A NOTE ON THE ORDINAL POWER AND THE LEXICOGRAPHIC PRODUCT OF PARTIALLY ORDERED SETS

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Introduction.

The ordinal power of partially ordered sets, which will be mentioned later on, has been defined by G.Birkhoff [1] . But the usual definition contains some essential difficulty, and on account of it, some restrictive condition on index set is necessary for this definition.

The object of the present note is to give some other definition of the ordinal power, which is a slight extension of usual one, yet is adoptable without any restriction on the sets concerned.

In §1, a new definition is introduced.

§2 is devoted to some identities.

In §3, a new definition of the lexicographic product is given, and we shall consider especially the case when the factor sets are homogeneous.

Concerning applications of that definition, see the author's next paper [3] .

1. Definitions.

With respect to the partially ordered set (abbr. poset), the chain, the dual, the descending chain condition, and other terms concerning partially ordered sets, we use the usual definitions (cf. [2]), unless different definitions are mentioned.

The following definitions are usually given.

Definition 1. The ordinal sum $X \oplus Y$ of two posets X and Y is the set of all $x \in X$ and $y \in Y$, where $x \leqslant x'$ in X and $y \leqslant y'$ in Y preserve their original meaning, and $x < y$ for all $x \in X$ and $y \in Y$.

The ordinal product $X \circ Y$ is the set of all pairs (x, y), $x \in X$, $y \in Y$, where $(x, y) \leqslant (x', y')$ is defined to mean that either $x < x'$, or $x = x'$ and $y \leqslant y'$.

Definition 1'. The ordinal power ${}^{X}Y$ consists of all functions $y = f(x)$ from X to Y, where $f \leqslant g$ means that for every x such that $f(x) \nleqslant g(x)$, there exists an $x' < x$ such that $f(x') < g(x')$.

The definitions of ordinal sum and ordinal product are always adequate, and the following identities are known:

associative law:

$$(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z),$$

$$(X \circ Y) \circ Z = X \circ (Y \circ Z);$$

right distributive law:

$$(X \oplus Y) \circ Z = (X \circ Z) \oplus (Y \circ Z).$$

But this definition for the ordinal power is often inadequate. Indeed, let 2 be the 2nd ordinal number, and let J be the chain of all integers with natural order, then ${}^{J}2$ is not a poset, that is, the order defined by definition 1', satisfies neither the antisymmetric law nor the transitive law. G.Birkhoff showed that the definition has meaning if and only if X satisfies the descending chain condition, unless Y is totally disordered. [1] , [2] , [4].

To avoid this difficulty, and the restrictions on the index set, a new definition of the ordinal power will now be introduced.

Definition 2. Let X and Y be posets, and y_0 be a fixed element (arbitrary chosen) of Y. The ordinal power ${}^{X}Y\langle y_0\rangle$ consists of all functions $f(x) = y$ from X to Y such that the set $\{x \mid f(x) \neq y_0\}$ satisfies the descending chain condition ($\{x \mid P\}$ means the set of all elements which satisfy the condition P), where the order is as usual, that is, $f \leqslant g$ means that for every x such that $f(x) \nleqslant g(x)$, there exists an element $x' < x$ such that $f(x') < g(x')$.

This restriction on the functions of ${}^{X}Y\langle y_0\rangle$ excludes the restrictions on the original sets. In fact, this definition is always proper, as we shall see later.

The set $\{x \mid f(x) \neq y_0\}$ be denoted by M_f , and the set $\{x \mid f(x) \neq g(x)\}$ by $M_{f,g}$ (throughout this paper we use those notations), then $M_{f,g} \subset M_f \cup M_g$, so $M_{f,g}$ satisfies the descending chain condition as well as M_f and M_g , because the family of all subsets which satisfy the descending chain condition, is an ideal in the Boolean algebra of all subsets of X, as easily seen.

The set of all minimal elements

of a set M is denoted by min(M). Then $f \leqslant g$ is equivalent to the fact that $f(x) < g(x)$ for every $x \in \min(M_{f,g})$.

In fact, if $f(x) < g(x)$ for every $x \in \min(M_{f,g})$, and $f(x') \nleqslant g(x')$ for some $x' \in X$, then there is an $x'' \in \min(M_{f,g})$ such that $x'' < x'$, on account of the descending chain condition of $M_{f,g}$, and for this x'', $f(x'') < g(x'')$. Conversely, if $f(x'') \nless g(x'')$ for some $x'' \in \min(M_{f,g})$, then $f(x'') \nleqslant g(x'')$ and for every $x' < x''$, $f(x') = g(x')$, that is, $f(x') \nless g(x')$, so $f \nleqslant g$.

