi+0) - W(\xi-0)\} = 0, \quad (\xi \neq 0)$$

These are also special cases of Theorem 18 and 20. For

$$E\{W(+0) - W(-0)\}$$

$$= E\{\underset{T\to\infty}{l.i.m}\ \frac{1}{T}\int_0^T \overline{X(t)}\, X(t+u)\, dt\}$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_0^T E\{\overline{X(t)}\, X(t+u)\} dt$$

$$= \rho(u).$$

Similarly we have

$$E\{W(\xi+0)-W(\xi-0)\}$$

$$= E\{\underset{T\to\infty}{l.i.m.}\ \frac{1}{T}\int_0^T \overline{X(t)}\, X(t+u)\, e^{-i\xi t} dt\}$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_0^T E\{\overline{X(t)}\, X(t+u)\} e^{-i\xi t} dt$$

$$= \rho(u) \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{-i\xi t} dt = 0, \ (\xi \neq 0).$$

§6. Concentration of a random spectral function.

6.1. Let $X(t) \in \mathbf{S}$. Since

$$E\{\int_{-\infty}^{\infty} \frac{|X(t)|^2}{1+t^2} dt\} = \int_{-\infty}^{\infty} \frac{E\{|X(t)|^2\}}{1+t^2} dt$$

$$= \sigma^2 \int_{-\infty}^{\infty} \frac{dt}{1+t^2} < \infty,$$

we have

$$(6.1) \quad \int_{-\infty}^{\infty} \frac{|X(t)|^2}{1+t^2} dt < \infty$$

with probability 1.(9) Hence $X(t)\frac{\sin ht}{t}$ has, for fixed $h>0$, a Fourier transform in $L_2(-\infty,\infty)$ with probability 1. We shall prove the Fourier transform to be $(\pi/2)^{1/2}\{Z(\alpha+h) - Z(\alpha-h)\}$, $Z(\alpha)$ being the random spectral function of $X(t)$. Before it, it is convenient to state a lemma.

Lemma 5. Suppose that $\varphi(t) \in L_1(-\infty,\infty)$ and

$$(6.2) \quad \underset{A\to\infty}{l.i.m.}\ X_A(t) = X(t)$$

boundedly in mean $-\infty < t < \infty$. Then

$$(6.3) \quad \underset{A\to\infty}{l.i.m.} \int_{-\infty}^{\infty} \varphi(t) X_A(t)\, dt = \int_{-\infty}^{\infty} \varphi(t) X(t) dt.$$

From assumptions, we have

$$(6.4) \quad E\{|X_A(t) - X(t)|^2\} \leqq K,$$

$K$ being a constant independent of $t$, and

$$(6.5) \quad \lim_{A\to\infty} E\{|X_A(t) - X(t)|^2\} = 0.$$

Now

$$E\{|\int_{-\infty}^{\infty}\varphi(t)\{X_A(t)-X(t)\}dt|^2\}$$

$$\leqq E[\{\int_{-\infty}^{\infty}|\varphi(t)|dt\}\cdot\{\int_{-\infty}^{\infty}|\varphi(t)|\cdot |X_A(t)-X(t)|^2 dt\}]$$

$$= \int_{-\infty}^{\infty}|\varphi(t)|dt \cdot \int_{-\infty}^{\infty}|\varphi(t)| E\{|X_A(t)-X(t)|^2\} dt.$$

By (6.4) and (6.5), the second integral of the last term converges to zero.

