# ON THE CONVERGENCE OF A MULTIPLE POWER SERIES

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§1. The notion of convergence of a multiple series is somewhat complicated; we can consider several kinds of convergences. Here we treat mainly double sequences or double series for simplicity's sake, but the same holds for general multiple ones.

Usually the convergence of a double sequence $s_{mn}$ is defined as follows: a double sequence

$\{s_{mn}\}_{m,n=0}^{\infty}$ converges to $s$, if

for any given $\varepsilon > 0$, we can find a number $\ell_0 = \ell_0(\varepsilon)$ such that for every $m, n \geqq \ell_0$, we have $|s_{mn} - s| < \varepsilon$. The convergence of a double series

$$(1) \qquad \sum_{m,n=0}^{\infty} a_{mn}$$

whose sum is $s$, is defined by the convergence of its partial sums

$$(2) \qquad s_{mn} \equiv \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} a_{\mu\nu}$$

to $s$ in the above sense. We call this the P-convergence (P means the "partial sum") in this paper.

On the other hand, we say a double series (1) is A-convergent (A means the "arrangement"), if at least one of the simple series in which the original series has been arranged is convergent.[1] In this case, the sum has no mean generally, because it depends on the arrangement, unless (1) is absolutely convergent.

It is evident that these two notions coincide with each other for the series with positive terms, and that the absolutely convergent series is A- and P- convergent. But a series which is A- and P-convergent is not always absolutely convergent as easily shown by:

Example 1.

$$\begin{array}{l} 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \\ + (-1) + (-\frac{1}{2}) + (-\frac{1}{3}) + (-\frac{1}{4}) + \cdots \\ + 0 + 0 + 0 + 0 + \cdots \\ + 0 + 0 + 0 + 0 + \cdots \\ + \cdots\cdots\cdots\cdots . \end{array}$$

Also these two convergences are not the same in general cases. In fact:

Example 2.

$$\begin{array}{l} 0 + 1 + \frac{1}{3} + \frac{1}{5} + \cdots \\ + (-\frac{1}{2}) + 0 + 0 + 0 + \cdots \\ + (-\frac{1}{4}) + 0 + 0 + 0 + \cdots \\ + (-\frac{1}{6}) + 0 + 0 + 0 + \cdots \\ + \cdots\cdots\cdots\cdots \end{array}$$

is A-convergent but not P-convergent. Conversely,

Example 3.

$$\begin{array}{l} -1 + 1 + 2 + 3 + 4 + \cdots \\ +1 + (-1) + (-2) + (-3) + (-4) + \cdots \\ +2 + (-2) + 0 + 0 + 0 + \cdots \\ +3 + (-3) + 0 + 0 + 0 + \cdots \\ + \cdots\cdots\cdots\cdots \end{array}$$

is P-convergent but not A-convergent.

We remark that the terms of an A-convergent series are bounded, but this is not true for P-convergent series as has already been shown in Example 3. However, we have from the definition,

Lemma 1. If the double series

$$(1) \qquad \sum_{m,n=0}^{\infty} a_{mn}$$

is P-convergent, there exists a number $\ell$ such that its partial sums

$$(2) \qquad s_{mn} = \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} a_{\mu\nu}$$

are uniformly bounded for $m, n$, provided that both suffixes $m$ and $n$ are $\geqq \ell$.

Corollary. Since we have

$$(3) \qquad a_{mn} = s_{mn} - s_{m\,n-1} - s_{m-1\,n} + s_{m-1\,n-1} \qquad (m, n \geqq 1),$$

$a_{mn}$ are also uniformly bounded for $m, n \geqq \ell+1$.

Definition 1. The minimal integer $\ell$ satisfying the conclusion of Lemma 1 is called the limit of boundedness of the series (1).

§2. For a double power series

$$(4) \qquad \sum_{m,n=0}^{\infty} \alpha_{mn} x^m y^n$$

the following result is very well-known [2]:

Lemma 2. If the terms of a double power series

$$(4) \qquad \sum_{m,n=0}^{\infty} \alpha_{mn} x^m y^n$$

are uniformly bounded at $x = x_0$, $y = y_0$, or especially if (4) is A-convergent at $x = x_0$, $y = y_0$, then (4) converges uniformly and absolutely in every compact subset contained in $|x| < |x_0|$, $|y| < |y_0|$.