Now we shall see that the order in ${}^{X}Y\langle y_0\rangle$ satisfies the axioms of order.

The reflexive law:— There is no element $x \in X$ such that $f(x) \nleqslant f(x)$, so $f \leqslant f$.

The antisymmetric law:— If $f \leqslant g$ and $g \leqslant f$, then $f = g$. In fact, unless $f = g$ then the set $M_{f,g}$ is non-void. $f \leqslant g$ implies that for every $x \in \min(M_{f,g})$, $f(x) < g(x)$, but $g \leqslant f$ implies the opposite order. This is a contradiction.

The transitive law:— Let $f \leqslant g$, and $g \leqslant h$. Because $f(x) = g(x)$ and $g(x) = h(x)$ implies $f(x) = h(x)$, $M_{f,h} \subset M_{f,g} \cup M_{g,h}$. So, if $f(x') \nleqslant h(x')$, then there exists an $x'' < x'$ such that $x'' \in \min(M_{f,g} \cup M_{g,h}) = \min(\min(M_{f,g}) \cup \min(M_{g,h}))$. Obviously $f(x'') < h(x'')$ for this $x'' < x'$.

This definition of the ordinal power seems somewhat artificial. But it is not so unnatural as it appears, because the following consideration is possible.

Let M be a subset of a poset X, and satisfy the descending chain condition. Let $\mathfrak{M}_M$ be the subset of ${}^{X}Y\langle y_0\rangle$ which consists of all functions such that $f(x) = y$ for every $x \notin M$. Then $\mathfrak{M}_M$ is isomorphic to ${}^{M}Y$, which has meaning in the old definition. The family Λ of all subsets M with the descending chain condition, is a distributive lattice with the order of set-inclusion. The family $\Delta = \{\mathfrak{M}_M\}$ whose element is a set of functions in ${}^{X}Y\langle y_0\rangle$ such that they takes the constant value y_0 outside some $M \in \Lambda$, is also a distributive lattice with the order of set-inclusion and is isomorphic to Λ. Those lattice may not be complete, but if we complete them by cut, the greatest elements will correspond to each other, —— the one is X, and the other is nothing but ${}^{X}Y\langle y_0\rangle$, and is not the set of all functions from X to Y.

2. Some identities.

We shall see that the following identities hold:

$$\text{I)}\quad {}^{X\oplus Y}Z\langle z_0\rangle = {}^{X}Z\langle z_0\rangle \circ {}^{Y}Z\langle z_0\rangle,$$

$$\text{II)}\quad {}^{X}({}^{Y}Z\langle z_0\rangle)\langle f_0\rangle = {}^{X\circ Y}Z\langle z_0\rangle,$$

where $f_0 \in {}^{Y}Z\langle z_0\rangle$ is such a function that for every $y \in Y$, $f_0(y) = z_0$.

Proof of I):

Let $f \in {}^{X\oplus Y}Z\langle z_0\rangle$. We denote f by f_X, when we regard f as a function from X to Z, and for Y similarly by f_Y. The set $M_f = \{u \in X \oplus Y \mid f(u) \neq z_0\}$ and so the sets $M_{f_X} = \{x \in X \mid f_X(x) \neq z_0\}$ and $M_{f_Y} = \{y \in Y \mid f_Y(y) \neq z_0\}$ satisfy the descending chain condition, that is, $f_X \in {}^{X}Z\langle z_0\rangle$ and $f_Y \in {}^{Y}Z\langle z_0\rangle$, and f is a couple of those functions, so $f \in {}^{X}Z\langle z_0\rangle \circ {}^{Y}Z\langle z_0\rangle$.

Conversely, if $f_X \in {}^{X}Z\langle z_0\rangle$ and $f_Y \in {}^{Y}Z\langle z_0\rangle$, then the function f from $X \oplus Y$ to Z such that $f(x) = f_X(x)$ for $x \in X$, and $f(y) = f_Y(y)$ for $y \in Y$ is contained in ${}^{X\oplus Y}Z\langle z_0\rangle$.