Theorem 20. Let $X(t)$ a stationary process of $\mathbf{S}$. Then with probability 1, the Fourier transform (in $L_2(-\infty\ \infty)$) of

$$\left(\frac{2}{\pi}\right)^{1/2} X(t)\cdot \frac{\sin ht}{t}$$

is $Z(\alpha+h) - Z(\alpha-h)$, $h>0$. And with probability 1, we have

$$(6.6) \quad \frac{1}{\pi h}\int_{-\infty}^{\infty} \overline{X(u)}\, X(t+u) \frac{\sin hu}{u}\cdot\frac{\sin h(t+u)}{t+u} du$$

$$= \frac{1}{2h}\int_{-\infty}^{\infty} e^{it\alpha}|Z(\alpha+h) - Z(\alpha-h)|^2 d\alpha,$$

especially

$$(6.7) \quad \frac{1}{\pi h}\int_{-\infty}^{\infty} |X(t)|^2 \frac{\sin^2 ht}{t^2} dt$$

$$= \frac{1}{2h}\int_{-\infty}^{\infty} |Z(\alpha+h) - Z(\alpha-h)|^2 d\alpha,$$

with probability 1.

The right hand side of (6.7) can be considered as the mean concentration of the random spectral function $Z(\alpha)$ and we denote as $C(h)$.

Proof. Let $F(x)$ be the spectral function of $X(t)$ .

$$\underset{A\to\infty}{l.i.m.}\int_{-A}^{A} e^{it\alpha} dZ(\alpha) = X(t)$$

holds boundedly, since

$$E\{|\int_{-A}^{A} e^{it\alpha} dZ(\alpha)|^2\} = \int_{-A}^{A} dF(x) \le \sigma^2,$$

$$E\{|X(t)|^2\} = \sigma^2.$$

Therefore by Lemma 5,

$$\frac{1}{\sqrt{2\pi}}\int_{-T}^{T} X(t)\frac{\sin ht}{t} e^{-ixt} dt$$

$$= \underset{A\to\infty}{l.i.m}\frac{1}{\sqrt{2\pi}}\int_{-T}^{T}\frac{\sin ht}{t} e^{-ixt} dt \cdot \int_{-A}^{A} e^{it\alpha} dZ(\alpha)$$

$$= l.i.m. \frac{1}{\sqrt{2\pi}}\int_{-A}^{A} dZ(\alpha)\int_{-T}^{T}\frac{\sin ht}{t} e^{it(\alpha-x)} dt$$

which is denoted as

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dZ(\alpha)\int_{-T}^{T}\frac{\sin ht}{t} e^{it(\alpha-x)} dt.$$

Put

$$D(\alpha,x) = \begin{cases} 1, & x-h<\alpha<x+h, \\ \frac{1}{2}, & \alpha = x-h,\ x+h, \\ 0, & \text{otherwise}. \end{cases}$$

Then $\frac{1}{\pi}\int_{-T}^{T}\frac{\sin ht}{t} e^{it(\alpha-x)} dt$ converges boundedly to $D(\alpha,x)$ as $T\to\infty$ .

Now

$$E\{|\sqrt{\tfrac{\pi}{2}}\int_{-\infty}^{\infty} dZ(\alpha)\{\frac{1}{\pi}\int_{-T}^{T}\frac{\sin ht}{t} e^{it(\alpha-x)} dt - D(\alpha x)\}|^2\}$$

$$= \frac{\pi}{2}\int_{-\infty}^{\infty}|\frac{1}{\pi}\int_{-T}^{T}\frac{\sin ht}{t} e^{it(\alpha-x)} dt - D(\alpha,x)|^2 dF(\alpha).$$

If $x\pm h$ is continuity points of $F(\alpha)$ , then the above integral tends to zero.

Furthermore

$$\int_{-\infty}^{\infty} E\{|\sqrt{\tfrac{1}{2\pi}}\int_{-T}^{T} X(t)\frac{\sin ht}{t} e^{-ixt} dt - \int_{-\infty}^{\infty} D(\alpha,x) dZ(\alpha)|^2\} dx$$

$$= \frac{\pi}{2}\int_{-\infty}^{\infty} dF(\alpha)\int_{-\infty}^{\infty}|\frac{1}{\pi}\int_{-T}^{T}\frac{\sin ht}{t} e^{it(\alpha-x)} dt - D(\alpha,x)|^2 dx$$

$$= \int_{-\infty}^{\infty} dF(\alpha)\int_{|t|>T}\frac{\sin^2 ht}{t^2} dt \to 0,$$

as $T\to\infty$ .

