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NOTE ON ARCHIMEDEAN VALUATIONS

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Ostrowski[1)] proved that any archimedean valuation of a field k can be obtained by embedding k in the field K_1 of complex numbers. Professor Iwasawa remarked in his lecture that this would also be proved easily if the following lemma were proved:

Lemma. Let K be the field of real numbers and $K(\alpha)$ be a simple extension of it. If $K(\alpha)$ has an archimedean valuation φ, then α is algebraic over K.

In this paper I shall give a proof[*)] of it and explain briefly Iwasawa s way of reduction.

Proof of the lemma. When ξ is a complex number, $\alpha^2-(\xi+\bar{\xi})\alpha+\xi\bar{\xi}$ belongs to $K(\alpha)$. We shall define a function $\sigma(\xi)=\varphi(\alpha^2-(\xi+\bar{\xi})\alpha+\xi\bar{\xi})$ on K_1. Then it is readily seen that $\sigma(\xi)$ is non-negative and a continuous function of ξ, and tends to infinity with $|\xi|$. Hence $\sigma(\xi)$ attains its greatest lower bound M. Put $\mathfrak{M}=\{\xi \mid \sigma(\xi)=M\}$ then $\mathfrak{M}$ is a non-null, closed and bounded set.

Now if $M=0$, then there is a $\xi_0\in K_1$ such that $\sigma(\xi_0)=0$, which means that $\alpha^2-(\xi_0+\bar{\xi}_0)\alpha+\xi_0\bar{\xi}_0=0$, namely α is algebraic over K. Hence we have only to deduce a contradiction, supposing $M>0$.

If φ_0 is the projection of φ on K, then $\varphi(a)=|a|^\varepsilon$, $0<\varepsilon\leqq 1$. There must be a negative number c such that

$$(1)\qquad M>|c|^\varepsilon>0.$$

From $\mathfrak{M}$ we take a number ξ_1 whose distance from the origin is largest. Since c is negative, at least one root η_1 of the equation

$$x^2-(\xi_1+\bar{\xi}_1)x+\xi_1\bar{\xi}_1-c=0$$

has a greater absolute value than $|\xi_1|$, and does not belong to $\mathfrak{M}$. We shall consider an algebraic equation

$$\{x^2-(\xi_1+\bar{\xi}_1)x+\xi_1\bar{\xi}_1\}^n-c^n=0,$$

and denote its roots by $\eta_1,\eta_2,\cdots\eta_{2n}$. Then

$$\begin{aligned}(M^n+|c|^{n\varepsilon})^2 &\geqq \varphi\{[\alpha^2-(\xi_1+\bar{\xi}_1)\alpha+\xi_1\bar{\xi}_1]^n-c^n\}^2\\ &=\varphi\Big\{\prod_{i=1}^{2n}(\alpha-\eta_i)\prod_{i=1}^{2n}(\alpha-\bar{\eta}_i)\Big\}\\ &=\varphi\Big\{\prod_{i=1}^{2n}(\alpha-\eta_i)(\alpha-\bar{\eta}_i)\Big\}\\ &=\prod_{i=1}^{2n}\varphi\{(\alpha-\eta_i)(\alpha-\bar{\eta}_i)\}\\ &=\prod_{i=1}^{2n}\sigma(\eta_i)\geqq M^{2n-1}\cdot\sigma(\eta_1)\end{aligned}$$

Dividing by M^{2n}, we get

$$\Big\{1+\Big(\frac{|c|^\varepsilon}{M}\Big)^n\Big\}^2\geqq\frac{\sigma(\eta_1)}{M}$$

Since n can be arbitrarily large, it follows from (1) that $\sigma(\eta_1)\leqq M$. But this means $\sigma(\eta_1)=M$, i.e. $\eta_1\in\mathfrak{M}$. That is a contradiction.

Way of Reduction. Since the lemma is assured, we deduce as follows. Let φ be an archimedean valuation of k. It is evident first of all that the characteristic of k must be zero. If φ_0 is the projection of φ on the prime field R, then $\varphi_0(a)=|a|^\varepsilon$, $0<\varepsilon\leqq 1$, $a\in R$, as is well-known. When we complete k to k' with respect to φ, R is completed automatically to the field K of real numbers with respect to φ_0. φ is extended uniquely to an archimedean valuation φ' of k', and φ_0 to φ_0' of K.

Take an element arbitrarily from k'. Then the subfield $K(\alpha)$ of k' has an archimedean valuation φ'' which is the projection of φ' on $K(\alpha)$. By the lemma, α is algebraic over K. Since α is an arbitrary element of k', k' is algebraic over K. Therefore k' and its subfield k may be looked on as contained in K_1. Or more precisely, there is an isomorphism S from k in K_1 and $\varphi(\alpha)=|\alpha^S|^\varepsilon$, $\alpha\in k$, $0<\varepsilon\leqq 1$.

Our final result is: a set of equivalent archimedean valuations of k (which gives the same topology of k) corresponds one-to-one to a pair $(S,\bar{S})$ of isomorphisms of k in K_1 (bar indicates the complex conjugate).

(*) Received October 16, 1949.

(1) Ostrowski: Ueber einige Loesungen der Funktionalgleichung $\varphi(x)\varphi(y) = \varphi(xy)$. Acta math. Bd. 41 (1918) S.271-284.

(2) The lemma was also proved by T. Asatani, using the theory of normed rings.

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