NOTE ON LAPLACE-TRANSFORMS (VII)

ON THE OVERCONVERGENCE AND SINGULARITIES OF LAPLACE-TRANSFORMS

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(Communicated by Y. Komatu)

(1) THEOREM I. Let $\alpha(x)$ be of bounded variation in any finite interval $0 \leqq x \leqq X$, X being arbitrary. Put

$$(1.1) \qquad F(s) = \int_0^\infty \exp(-sx)\, d\alpha(x)$$

$$(s = \sigma + it, \quad \alpha(0) = 0)$$

In this present Note, we shall discuss the relation between the overconvergence and singularities of (1.1). We shall begin with

THEOREM I. Let $F(s)$ be simply convergent for $\sigma > 0$. If $d\alpha(x) = 0$ in $x \in (\lambda_\nu, \tau_\nu)$ $(\nu = 1, 2, \cdots)$ with $\tau_{\nu-1} < \lambda_\nu < \tau_\nu$ and $\lim_{\nu\to\infty} \tau_\nu/\lambda_\nu > 1$, then in the sufficiently small neighbourhood of the regular point on $\sigma = 0$, the sequence of partial sums $\int_0^{\lambda_\nu} \exp(-sx)\, d\alpha(x)$ $(\nu = 1, 2, \cdots)$ is uniformly convergent, i.e. $\sum_{\nu=1} \int_{\tau_{\nu-1}}^{\lambda_\nu} \exp(-sx)\, d\alpha(x)$ $(\tau_0 = 0)$ is uniformly convergent.

If we apply Theorem 1 to Dirichlet series, putting $\alpha(x) = \sum_{\lambda_n < x} a_n$ $(0 \leqq \lambda_1 < \lambda_2 < \cdots \lambda_n \to \infty)$, we get easily

COROLLARY I (A.Ostrowski), ([1], [2] p.39). Let $F(s) = \sum a_n \exp(-\lambda_n s)$ be simply convergent for $\sigma > 0$. If in $\{\lambda_n\}$, there exists a subsequence $\{\lambda_{n_\nu}\}$ with $\lim_{\nu\to\infty} \lambda_{1+n_\nu}/\lambda_{n_\nu} > 1$, then the sequence of partial sums $\sum^{n_\nu} a_n \exp(-\lambda_n s)$ $(\nu = 1, 2, \cdots)$ is uniformly convergent in the sufficiently small neighbourhood of the regular point on $\sigma = 0$.

(2) LEMMA. For the proof of Theorem 1, we need next Lemma.

LEMMA. Let (1.1) be simply convergent for $\sigma > 0$. Then, we have

$$(2.1) \qquad \frac{1}{y} \log |R_y(s)|_{s \in D} < -\sigma + \varepsilon_y(s),$$

where (i) $R_y(s) = F(s) - S_y(s)$, $S_y(s) = \int_0^y \exp(-sx)\, d\alpha(x)$,

(ii) D : bounded domain, in which $F(s)$ is regular.

(iii) $\varepsilon_y(s) \to 0$ as $y \to +\infty$ uniformly in D.

Proof. Put $\eta = \frac{1}{2}(\sigma_0 + \sigma_0^2)$, $(0 < \sigma_0 < 1)$, $M = \max_{s \in D} |F(s)|$. By Lemma 1 of the previous Note ([3]),

$$(2.2) \qquad |R_y(s)|_{s \in D,\, \sigma \leqq \eta} \leqq M + |S_y(s)|$$

$$< M + K(\sigma_0) \exp(-(\sigma - \sigma_0)y) \left\{1 + \frac{d(\sigma_0, D)}{\sigma_0 - \eta}\right\},$$

where (i) $d(\sigma_0, D) = \max_{s \in D} |s - \sigma_0|$.

(ii) $|S_y(\sigma_0)| < K(\sigma_0)$ $(0 \leqq y < +\infty)$

Since $\sigma - \sigma_0 < 0$, for sufficiently large y, by (2.2) we get

$$R_y(s)_{s \in D,\, \sigma \leqq \eta} < 2K(\sigma_0) \exp(-(\sigma - \sigma_0)y) \left\{1 + \frac{d(\sigma_0, D)}{\sigma_0 - \eta}\right\}$$

Hence

$$(2.3) \qquad \frac{1}{y} \log |R_y(s)|_{s \in D,\, \sigma \leqq \eta} < -\sigma + \sigma_0 + \frac{1}{y} C_1(\sigma_0, \eta, D),$$

where $C_1 = \log\left\{2K(\sigma_0)\left(1 + \frac{d(\sigma_0, D)}{\sigma_0 - \eta}\right)\right\}$.