Hence with probability 1,

$$\int_{-\infty}^{\infty}|\sqrt{\tfrac{1}{2\pi}}\int_{-T_n}^{T_n} X(t)\frac{\sin ht}{t} e^{-ixt} dt - \int_{-\infty}^{\infty} D(\alpha,x) dZ(\alpha)|^2 dx$$

tends to zero as $n\to\infty$ for some sequence $T_n\to\infty$ . On the other hand $X(t)\frac{\sin ht}{t}$ is known to have a Fourier transform (in $L_2$ ) with probability 1 and hence the Fourier transform is

$$\int_{-\infty}^{\infty} D(\alpha,x) dZ(\alpha) = Z(x+h) - Z(x-h).$$

Thus the former part is proved. (6.6) is a result of Parseval relation.

6.2. The object of this section is to prove the following theorem concerning the concentration

$$C(h) = \frac{1}{2h}\int_{-\infty}^{\infty}|Z(\alpha+h) - Z(\alpha-h)|^2 d\alpha$$

$X$ being a stationary process of $\mathbf{S}$ .

Theorem 21.

$$(6.8)\qquad \underset{h\to\infty}{l.i.m.}\ C(h) = |X(0)|^2,$$

$$(6.9)\qquad \underset{h\to+0}{l.i.m.}\ C(h) = \underset{T\to\infty}{l.i.m.}\frac{1}{T}\int_0^T |X(t)|^2 dt.$$

(6.9) is nothing but (5.21) in Theorem 19 which is not yet proved.

The proof of (6.8) is not difficult.

We have now

$$E\left\{\left|\frac{1}{\pi h}\int_{-\infty}^{\infty}|X(t)|^2\frac{\sin^2 ht}{t^2}dt-|X(0)|^2\right|^2\right\}$$

$$=E\left\{\left|\frac{1}{\pi h}\int_{-\infty}^{\infty}\{|X(t)|^2-|X(0)|^2\}\frac{\sin^2 ht}{t^2}dt\right|^2\right\}$$

$$\leqq E\left\{\frac{1}{\pi}\int_{-\infty}^{\infty}\{|X(t)|^2-|X(0)|^2\}^2\frac{\sin^2 ht}{ht^2}dt\right\}$$

(6.10)

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}E\{(|X(t)|^2-|X(0)|^2)^2\}\frac{\sin^2 ht}{ht^2}dt$$

Since

$$E\{(|X(t)|^2-|X(0)|^2)^2\}$$

$$=E\{(|X(t)|-|X(0)|)^2(|X(t)|+|X(0)|)^2\}$$

$$\leqq E\{(|X(t)-X(0)|^2)(|X(t)|+|X(0)|)^2\}$$

$$\leqq [E\{|X(t)-X(0)|^4\}]^{\frac{1}{2}}\cdot[E\{|X(t)|+|X(0)|\}^4]^{\frac{1}{2}}$$

$$\leqq[E\{|X(t)-X(0)|^4\}]^{\frac{1}{2}}\cdot\{8\cdot E\{|X(t)|^4+|X(0)|^4\}\}$$

$$\leqq C\cdot[E\{|X(t)-X(0)|^4\}],$$

by Lemma 3, this tends to zero as $t\to 0$. Hence $E\{|X(t)|^2-|X(0)|^2\}$ is continuous at $t=0$. Thus by the well known property of Fejér's integral, (6.10) converges to zero, which proves (6.8).