Now let $f = (f_X, f_Y)$ and $g = (g_X, g_Y)$ be contained in ${}^{X}Z\langle z_0\rangle \circ {}^{Y}Z\langle z_0\rangle$, and let $f \leqslant g$ in this ordinal product. This is equivalent to the fact that $f_X < g_X$, or $f_X = g_X$, $f_Y \leqslant g_Y$, that is, either $f(x) < g(x)$ for all $x \in \min(M_{f_X, g_X})$, or $f_X(x) = g_X(x)$ for all $x \in X$ and $f_Y(y) < g_Y(y)$ for all $y \in \min(M_{f_Y, g_Y})$. If we consider the function f from $X \oplus Y$ to Z, such that $f(x) = f_X(x)$ for all $x \in X$, and $f(y) = f_Y(y)$ for all $y \in Y$, the above statement is equivalent to the fact that either $f(x) < g(x)$ for all $x \in \min(M_{f,g} \cap X)$, or $f(x) = g(x)$ for all $x \in X$, and $f(y) < g(y)$ for all $y \in \min(M_{f,g} \cap Y)$, that is, $f(u) < g(u)$ for all $u \in \min(M_{f,g})$. This is nothing but the definition of $f \leqslant g$ in ${}^{X\oplus Y}Z\langle z_0\rangle$.

Proof of II):

Let ${}^{X}({}^{Y}Z\langle z_0\rangle)\langle f_0\rangle \ni \Phi$, then for $x \in X$, $\Phi(x) \in {}^{Y}Z\langle z_0\rangle$. We denote $\Phi(x)$ by Φ_x, which is a function from Y to Z. So for a $y \in Y$, $\Phi_x(y) \in Z$. This shows that Φ is a function from the set product (X, Y) to Z.

Now the set $M_\Phi = \{x \mid \Phi_x \neq f_0\}$ satisfies the descending chain condition, and if $\Phi_x = f_0$, then $\Phi_x(y) = z_0$ for all $y \in Y$, and if $\Phi_x \neq f_0$, then the set $M_{\Phi_x} = \{y \mid \Phi_x(y) \neq z_0\}$ satisfies the descending chain condition. So the set $M'_\Phi = \{(x, y) \mid \Phi_x(y) \neq z_0\}$ satisfies the descending chain condition in $X \circ Y$, that is, $\Phi \in {}^{X \circ Y}Z\langle z_0\rangle$.

Conversely, if $\Psi \in {}^{X \circ Y}Z\langle z_0\rangle$, then the set $M'_\Psi = \{(x, y) \mid \Psi(x, y) \neq z_0\}$ satisfies the descending chain condition. So the set $M_\Psi = \{x \in X \mid (x, y) \neq z_0$ for some $y \in Y\}$, being the homomorphic image of M'_Ψ, does also.
The function $\Psi(x, y)$ with a fixed x, is considered as a function from Y to Z, which we denote by Ψ_x. The above statement implies that the set $M_\Psi = \{x \mid \Psi_x \neq f_0\}$ satisfies the descending chain condition.

Still more, the set $M_{\Psi_x} = \{y \mid \Psi_x(y) \neq z_0$ for a fixed $x \in M_\Psi\}$, being a subset of M'_Ψ, must satisfy the descending chain condition. So $\Psi_x \in {}^{Y}Z\langle z_0\rangle$ and $\Psi \in {}^{X}({}^{Y}Z\langle z_0\rangle)\langle f_0\rangle$.

Now let $\Phi \leqslant \Psi$ in ${}^{X}({}^{Y}Z\langle z_0\rangle)\langle f_0\rangle$, then for any x in $\min(M_{\Phi,\Psi})$ ($M_{\Phi,\Psi} = \{x \mid \Phi_x \neq \Psi_x\}$ satisfies the descending chain condition) $\Phi_x < \Psi_x$, that is, for any y in $\min(M_{\Phi_x,\Psi_x})$ ($M_{\Phi_x,\Psi_x} = \{y \mid \Phi_x(y) \neq \Psi_x(x)\}$) $\Phi_x(y) < \Psi_x(y)$. But the element (x, y) in which $x \in \min(M_{\Phi,\Psi})$ and $y \in \min(M_{\Phi_x,\Psi_x})$ is a minimal element of $M'_{\Phi,\Psi} = \{(x, y) \mid \Phi_x(y) \neq \Psi_x(y)\}$, and moreover it ranges over all $\min(M'_{\Phi,\Psi})$.

After all, $\Phi \leqslant \Psi$ in ${}^{X}({}^{Y}Z\langle z_0\rangle)\langle f_0\rangle$ implies $\Phi_x(y) < \Psi_x(y)$ for every $(x, y) \in \min(M'_{\Phi,\Psi})$, which means $\Phi \leqslant \Psi$ in ${}^{X \circ Y}Z\langle z_0\rangle$.