The assumption of Lemma 2 cannot be replaced by the P-convergence at $x = x_0$, $y = y_0$, for P-convergence does not imply the boundedness of the terms of (4). It seems to me that the Theorem 1 on p.12 in the book of Prof. M.Tsuji[3] saying as follows is inexact: "If a power series

$$(*) \qquad \sum_{m_1,\cdots,m_n}^{0,\cdots,\infty} a_{m_1\cdots m_n} z_1^{m_1} \cdots z_n^{m_n}$$

is convergent [which means the P-convergence in our terminology] at $z_k = z_k^0 \ (\neq 0)\ (k=1,\cdots,n)$, then (*) converges uniformly and absolutely in $|z_k| \leqq r_k < |z_k^0|$, $(k=1,\cdots,n)$, where $r_k$ are arbitrary positive numbers less than $|z_k^0|$."

Indeed, the following example shows that the P-convergence is not convenient for the convergence of power series.

Example 4. The power series with

$$\alpha_{mn} = \begin{cases} 4 & m=n=0, \\ 1 & m=0, n=1 \text{ and } m=1, n=0, \\ 2 & m=0, n\geqq 2 \text{ and } m \geqq 2, n=0, \\ -2 & m=n=1, \\ -1 & m=1, n\geqq 2 \text{ and } m\geqq 2, n=1, \\ 0 & m, n \geqq 2, \end{cases}$$

i.e.,

$$\sum_{m,n=0}^{\infty} \alpha_{mn} x^m y^n = (2-y)\sum_{m=0}^{\infty} x^m + (2-x)\sum_{n=0}^{\infty} y^n$$

is P-convergent (but not A-convergent) at $x=2$, $y=2$ yet its absolute convergence region is not $|x|<2$, $|y|<2$, but is $|x|<1$, $|y|<1$.

To avoid such cases, the absolute convergence is assumed for Lemma 2 in the books of Bochner-Martin[2] and Severi[4], but it seems to me that A-convergence is enough for Lemma 2. However, what will happen if we dare take the P-convergence to the last? We shall show in the next section, that such singular phenomenon as in Example 4 occurs only on some singular sets, or more exactly, the point-set on which (4) is P-convergent but not absolutely convergent has no inner point.

§3. Theorem. Let the power series

$$(4) \qquad \sum_{m,n=0}^{\infty} \alpha_{mn} x^m y^n$$

be P-convergent at every point of a neighborhood $U$ of a point $(x_0, y_0)$. Then (4) converges absolutely and uniformly in $|x| \leqq |x_0|$, $|y| \leqq |y_0|$.

Proof. Using Lemma 2, we may assume that $x_0 \neq 0$, $y_0 \neq 0$ and the neighborhood

$$(5) \qquad U: |x-x_0| < r, \ |y-y_0| < r$$

has no common point with the planes $x=0$ and $y=0$. Put

$$(6) \qquad \sigma_{mn}(x,y) \equiv \sum_{\mu=0}^{m}\sum_{\nu=0}^{n} \alpha_{\mu\nu} x^\mu y^\nu .$$

We shall first give a definition and a lemma for the later use.

Definition 2. We say that a system of real-valued functions $\{f_1(x_1,\cdots,x_n), \cdots, f_k(x_1,\cdots,x_n)\}$ defined in a set $W$ of the real Cartesian $n$-space is lower semi-continuous, if for any given real values $\gamma_1,\cdots,\gamma_k$, the point set

$$(7) \qquad A(\gamma_1,\cdots,\gamma_k) \equiv \{(x_1,\cdots,x_n) \mid f_i(x_1,\cdots,x_n) \leqq \gamma_i, \ (i=1,\cdots,k)\}$$

is always closed in $W$.

This definition does not seem to imply the lower semi-continuity of each component $f_i(x_1,\cdots,x_n)$.