Next we shall prove (6.9) by Theorem 19 (5.20)

$$\underset{T\to\infty}{l.i.m.}\frac{1}{T}\int_0^T|X(t)|^2dt$$

$$=\underset{v\to+0}{l.i.m}\int_{-\infty}^{\infty}\{Z(\alpha+v)-Z(\alpha-v)\}d\overline{Z(\alpha)}.$$

The right hand side is

$$\underset{v\to+0}{l.i.m}\int_{-\infty}^{\infty}d\overline{Z(\alpha)}\int_{\alpha-v}^{\alpha+v}dZ(\beta)$$

(6.11)

$$=\underset{v\to+0}{l.i.m.}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}p(v;\alpha,\beta)d\overline{Z(\alpha)}dZ(\beta),$$

where

$$p(v;\alpha,\beta)=1,\quad \alpha-v\leqq\beta\leqq\alpha+v,$$
$$=0,\quad \text{otherwise},$$

while

$$C(v)=\frac{1}{2v}\int_{-\infty}^{\infty}|Z(\gamma+v)-Z(\gamma-v)|^2d\gamma$$

$$=\frac{1}{2v}\int_{-\infty}^{\infty}d\gamma\left|\int_{\gamma-v}^{\gamma+v}dZ(\beta)\right|^2$$

$$=\frac{1}{2v}\int_{-\infty}^{\infty}d\gamma\int_{\gamma-v}^{\gamma+v}dZ(\beta)\int_{\gamma-v}^{\gamma+v}d\overline{Z(\alpha)}$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}q(v;\alpha,\beta)d\overline{Z(\alpha)}dZ(\beta)d\gamma$$

(6.12)

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}q(v;\alpha,\beta,\gamma)d\gamma\right)d\overline{Z(\alpha)}dZ(\beta),$$

where

$$q(v;\alpha,\beta,\gamma)=\frac{1}{2v},\quad \gamma-v\leqq\alpha\leqq\gamma+v,\ \gamma-v\leqq\beta\leqq\gamma+v,$$
$$=0,\quad \text{otherwise}.$$

We have

$$\gamma(v;\alpha,\beta)=\int_{-\infty}^{\infty}q(v;\alpha,\beta,\gamma)d\gamma$$

$$=1-\frac{|\alpha-\beta|}{2v},\quad |\alpha-\beta|\leqq 2v$$

(6.13)

$$=0,\quad \text{otherwise}.$$

Therefore (6.11) and (6.12) shows that

(6.14)

$$\underset{v\to+0}{l.i.m.}C(v)-\underset{T\to\infty}{l.i.m.}\frac{1}{T}\int_0^T|X(t)|^2dt$$

$$=\underset{v\to+0}{l.i.m.}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\gamma(v;\alpha,\beta)-p(v;\alpha,\beta)\}d\overline{Z(\alpha)}dZ(\beta).$$

$\gamma(v;\alpha,\beta)$ and $p(v;\alpha,\beta)$ depend only on $v$ and $\alpha-\beta$ and we put

$$l(v;\alpha-\beta)=\gamma(v;\alpha,\beta)-p(v;\alpha,\beta).$$

(6.14) is

$$\underset{v\to+0}{l.i.m.}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} l(v;\alpha-\beta)\, d\overline{Z(\alpha)}\, dZ(\beta)$$

$$=\underset{v\to+0}{l.i.m.}\int_{-\infty}^{\infty} l(v;y)\, d\Big(\int_{-\infty}^{\infty}\overline{Z(\beta+y)}\, dZ(\beta)\Big)$$

$$=\underset{v\to+0}{l.i.m.}\int_{-\infty}^{\infty} l(v;y)\, dW_1(y).$$

Here $W_1(y)$ is the one putting $u=0$ in $W(y)$ in Theorem 17, and is the random spectral function of $|X(t)|^2$. If $G(x)$ is the spectral function of $|X(t)|^2$, then

$$E\Big\{\Big|\underset{v\to+0}{l.i.m.}\int_{-\infty}^{\infty} l(v;y)\, dW_1(y)\Big|^2\Big\}$$

$$=\lim_{v\to+0} E\Big\{\Big|\int_{-\infty}^{\infty} l(v;y)\, dW_1(y)\Big|^2\Big\}$$