Conversely, let $\Phi \leqslant \Psi$ in ${}^{X \circ Y}Z\langle z_0\rangle$. We denote $\Phi(x,y)$ by $\Phi_x(y)$, where Φ_x is a function from Y to Z. $\Phi \leqslant \Psi$ in ${}^{X \circ Y}Z\langle z_0\rangle$ implies $\Phi_x(y) < \Psi_x(y)$ for any $(x, y) \in \min(M'_{\Phi,\Psi})$ ($M'_{\Phi,\Psi} = \{(x, y) \mid \Phi_x(y) \neq \Psi_x(y)\}$. If we fix some x, such that $(x, y) \in \min(M'_{\Phi,\Psi})$ for some y, then the above condition means that $\Phi_x(y) < \Psi_x(y)$ for any $y \in \min(M_{\Phi_x,\Psi_x})$. This implies $\Phi_x \leqslant \Psi_x$ in ${}^{Y}Z\langle z_0\rangle$. If $x \in \min(M_{\Phi,\Psi})$ ($M_{\Phi,\Psi} = \{x \mid \Phi_x \neq \Psi_x\}$), then $(x, y) \in \min(M'_{\Phi,\Psi})$ for some $y \in Y$, so the above relation implies $\Phi \leqslant \Psi$ in ${}^{X}({}^{Y}Z\langle z_0\rangle)\langle f_0\rangle$. This completes the proof.

3. The homogeneous case.

The new definition of ordinal power is very convenient, because any restriction on the original sets is unnecessary, but on the other hand, it has an inconvenient point that the type of resultant system of an ordinal power depends on the choice of a fixed element in the base poset. In fact, let S^+ be the chain of real numbers, which are equal to or greater than zero, and let J be the chain of integers, then ${}^{J}S^+\langle 0\rangle$ has a least element f such that $f(n) = 0$ for all $n \in J$, but ${}^{J}S^+\langle 1\rangle$ has no least element.

However, this difficulty is avoidable in special cases. Indeed, when X in ${}^{X}Y\langle y_0\rangle$ itself satisfies the descending chain condition, any function from X to Y is admissible regardless of the choice of a fixed element in Y, and in this case, the new definition of the ordinal power is equivalent to the old one.

On the other hand, if the base poset Y is homogeneous, then the type of resultant system of ${}^{X}Y\langle y_0\rangle$ does not depend on the choice of $y_0 \in Y$, regardless of the type of X.

As to this fact, we will take a more general standpoint.

<u>Definition 3.</u> Let X be a poset, and for each $x \in X$, there be a corresponding poset Y_x. Let y_{0x} be a fixed element in Y_x. The <u>lexicographic product</u> $\prod_x Y_x\langle y_{0x}\rangle$ is defined as the set of all functions which select for each $x \in X$, a $y = f(x) \in Y_x$, and 'make the sets $\{x \mid f(x) \neq y_{0x}\} = M_f$ satisfy the descending chain condition', where $f \leqslant g$ means that for every $x \in X$ such that $f(x) \nleqslant g(x)$, there exists an $x' < x$ such that $f(x') < g(x')$.

It is all the same as in the case of ordinal power, that, based on this definition, the axioms of the order are satisfied, and that $f \leqslant g$ is equivalent to the fact that $f(x) < g(x)$ for all $x \in \min(M_{f,g})$, $M_{f,g}$ being the set $\{x \mid f(x) \neq g(x)\}$.

Especially, let $Y_x = Y$ and $y_{0x} = y_0$ for all $x \in X$, then $\prod_x Y_x\langle y_{0x}\rangle$ is reduced to ${}^{X}Y\langle y_0\rangle$. So, we will study especially the case of lexicographic product in the following lines.

<u>Definition 4.</u> Let Y be a poset, and $y_0, y_1 \in Y$. y_0 is called <u>transitive</u> to y_1, if and only if <u>there exists</u> an automorphism φ of Y which maps y_0 to y_1. If any two elements of Y are mutually transitive, then we call the set Y <u>homogeneous</u>.

<u>Lemma I.</u> $\prod_x Y_x\langle y_{0x}\rangle$ and $\prod_x Y_x\langle y_{1x}\rangle$ are isomorphic to each other, if every y_{0x} is transitive to y_{1x}.