Lemma 3. Let a system of functions $\{f_1(x_1,\cdots,x_n),\cdots, f_k(x_1,\cdots,x_n)\}$ defined in an open set $W$, be lower semi-continuous. Suppose that each component is non-negative and finite at every point of $W$. Then there exists an open subset of $W$ in which $f_i(x_1, \cdots, x_n)$ $(i=1,\cdots,k)$ are bounded.[5]

Proof. Putting

$$A_\gamma \equiv A(\gamma,\cdots,\gamma) = \{(x) \mid f_i(x) \leqq \gamma, \quad (i=1,\cdots,k)\}, \quad (\gamma=1,2,3,\cdots),$$

these sets are all closed and we have

$$W = \bigcup_{\gamma=1}^{\infty} A_\gamma$$

by our hypotheses. If non of the $A_\gamma$ $(\gamma=1,2,\cdots)$ has inner points, $A_\gamma$ is nowhere dense and then $W$ is a point-set of first-category, contradicting that $W$ is a non-empty open set. Hence there must exist a $\gamma$ such that $A_\gamma$ has inner points, which proves our Lemma.//

We now return the proof of our Theorem and proceed on. For every point $(x,y) \in U$, we denote by $\ell(x,y)$ the limit of boundedness of $\sum \alpha_{mn} x^m y^n$, and put

$$(8)\quad M(x,y) \equiv \sup\{|\sigma_{mn}(x,y)|\,; \quad \text{for} \quad m,n \geqq \ell(x,y)\}.$$

By our assumptions, $\ell(x,y)$ and $M(x,y)$ are non-negative and finite at every point $(x,y)$ in (5). Next we shall show that the system of functions $\{\ell(x,y), M(x,y)\}$ is <u>lower semi-continuous</u> in the sense of Definition 2. In fact, take two positive numbers $\beta$ and $\gamma$ and put

$$(7)\quad A(\beta,\gamma) \equiv \{(x,y) \mid \ell(x,y) \leqq \beta, \quad \text{and} \quad M(x,y) \leqq \gamma\}.$$

We remark that $\ell \leqq \beta$ is equivalent to $\ell \leqq [\beta]$ where $[\;]$ means the Gauss' notation, for $\ell$ takes only the integral values. Taking a sequence $\{(x_\lambda, y_\lambda)\}_{\lambda=1}^{\infty} \in A(\beta,\gamma)$ converging to a point $(\xi,\eta)$ in $U$, these assumptions (7), (8) tell us that

$$|\sigma_{mn}(x_\lambda, y_\lambda)| \leqq \gamma, \quad \text{for every} \quad m,n \geqq [\beta], \quad \lambda=1,2,\cdots.$$

But since all $\sigma_{mn}(x,y)$ are polynomials of $x$, $y$, and since $\gamma$ does not depend upon $m,n$ and $\lambda$, we have, by tending $\lambda$ to $\infty$,

$$|\sigma_{mn}(\xi,\eta)| \leqq \gamma, \quad \text{for every} \quad m,n \geqq [\beta].$$

This means that $\ell(\xi,\eta) \leqq [\beta] \leqq \beta$ and $M(\xi,\eta) \leqq \gamma$, which proves that (7) is closed in $U$.//

Therefore our system of functions $\{\ell(x,y), M(x,y)\}$ satisfies the conditions of Lemma 3 in a open set

$$(9)\quad W \equiv \{(x,y) \mid |x-x_0|<r,\ |x|>|x_0|;\ |y-y_0|<r,\ |y|>|y_0|\},$$

and, by Lemma 3, we have an open neighborhood

$$(10)\quad V:\ |x-x_1|<\rho,\ |y-y_1|<\rho$$

contained in $W$, in which $\ell(x,y)$ and $M(x,y)$ are bounded. Thus, we have obtained a positive integer $\ell$, and a positive number $M$ such that

$$(11)\quad |\sigma_{mn}(x,y)|<M \quad \text{for} \quad m,n \geqq \ell, \text{ and } \quad (x,y)\in V,$$

and then we have from (3), (6), (9) and (10),

$$(12)\quad |\alpha_{mn}| < 4M/|x_1|^m |y_1|^n, \quad \text{for} \quad m,n \geqq \ell+1,$$

where $|x_0|<|x_1|$ and $|y_0|<|y_1|$.

