ON THE GROUP OF FORMAL ANALYTIC TRANSFORMATIONS

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1. Let K be an algebraic field. Under a (k-dimensional) formal analytic transformation[1] we mean a k-ple of integral formal power series in k variables $x_1, \ldots, x_k$ over K without constant terms. Let a and b be formal analytic transformations;

$$a: f_i(x) = f_i(x_1, \ldots, x_k) = \sum a^i_{j_1 \cdots j_k} x_1^{j_1} \cdots x_k^{j_k},$$

$$b: g_i(x) = g_i(x_1, \ldots, x_k) = \sum b^i_{j_1 \cdots j_k} x_1^{j_1} \cdots x_k^{j_k}.$$

$$(j_1 \geq 0, \ \ldots, \ j_k \geq 0, \ j_1 + \ldots + j_k \geq 1)$$

The product ab can be expressed as follows;

$$ab: h_i(x) = f_i(g_1(x), \ldots, g_k(x)), \quad (i=1, \ldots, k)$$

where it is to be noticed that the coefficients of ab can be determined formally as polynomials of those of a and b. The associativity of this multiplication is easy to verity and we obtain a semi-group F_k composed of all formal analytic transformations, whose identity is

$$e: \ e_i(x) = x_i \ . \quad (i=1, \ldots, k)$$

Next, letting correspond to any element a of F_k the linear part

$$La: f_i(x) = a^i_{1 0 \cdots 0} x_1 + \cdots + a^i_{0 \cdots 0 1} x_k, \quad (i=1, \ldots, k)$$

we have a linear representation of F_k;

$$(1) \qquad La \cdot Lb = L(ab).$$

Now let E_k be the group composed of all elements having inverses in F_k. E_k may be called the group of k-dimensional formal analytic transformations. An element a of F_k belongs to E_k if and only if La is a non-singular linear transformation. Now we define two subgroups of E_k in the following manner;

$$L_k = \{a \ : \ La = a\},$$

$$R_k = \{a \ : \ La = e\}.$$

Then (1) implies that R_k is an invariant subgroup such that

$$(2) \qquad E_k = R_k L_k, \quad R_k \cap L_k = e.$$

Now, let G be a group, and G, D(G), ..., $D_n(G) = D(D_{n-1}G)$, ... the descending series of subgroups of G, where D(G) denotes the commutator subgroup of G. When $\cap D_n(G) = e$, we shall call G solvable.

PROPOSITION 1. <u>R is a solvable group.</u>

Proof. Let

$$a: f_i = x_i + \sum_{n=2}^{\infty} A^i_n(x) \quad (i=1, \ldots, k)$$

be an element of R_k, where $A^i_n(x)$ denotes the homogeneous part of degree n. If a is not the identity, there exists $A^i_n(x) \neq 0$. The smallest number r such that there exists $A^i_r(x) \neq 0$ for some i is called the rank of a: r(a) =r. The rank of e is ∞.

Now let a and b be elements of R_k, of rank r and s respectively;

$$a: f_i = x_i + A^i_r + \text{higher terms},$$

$$b: g_i = x_i + B^i_s + \text{higher terms}. \quad (i=1, \ldots, k)$$

Then clearly we have

(3) ab:

$$h_i = \begin{cases} x_i + (A^i_r + B^i_r) + \text{higher terms}, \\ x_i + A^i_r \qquad + \text{higher terms}, \\ x_i + B^i_s \qquad + \text{higher terms}, \end{cases} \quad (i=1, \ldots, k)$$

according as r=s, r<s, or r>s respectively. Hence in the expressions of ab and ba the terms of degree Min(r,s) concide, and this readily leads to the following inequality;

$$(4) \qquad r(aba^{-1}b^{-1}) > \mathrm{Min}(r(a), r(b)),$$

which is valid except for a=b=e.

Let R^t_k be the subset of R_k composed of all elements of rank at least t(t≥2). By (3) R^t_k is a subgroup, and we have that $\cap R^t_k = e$. On the other hand we can conclude from (4) that

$$D(R_k) \subseteq R^3_k, \quad D_2(R_k) \subseteq R^4_k, \ \ldots,$$

whence R_k is solvable.