On the other hand, by Lemma 2 of the previous Note ([3]),

$$\left|\int_y^{y'} \exp(-sx)\, d\alpha(x)\right| < \exp(-(\sigma - \sigma_0^2)y)\, K_1(\sigma_0) \frac{d(\sigma_0^2, D)}{\eta - \sigma_0^2},$$

where (i) $\left|\int_y^{y'} \exp(-\sigma_0^2 x)\, d\alpha(x)\right| < K_1(\sigma_0)$, $y' > y \geqq$

(ii) $d(\sigma_0^2, D) = \max_{s \in D} |s - \sigma_0^2|$

Letting $y' \to +\infty$,

$$|R_y(s)|_{s \in D,\, \sigma \geqq \eta} < \exp(-(\sigma - \sigma_0^2)y)\, K_1(\sigma_0) \frac{d(\sigma_0^2, D)}{\eta - \sigma_0^2},$$

so that

$$(2.4) \qquad \frac{1}{y} \log |R_y(s)|_{s \in D,\, \sigma \geqq \eta} < -\sigma + \sigma_0^2 + \frac{1}{y} C_2(\sigma_0, \eta, D),$$

where $C_2 = \log\left\{K_1(\sigma_0) \frac{d(\sigma_0^2, D)}{\eta - \sigma_0^2}\right\}$

Finally, by (2.3) and (2.4),

$$\frac{1}{y} \log |R_y(s)|_{s \in D} < -\sigma + \sigma_0 + \sigma_0^2 + \frac{1}{y}(C_1 + C_2)$$

Since σ_0 is arbitrary, we can put

$$\varepsilon_y(s) = \sigma_0 + \sigma_0^2 + \frac{1}{y}(C_1 + C_2) \qquad \text{q. e. d.}$$

(3) PROOF OF THEOREM I. By the assumption $\varliminf_{\nu\to\infty} \tau_\nu/\lambda_\nu > 1$, there exists $\vartheta\ (>0)$, such that

$$(3.1) \qquad \tau_\nu/\lambda_\nu > 1 + \vartheta \qquad \text{for } \nu > N(\vartheta)$$

Putting $\xi_\nu = \tau_\nu - \varepsilon\ (\varepsilon > 0)$, for sufficiently large ν, we have evidently

$$\lambda_\nu < \xi_\nu < \tau_\nu$$

Hence, by Lemma and the assumption $d\alpha(x) = 0$, $x \in (\lambda_\nu, \tau_\nu)$,

$$\frac{1}{\xi_\nu} \log |R_{\xi_\nu}(s)|$$
$$= \frac{1}{\xi_\nu} \log |R_{\lambda_\nu}(s)|$$
$$< -\sigma + \varepsilon_{\xi_\nu}(s)$$

In particular

$$(3.2) \qquad \frac{1}{\xi_\nu} \log |R_{\lambda_\nu}(s)|_{\sigma \leq 0}$$
$$< -\sigma + \varepsilon_{\xi_\nu}(s)$$

Again, by Lemma,

$$\frac{1}{\xi_\nu} \log |R_{\lambda_\nu}(s)|_{\sigma<0}$$
$$< \left(\frac{\lambda_\nu}{\xi_\nu}\right)(-\sigma) + \left(\frac{\lambda_\nu}{\xi_\nu}\right) \varepsilon_{\lambda_\nu}(s)$$
$$< \left(\frac{\lambda_\nu}{\tau_\nu}\right) \frac{1}{1-\varepsilon/\tau_\nu} (-\sigma) + \varepsilon_{\lambda_\nu}(s)$$
$$< (-\sigma) \frac{1}{1+\vartheta} \frac{1}{1-\varepsilon/\tau_\nu} + \varepsilon_{\lambda_\nu}(s)$$

by (3.1), whence, for sufficiently large ν,

$$(3.3) \qquad \frac{1}{\xi_\nu} \log |R_{\lambda_\nu}(s)|_{\sigma<0}$$
$$< (-\sigma) \frac{1}{1+\vartheta/2} + \varepsilon_{\lambda_\nu}(s)$$

Let s_1 be the regular point on $\sigma = 0$, and $F(s)$ be regular in $|s - s_1| \leq \delta$. We define the harmonic function $h(s)$ such that

$$\text{(i)} \quad h(s) = -\sigma \qquad \text{on } |s - s_1| = \delta,\ \sigma \geq 0,$$

$$\text{(ii)} \quad h(s) = (-\sigma) \frac{1}{1+\vartheta/2} \qquad \text{on } |s - s_1| = \delta,\ \sigma < 0$$