(6.15)

$$=\lim_{v\to+0}\int_{-\infty}^{\infty}|l(v;y)|^2\, dG(y),$$

where

$$l(v;y)=\frac{|y|}{2v},\quad |y|\leq v,$$

$$=1-\frac{|y|}{2v},\quad v<|y|\leq 2v,$$

$$=0,\quad \text{otherwise}.$$

Hence (6.15) becomes

$$\lim_{v\to+0}\Big\{\int_{-v}^{v}\frac{|y|^2}{(2v)^2}\, dG(y)+\int_{2v\geq|y|\geq v}\Big(1-\frac{|y|}{2v}\Big)^2 dG(y)\Big\}$$

$$(6.16)\quad =\lim_{v\to0}\Big\{\frac{1}{4v^2}\int_0^{v} y^2\, d(G(y)-G(-y))+\int_v^{2v}\Big(1-\frac{y}{2v}\Big)^2 d(G(y)-G(-y)\Big)\Big\}.$$

But

$$\frac{1}{v^2}\int_0^{v} y^2\, d(G(y)-G(-y))$$

(6.17)

$$=\Big[\frac{y^2}{v^2}H(y)\Big]_0^{v}-\frac{2}{v^2}\int_0^{v} y H(y)\, dy,$$

$$H(y)=G(y)-G(-y).$$

Letting $v\to+0$, (6.17) tends to $H(+0)-H(+0)=0$. The second integral in (6.16) does not exceed

$$\int_v^{2v} d(G(y)-G(-y))=H(2v)-H(+0)=0$$

which tends to zero and the theorem is proved.

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References

(1) [1] A. Branc-Lapierre et R. Brard, Les functions aléatoires stationaires et la loi grands nombres, Bull. dé la sóc. math. de France, t. 74, (1946).

[2] A. Branc-Lapierre, Remarques sur l'analyse harmonique des functions aléatoires, Revue Scientifique 85 Année, 1027-1040

[3] H. Cramér, On the theory of stationarў procèsses, Ann. of Math., s. 2, 41 (1940).

[4] H. Cramér, On harmonic analysis in certain functional spaces, Arkiv of mat. astron. och fysik, 28 (1942).

[5] J. L. Doob, Time series and harmonic analysis, Proc. of the 1st Barkeley Symposium on Math. Stat. and Prob.

[6] K. Karhunen, Über lineare Methoden in der Wahrscheinlichkeitsrechnung, Ann. Ac. Sci. Fennicae, 37 (1947).

[7] K. Karhunen, Über die Struktur stationärer zufälliger Funktionen, Arkiv für Math. 1 (1949).

[8] A. Khintchine, Korrelationstheorie der stationären stochastichen Prozesse, Math. Ann. 109 (1934).

[9] M. Loeve, Functions aléatoires du second ordre, Notes in P. Lévy, Processus stochastiques et mouvement Brownieu, Paris (1948).

[10] N. Wiener, Extrapolation, interpolation and smoothing of stationary time series, New York, (1949).

(2) K. Karhunen, (1), [6] [7].

(3) H. Cramér, A contribution to the theory of stochastic processes, Proc. 2nd Barkeley Symposium, (1951).

(4) cf. K. Karhunen, (1), [6].
(5) N. Winer, (1), [10].
(6) N. Winer, (1), [10].
(7) R.E.A.C. Paley and N. Winer, The Fourier transforms in the complex domain, Amer. Math. Col. XIX. Chap. 1.
(8) A. Blanc-Lapierre, (1), [2].

(9) Since X(t) is mean continuons, it is a measurable function of (t, $\omega$), $\omega$ being an element of a probability field a domain of X(t).

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