NOTES ON AN OPEN RIEMANN SURFACE (II)

By Tadashi KURODA

In this note, we shall give the proofs of well-known results on an open Riemann surface. In author's previous paper [4] we discussed the existence of bounded harmonic functions on an open Riemann surface. In the present paper we shall discuss the existence of harmonic functions whose Dirichlet integral is finite.

1. Let F be an open abstract Riemann surface, Γ be its ideal boundary and F_n $(n=0,1,\cdots)$ be an exhaustion of F, that is, F_n is a compact subdomain of F such that $\overline{F}_n \subset F_{n+1}$ $(n=0,1,\cdots)$, $\bigcup_0^\infty F_n = F$ and the boundary Γ_n of F_n consists of a finite number of analytic closed curves. We denote by $\mathfrak{G}$ the class of non-compact domains G on F whose non-empty relative boundary C consists of an enumerable number of analytic curves which are compact or non-compact and do not cluster in F. Putting $F_n \cap G = G_n$, $F_n \cap C = C_n$ and $\Gamma_n \cap G = H_n$, G_n is not empty for all sufficiently large $n(\geq n_0)$ and is bounded by C_n and H_n. We can consider that the sequence H_n $(n = n_0, n_0+1, \cdots)$ clusters to the ideal boundary $\Gamma_G (\subset \Gamma)$ of G.

2. Suppose that G is a domain on F belonging to the class $\mathfrak{G}$. If in G there exist no non-constant harmonic function whose Dirichlet integral on G is finite and which equals to zero on C, then we shall say that G belongs to the class C_D. We shall prove

Theorem 1 (Bader-Parreau [2]). Suppose that, for two domains G and G' which belong to $\mathfrak{G}$, G is contained in G'. If G does not belong to the class C_D, then G' does not belong to C_D.

Proof. From the assumption there exists a non-constant harmonic function $u(p)$ whose Dirichlet integral $D_G(u)$ over G is finite and which equals to zero on C. By Mori's lemma [5], we may assume that $|u| < 1$. Let V be a continuous function in G' which equals to $u(p)$ in G and to zero in $G'-G$. Then obviously

$$D_{G'}(V) = D_G(u) < \infty.$$

Next let v_n be the harmonic function in G' such that $v_n = u$ on H_n and $v_n = 0$ on $H_n'-H$ and on C_n'. Then by the Dirichlet principle we have

$$D_{G_n'}(v_n) \leqq D_{G_n'}(V).$$

Since $|u| < 1$, we get $|v_n| < 1$. Hence we can choose a suitable subsequence $\{v_{n_k}\}$ $(k=1,2,\cdots)$ such that this sequence converges to the harmonic function v in G' uniformly in the wider sense. For $h < k$ we get

$$D_{G'_{n_h}}(v_{n_k}) \leqq D_{G'_{n_k}}(v_{n_k}).$$

From above three formulae we obtain

$$D_{G'_{n_h}}(v_{n_k}) \leqq D_G(u) < \infty.$$

Making first $k \to \infty$ and next $h \to \infty$, it follows that

$$D_{G'}(v) < \infty.$$

Moreover $u(p) \leqq v(p)$ at any point p in G, since, by the maximum principle, $u(p) \leqq v_n(p)$ at the point of G_n. On the other hand $v(p) = 0$ on C'. Hence $v(p)$ is non-constant. Thus our theorem is established.

We may say that the ideal boundary Γ is (u, D)-removable if there exist no harmonic function whose Dirichlet integral over F is finite and which is non-constant. The class of Riemann surfaces with (u, D)-removable boundary is identical to the class O_{HD} in Ahlfors' sense [1].

The class O_{HB} in his sense is equivalent to the class of Riemann surfaces with (u, M)-removable boundary, which was discussed in author's previous paper [4] .

By the same method as the above proof, we can show the following

Theorem 2 (Bader-parreau [2] , Mori [5]). The ideal boundary of F is not (u, D)-removable, if and only if there exist two domains G' and G'' on F belonging to $\mathcal{O}\!\mathcal{J}$ such that these both domains do not belong to the class C_D and have no point in common.

Proof. First we shall prove the necessity. By our assumption there exists a non-constant harmonic function $u(p)$ whose Dirichlet integral on F is finite. Taking an inner point p_0 of F , we choose two components G' and G'' whose points p satisfy the condition $u(p) > u(p_0)$ in G' or $u(p) < u(p_0)$ in G'' respectively. Then it is easy to see that these two domains G' and G'' do not belong to the class C_D .

Next we shall prove the sufficiency. Since G' does not belong to C_D , there exists a non-constant harmonic function $u(p)$ whose Dirichlet integral over G is finite and which equals to zero on the relative boundary C' of G' . Again by Mori's lemma we may suppose that $|u| < 1$. Hence the domain G' does not belong to the class C_0 and this holds also to the other domain G'' .

Let V be the continuous function which equals to u in G' and to zero in $F - G'$. Obviously

$$D_F(V) = D_{G'}(u) .$$

We denote by v_n the harmonic function which equals to u on $H_n'(= G' \cap \Gamma_n)$ and to zero on $\Gamma_n - H_n'$. Then by the Dirichlet principle we have

$$D_{F_n}(v_n) \leqq D_{F_n}(V).$$

Since $|u| < 1$, we can find a suitable subsequence $\{v_{n_k}\}$ $(k=1,2,\cdots)$ which converges to the harmonic function v uniformly in the wider sense. For $h < k$,

$$D_{F_{n_h}}(v_{n_k}) \leqq D_{F_{n_k}}(V_{n_k}) .$$

Hence it follows that

$$D_{F_{n_h}}(v_{n_k}) \leqq D_{G'}(u) < \infty .$$

Making first $k \to \infty$ and next $h \to \infty$, we obtain

$$D_F(v) < \infty .$$

Moreover it is immediate that $v \geqq u > 0$ at the point p_0 in G' where $u(p_0) > 0$, for $v_n \geqq u$ in G_n' . On the other hand, using the fact that G'' and hence $F - G' (\supset G'')$ do not belong to the class C_0 , we can easily see that $\lim_{p \in F-G'} v = 0$. Hence the function v is non-constant.