2. In this section we consider the case where K is the field of complex (or real) numbers. Then we can introduce a topology (the so-called weak topology) in F, namely the sequence $\{a(n)\}$;

$$a(n):\ f_i(n)=\sum a^{i}_{j_1\cdots j_k}(n)x_1^{j_1}\cdots x_k^{j_k}$$
$$(i=1,\ \ldots,\ k)$$

converges to

$$a(\infty):\ f_i(\infty)=\sum a^{i}_{j_1\cdots j_k}(\infty)x_1^{j_1}\cdots x_k^{j_k},$$
$$(i=1,\ \ldots,\ k)$$

if and only if every $a^{i}_{j_1\cdots j_k}(n)$ converges to $a^{i}_{j_1\cdots j_k}(\infty)$. E_k can thus be considered as a topological group. It is clear that L_k, R_k, and R_k^t are all closed subgroups.

Let us now define the topological commutator group $C(G)$ of a topological group G as the closure of $D(G)$; $\overline{D(G)}=C(G)$. Then we get the descending series of subgroups $\{C_n(G)\}$, where $C_n(G)=C(C_{n-1}(G))$. When $\cap C_n(G)=e$, we call G topologically solvable. Then by a slight modification of the proof of Proposition 1 we obtain

PROPOSITION 2. _R_k is topologically solvable._

From this proposition follows readily the following

COROLLARY. _Let S be a local Lie group in E_k. If S is semi-simple, then $a\to La$ for $a\in S$ defines a faithful representation._

Now the following lemma, which is a generalization of the so-called uniqueness theorem of H. Cartan, is known.

LEMMA 1.[2] Let K be a compact subgroup of E_k. Then K is a Lie group. In detail there exists an element d of R_k such that

$$d^{-1}ad = La \quad \text{for every} \quad a\in K.$$

On the other hand K. Iwasawa called a locally compact group G an (L)-group if G can be approximated by Lie groups.[3] We owe to him the following lemmas.

LEMMA 2.[4] A connected locally compact solvable group is an (L)-group.

LEMMA 3.[5] A connected (L)-group is a Lie group if it is locally euclidean.

LEMMA 4.[6] The space of a connected (L)-group is a direct product of that of a (maximal) compact subgroup and a euclidean space.

LEMMA 5.[7] Let H be a locally compact group, and N a closed invariant subgroup of H. If N is a simply connected solvable Lie group and if the factor group H/N is compact, then there exists a compact subgroup K of H such that $H=KN$.

Using above lemmas we shall prove the following theorem.

THEOREM. _A locally compact subgroup G of E_k is a Lie group._

Proof. Let N_1 be the intersection of G and R_k, and N the connected component of N_1 containing e. Since R_k is solvable, so is N. Hence N is an (L)-group by Lemma 2. From Lemma 4 N is topologically a direct product of a compact subgroup and a euclidean space. On the other hand, from (2) and Lemma 1 N contains no compact subgroup but for the identity group. Therefore N is a (simply connected solvable) Lie group according to Lemma 3.

Next let H be an open subgroup of N_1 containing N such that H/N is compact. Then from Lemma 5 there is a compact subgroup K of H so that $H=KN$. Again by Lemma 1 we have $K=e$, $H=N$. Hence N is open in N_1. Therefore N_1 is a Lie group.

Now the correspondence $a\to La$ for $a\in G$ gives a faithful representation of G/N_1 into L_k. Hence G/N_1 is also a Lie group. Our theorem follows directly from the extension theorem of Lie groups due to K. Iwasawa and M. Kuranishi.[8]

(*) Received June 14, 1950.

(1) S.Bochner and W.T.Martin: "Several complex variables", Princeton, (1948). We owe the principal idea of the present note to this book. Our "formal analytic transformation" is called an "inner transformation" there.
(2) Bochner and Martin: loc. cit., Chapter I, THEOREM 8.
(3) K.Iwasawa: "On some types of topological groups", Annals of Mathematics, Vol. 50 (1949).
(4) K.Iwasawa: loc. cit., THEOREM 10.
(5) K.Iwasawa: loc. cit., THEOREM 12.
(6) K.Iwasawa: loc. cit., THEOREM 13.
(7) Directly from LEMMA 3. 8 of K. Iwasawa, loc. cit.
(8) K.Iwasawa: loc. cit., LEMMA 3. 18.

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