Hence, we have evidently

$$(3.4) \qquad h(s) < -\sigma,$$

in particular $h(s_1) < 0$

By (3.2) and (3.3),

$$\frac{1}{\xi_\nu} \log |R_{\lambda_\nu}(s)|_{|s-s_1|<\delta}$$
$$< h(s) + \max\{\varepsilon_{\lambda_\nu}(s), \varepsilon_{\tau_\nu}(s)\},$$

whence, by (3.4), in the sufficiently small neighbourhood of s_1, we can put

$$\frac{1}{\xi_\nu} \log |R_{\lambda_\nu}(s)| < \log \rho \qquad (0 < \rho < 1),$$

so that

$$|R_{\lambda_\nu}(s)| < \exp(-\xi_\nu \log(\tfrac{1}{\rho}))$$

for sufficiently large ν

Therefore, the sequence of $S_{\lambda_\nu}(s)$ converges uniformly in the neighbourhood of s_1, which is to be proved.

(4) THEOREM II. As an application of Theorem 1, we can establish the next gap-theorem of Hadamard's type.

THEOREM II. Under the same conditions as Theorem 1, if, for any given $\varepsilon\ (>0)$, $\int_{\tau_\nu}^{\lambda} |d\alpha(x)| = o(\exp(-\varepsilon\lambda_{\nu+1}))$ $(\tau_\nu \leq \lambda < \lambda_{\nu+1})$, $\sigma = 0$ is the natural boundary for (1.1).

Proof. If there would exist a regular point s_1 on $\sigma = 0$, by the Theorem 1, the sequence of partial sums $\int_0^{\lambda_\nu} \exp(-sx)\, d\alpha(x)$ $(\nu = 1, 2, \cdots)$ would uniformly convergent in the neighbourhood of s_1: $|s - s_1| < \delta$. Hence, in particular, the sequence of $\int_0^{\lambda_\nu} \exp(\varepsilon x)\, d\alpha(x)$ $(\nu = 1, 2 \cdots, 0 < \varepsilon < \delta)$ is convergent. For any given $\lambda\ (>0)$, we can put

$$(4.1) \qquad \int_0^{\lambda} \exp(\varepsilon x)\, d\alpha(x)$$
$$= \int_0^{\lambda_\nu} \exp(\varepsilon x)\, d\alpha(x) + \int_{\tau_\nu}^{\lambda} \exp(\varepsilon x)\, d\alpha(x),$$

where $\tau_\nu \leq \lambda < \lambda_{\nu+1}$ $(\nu(\lambda))$. By the assumption,

$$\left| \int_{\tau_\nu}^{\lambda} \exp(\varepsilon x)\, d\alpha(x) \right| \leq \exp(\varepsilon\lambda_{\nu+1}) \int_{\tau_\nu}^{\lambda} |d\alpha(x)| = o(1)$$

Therefore, (4.1) is convergent for $\sigma = -\varepsilon$, which is impossible, q.e.d.

From Theorem 2, we get easily

COROLLARY II. Let $F(s) = \sum_1^\infty a_n \exp(-\lambda_n s)$ be simply convergent for $\sigma > 0$. If $\varliminf_{n\to\infty} \lambda_{n+1}/\lambda_n > 1$, $\sigma = 0$ is the natural boundary for $F(s)$.

In this case, since $\lambda_\nu = \tau_{\nu-1}$, we have evidently $\int_{\tau_\nu}^{\lambda} |d\alpha(x)| = 0$, $(\tau_\nu \leq \lambda < \lambda_{\nu+1})$. Hence corollary immediately follows from Theorem 2.

(5) THEOREM III. In this section, we shall prove

THEOREM III. Let (1.1) be simply convergent for $\sigma > 0$. If $d\alpha(x) = 0$ in $x \in (\lambda_\nu, \tau_\nu)$ $(\nu = 1, 2, \cdots)$ with $\tau_{\nu-1} < \lambda_\nu < \tau_\nu$ and

$\lim_{\nu\to\infty} \tau_\nu/\lambda_\nu = \infty$, then the sequence of partial sums $\int_0^{\lambda_\nu} \exp(-sx)\,d\alpha(x)$ is convergent uniformly in the wider sense in the existence-domain E of $F(s)$.

REMARK. Since $\sum_{\nu=1}^{\infty} \int_{\tau_{\nu-1}}^{\lambda_\nu} \exp(-sx)\,d\alpha(x)$ $(\tau_0 = 0)$ is convergent uniformly in the wider sense in D, by well-known Weierstrass's theorem E is the simply connected domain.