Thus the proof of our theorem is complete.

By the same manner as stated already in [4] , we can get the following

Theorem 3 (Mori [3]). Let F have (u, D)-removable boundary, G be any domain on it belonging to the class $\mathcal{O}\!\mathcal{J}$ and u be any non-constant harmonic function in G whose Dirichlet integral over G is finite. There at least one of the maximum principle and minimum principle holds good, i.e., $\underline{\lim}_C\, u \leqq u$ or $\overline{\lim}_C\, u \geqq u$.

3. In general the following is well-known (c.f. Kuroda [3]).

Theorem. (Nevanlinna [6]). If the Riemann surface has a null boundary, its ideal boundary is (u, D)-removable.

In the case of a Riemann surface with finite genus we can prove

Theorem 4 (Nevanlinna [7]). Let F be a Riemann surface with finite genus. If F has a null boundary, its ideal boundary Γ is (u, D)-removable. Therefore, in the case of F with finite genus, F has a null boundary if and only if Γ is (u, D)-removable.

Proof. We shall show that there exists a non-constant harmonic function on F whose Dirichlet integral over F is finite, if F has a positive boundary. Since the genus of F is finite, the ideal boundary has the real sense. Hence we can find two closed subsets Γ_1 and Γ_2 such that these are disjoint each other and both have positive absolute harmonic measures. We put $F^* = F \cup \Gamma - (\Gamma_1 \cup \Gamma_2)$. Let $F_1 \subset F_2 \subset \cdots \subset F_n \subset \cdots$ be an exhaustion of F^* such that the boundary of F_n consists of two classes H_n and H_{n+1} of closed analytic curves on F^* and H_1, $H_3, \cdots, H_{2n-1}, \cdots$ converges to Γ_1 and $H_2, H_4, \cdots, H_{2n}, \cdots$ converges to Γ_2. Of course F_n $(n = 1, 2, \cdots)$ is compact in F^*. We denote by n' the odd number among n and $n+1$, by n'' the even number among n and $n+1$ and by ω_n the harmonic measure of H_n'' in F_n. Then we can easily see that for $m > n$

$$D_{F_n}(\omega_n - \omega_m) = D_{F_n}(\omega_n) + D_{F_n}(\omega_m) - 2\int_{H_{n''}} \frac{\partial \omega_m}{\partial \nu} ds .$$

Further it is easy to see that

$$\int_{H_{n''}} \frac{\partial \omega_m}{\partial \nu} ds = \int_{H_{m''}} \frac{\partial \omega_m}{\partial \nu} ds = D_{F_m}(\omega_m),$$

whence it follows that

$$(*) \quad D_{F_n}(\omega_n - \omega_m) \leqq D_{F_n}(\omega_n) - D_{F_m}(\omega_m) \leqq D_{F_m}(\omega_n) - D_{F_n}(\omega_m) .$$

On the other hand we can choose a suitable subsequence $\{\omega_{n_k}\}$ $(k = 1, 2, \cdots)$ such that this sequence converges to a harmonic function ω on F^* uniformly in the wider sense. Applying $(*)$ for this sequence, we see that the Dirichlet integral $D_F(\omega)$ of ω over F^* is finite. Moreover it is immediate that ω is non-constant. Thus our assertion is proved.

4. The following is still open:

Does a domain G of the class $\mathcal{G}$ belong to the class C_0 if G belongs to the class C_D ?

The converse of the above was proved by Mori [5] using the method owing to Virtanen [8] and was refered to as Mori's lemma in this note.

(*) Received Dec. 15, 1951.

[1] L.Ahlfors: Remarks on the classification of open Riemann surfaces, Ann. Acad. Sci. Fenn. A.I. 87 (1951) 8pp.

[2] R.Bader-M.Parreau: Domaines non-compacts et classification des surface de Riemann, C.R.Paris, 232 (1951), pp. 138-139.

[3] T.Kuroda: Some remarks on an open Riemann surface with null boundary, Tôhoku Math. Journ. Ser. 2, 3 (1951), pp.182-186.

[4] T.Kuroda: Notes on an open Riemann surface, Kōdai Sem. Rep. Nos. 3-4 (1951), pp. 61-63.

[5] A.Mori: On the existence of harmonic functions on a Riemann surface, To appear in Sci. Rep. Tokyo Univ.

[6] R.Nevanlinna: Quadratisch integrierbare Differentiale auf einer Riemannschen Mannigfaltigkeit, Ann. Acad. Sci. Fenn. A.I. 1 (1941), 34pp.

[7] R.Nevanlinna: Sur l'existence de certaines classes dé différentiales analytiques, C.R.Paris, 228 (1949), pp. 2002-2004.

[8] K.I.Virtanen: Über die Existenz von beschränkten harmonische Funktionen auf einer Riemannschen Fläche, Ann. Acad. Sci. Fenn. A. I. 75 (1950), pp.8.

Mathematical Institute, Nagoya University.