Next we consider the mixed terms $m \leqq \ell$, $n>\ell$ or $m>\ell$, $n \leqq \ell$. Putting

$$(13)\begin{cases} p_n(x) \equiv \sum_{m=0}^{\ell} \alpha_{mn} x^m, \\ q_m(y) \equiv \sum_{n=0}^{\ell} \alpha_{mn} y^n, \end{cases}$$

we have from (6),

$$\begin{cases} p_n(x)y^n = \sigma_{\ell n}(x,y) - \sigma_{\ell n-1}(x,y); \\ q_m(y)x^m = \sigma_{m\ell}(x,y) - \sigma_{m-1\,\ell}(x,y), \end{cases}$$

for $m,n \geqq 1$, and so by (11),

$$
(14)\quad
\begin{cases}
|p_n(x)| < 2M/|y_1|^n \\
\qquad \text{for} \quad n \geqq l+1, \quad \text{in } |x-x_1| < \rho; \\
|q_m(y)| < 2M/|x_1|^m \\
\qquad \text{for} \quad m \geqq l+1, \quad \text{in } |y-y_1| < \rho.
\end{cases}
$$

Now, by the Cauchy's coefficient-estimation, we easily have:

Lemma 4. If a polynomial

$$p(x) \equiv \sum_{i=0}^{l} a_i x^i ,$$

is $|p(x)| < 1$ in a circle $|x-x_1| < \rho$, we have $|a_i| < C$, where $C$ is a constant depending only on $l$, $\rho$ and $|x_1|$.

Proof. Putting

$$p(x) = \sum_{j=0}^{l} b_j (x-x_1)^j ,$$

we have

$$|b_j| \leqq \frac{1}{\rho^j} ,$$

and then

$$|a_i| = \left| \sum_{k=0}^{l-i} b_{i+k} \binom{i+k}{i} (-x_1)^k \right|$$

$$\leqq (l+1) \cdot l! \cdot \frac{\max(|x_1|^l, 1)}{\min(\rho^l, 1)} .//$$

Applying Lemma 4 to (13) with (14) we have

$$
(15)\quad
\begin{aligned}
&|\alpha_{mn}| < C \cdot 2M/|y_1|^n \\
&\qquad \text{for} \quad m \leqq l,\ n > l; \\
&|\alpha_{mn}| < C \cdot 2M/|x_1|^m \\
&\qquad \text{for} \quad m > l,\ n \leqq l.
\end{aligned}
$$

The terms with $m, n \leqq l$ in (4) are only finite numbers, and so, summing up (12) and (15), we finally obtain the estimation

$$(16)\quad |\alpha_{mn}| < K/|x_1|^m |y_1|^n$$

for all $m, n = 0, 1, 2, \cdots$, where $|x_0| < |x_1|$ and $|y_0| < |y_1|$. Therefore the original power series (4) converges absolutely and uniformly in $|x| \leqq |x_0|$, $|y| \leqq |y_0|$ by Lemma 2, which proves our Theorem completely.

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(1) W.F.Osgood: Lehrbuch der Funktionentheorie, II 1 (1923) Leipzig and Berlin; and H. Behnke- P.Thullen: Theorie der Funktionen mehrerer komplexer Veränderlichen, (1934) Berlin.

(2) See for example, S.Bochner - W.T.Martin: Several complex variables, (1948) Princeton, p.30.

(3) M.Tsuji: Ta-hukuso hensû kansûron. (Theory of functions of several complex variables), (1935) Tokyo.

(4) F.Severi (translated into Japanese by S.Iyanaga): Severi ta-hensû kaiseki kansûron kôgi, (Lectures on the theory of analytic functions of several variables by Severi), (1936) Tokyo.

(5) Cf. S.Bochner - W.T.Martin, 2), p.139, Theorem 3.

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