From Theorem 3 follows next corollary.

COROLLARY III (A.Ostrowski), ([2] p.43, [4]). Let $F(s) = \sum_{1}^{\infty} a_n \exp(-\lambda_n s)$ be simply convergent for $\sigma > 0$. If in $\{\lambda_n\}$, there exists a subsequence $\{\lambda_{n_\nu}\}$ with $\lim_{\nu\to\infty} \lambda_{1+n_\nu}/\lambda_{n_\nu} = \infty$, then the sequence of partial sums $\sum_{1}^{n_\nu} a_n \exp(-\lambda_n s)$ is convergent uniformly in the wider sense in the existence-domain of $F(s)$.

Proof. By the assumption $\lim_{\nu\to\infty} \tau_\nu/\lambda_\nu = \infty$, we can put

$$(5.1) \qquad (\tau_\nu - \lambda_\nu) > \vartheta_\nu \lambda_\nu, \quad \lim_{\nu\to\infty} \vartheta_\nu = +\infty$$

If $\sigma = 0$ is the natural boundary for $F(s)$, Theorem 3 is trivial, so that we can assume that E contains points with negative real parts. For any given bounded domain D $(\subset E)$, we can surround D by the closed analytic curve C, which is contained in E and contains points with negative real parts. Let us denote by Δ the bounded domain surrounded by C. By (3.2) and (3.3), we have

$$(5.2) \quad \begin{cases} \frac{1}{\xi_\nu} \log |R_{\lambda_\nu}(s)| < -\sigma + \varepsilon_{\xi_\nu}(s) & s \in \Delta,\ \sigma \geq 0 \\ \frac{1}{\xi_\nu} \log |R_{\lambda_\nu}(s)| < (-\sigma)\, \frac{1}{1+\vartheta_\nu/2} + \varepsilon_{\lambda_\nu}(s) & s \in \Delta,\ \sigma < 0 \end{cases}$$

Now, let us define the harmonic function $h(s)$ such that

$$(5.3) \quad \begin{cases} h(s) = \sigma & \text{on the boundary arc of } C \text{ contained in } \sigma \geq 0, \\ h(s) = 0 & \text{on the boundary arc of } C \text{ contained in } \sigma < 0, \end{cases}$$

so that

$$(5.4) \qquad \min_{s \in D} h(s) = m > 0$$

On the other hand, by (5.2) and (5.3),

$$\frac{1}{\xi_\nu} \log |R_{\lambda_\nu}(s)|_{s \text{ on } C} < (-\sigma) \cdot \frac{1}{1+\vartheta_\nu/2} - \left(1 - \frac{1}{1+\vartheta_\nu/2}\right) h(s) + \max\{\varepsilon_{\lambda_\nu}(s),\ \varepsilon_{\xi_\nu}(s)\}$$

Accordingly, in D,

$$(5.5) \qquad \frac{1}{\xi_\nu} \log |R_{\lambda_\nu}(s)|_{s \in D} < \frac{1}{1+\vartheta_\nu/2} \{h(s) - \sigma\} - h(s) + \max\{\varepsilon_{\lambda_\nu}(s),\ \varepsilon_{\xi_\nu}(s)\}$$

Since $h(s) - \sigma$ is bounded in D, by (5.1) and (5.4),

$$\frac{1}{\xi_\nu} \log |R_{\lambda_\nu}(s)|_{s \in D} < -m/2$$

for sufficiently large ν.

Hence,

$$\lim_{\nu\to\infty} |R_{\lambda_\nu}(s)|_{s \in D} = 0$$

uniformly in D.

Since D is arbitrary, the sequence of partial sums $\int_0^{\lambda_\nu} \exp(-sx)\,d\alpha(x)$ is convergent uniformly in the wider sense in E. q.e.d.

(*) Received July 28, 1951.

[1] A.Ostrowski: Über eine Eigenschaft gewisser Potenzreihen mit unendlich vielen verschwindenden Koeffizienten. Berl.Sitz.1921.

[2] V.Bernstein: Sur les progrès récents de la théorie des séries de Dirichlet. Paris. 1933.

[3] C.Tanaka: Note on Lapalace-transforms. (VI) On the distribution of zeros of partial sums of Laplace-transforms.

[4] A.Ostrowski: Über vollständige Gebiete gleichmässiger Konvergenz von Folgen analytischer Funktionen. Abh. Hamb.Math.Sem. t.1.1922.

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