Proof. For each x, there exists an automorphism φ_x of Y_x such that $\varphi_x(y_{ox}) = y_{1x}$. Let $f \in \Pi_x Y_x \langle y_{ox} \rangle$, and consider the function g of X such that $g(x) = \varphi_x f(x)$. If $f(x) = y_{ox}$, then $g(x) = y_{1x}$ and vice versa. So, the set $M'_g = \{ x \mid g(x) \neq y_{1x} \} = M_f = \{ x \mid f(x) \neq y_{ox} \}$ satisfies the descending chain condition, and so $g \in \Pi_x Y_x \langle y_{1x} \rangle$.

If we map $f \in \Pi_x Y_x \langle y_{ox} \rangle$ to $g \in \Pi_x Y_x \langle y_{1x} \rangle$ so that $g(x) = \varphi_x f(x)$, then it is obvious that this mapping is one-to-one and order-preserving, because every φ_x is an automorphism of Y_x. So $\Pi_x Y_x \langle y_{ox} \rangle$ is isomorphic to $\Pi_x Y_x \langle y_{1x} \rangle$.

Lemma II. $\Pi_x Y_x \langle y_{ox} \rangle$ coincides with $\Pi_x Y_x \langle y_{1x} \rangle$ if the set $N = \{ x \mid y_{ox} \neq y_{1x} \}$ satisfies the descending chain condition.

Proof. Let $f \in \Pi_x Y_x \langle y_{ox} \rangle$, then the set $M_f = \{ x \mid f(x) \neq y_{ox} \}$ satisfies the descending chain condition. On the other hand, the set $N = \{ x \mid y_{ox} \neq y_{1x} \}$ satisfies the descending chain condition, so does the set $M'_f = \{ x \mid f(x) \neq y_{1x} \} \subset M_f \cup N$ also. This implies $f \in \Pi_x Y_x \langle y_{1x} \rangle$.

We can see $\Pi_x Y_x \langle y_{ox} \rangle \supset \Pi_x Y_x \langle y_{1x} \rangle$ all the same, and this completes the proof.

The above two lemmas can be stated together as follows.

Theorem I. $\Pi_x Y_x \langle y_{ox} \rangle$ is isomorphic to $\Pi_x Y_x \langle y_{1x} \rangle$, if the set of $x \in X$, such that y_{ox} is not transitive to y_{1x} in Y_x, satisfies the descending chain condition.

Corollary I. If the set of all $x \in X$, such that Y_x are not homogeneous, satisfies the descending chain condition, then the type of $\Pi_x Y_x \langle y_{ox} \rangle$ does not depend on the choice of the fixed elements y_{ox}.

Corollary II. If Y is homogeneous, then the type of ${}^X Y \langle y_o \rangle$ does not depend on the choice of a fixed element y_o.

Theorem II. If all Y_x are homogeneous, then $\Pi_x Y_x \langle y_{ox} \rangle$ is homogeneous.

Proof. Let $f, g \in \Pi_x Y_x \langle y_{ox} \rangle$. Then the set $M_{f,g} = \{ x \mid f(x) \neq g(x) \}$ satisfies the descending chain condition. Now, for every Y_x is homogeneous, we can take automorphisms φ_x of Y_x such that $\varphi_x(f(x)) = g(x)$. The set M_φ of all x such that φ_x is not an identical mapping, satisfies the descending chain condition. Consider the following mapping of $\Pi_x Y_x \langle y_{ox} \rangle$:

$\Phi(h) = k$ for $h \in \Pi_x Y_x \langle y_{ox} \rangle$, where $k(x) = \varphi_x(h(x))$.

Then $k \in \Pi_x Y_x \langle y_{ox} \rangle$, because the $M_h = \{ x \mid h(x) \neq y_{ox} \}$ and the set M_φ satisfy the descending chain condition, so does the set $M_k = \{ x \mid k(x) \neq y_{ox} \} \subset M_h \cup M_\varphi$ also.

It is obvious that Φ is one-to-one, and order-preserving, because each φ_x is an automorphism of Y_x.

Still more, $\Phi(f) = g$ by the definition of Φ. So any two elements $f, g \in \Pi_x Y_x \langle y_{ox} \rangle$ are mutually transitive, that is, the resultant system is homogeneous.

Finally, we note the following obvious fact.

Theorem III. If X and all Y_x are chains, then $\Pi_x Y_x \langle y_{ox} \rangle$ is a chain.

In the case of corollary I, II, if only the type of the resultant system comes into question, we can omit writing the fixed elements of the factor posets. On account of it, the new definitions of ordinal power and lexicographic product seem useful especially in this homogeneous case.

With the application of the definitions and theorems of the present paper, the author continued his study on the type-problem of some homogeneous chain, which will be given in a subsequent paper.

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