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THE REAL PART OF DECOMPOSITION OF A POLYNOMIAL AND ITS DETERMINACY

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1. Introduction

Let $f(x,y)$, $g(x,y)$: $(\mathbb{R}^2,0) \to (\mathbb{R},0)$ be two C^∞ function-germs. Germs f and g are called to be r-jet equivalent if at $(0,0)$, their derivatives of degree not greater than r are identical. Denote this fact by $j^r(f) = j^r(g)$. Germ f is called to be C^0-r-determined if for each germ g with $j^r(f) = j^r(g)$, there exists a germ of homeomorphism $h : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ such that $f \circ h = g$. f is called to be C^0-finitely-determined if it is C^0-r-determined for some r. The degree of C^0-determinacy of f is the least number such that f is C^0-r-determined.

Germs f and g are called to be V-equivalent if germs $f^{-1}(0)$ and $g^{-1}(0)$ are homeomorphic.

Let $P_0(n,k;\mathbb{R})$ denote the set of topological equivalence classes of germs of real polynomials in n variables of degree $\leq k$, and $P_0(n,\mathbb{R})$ the set of those classes for all k. T. Fukuda [1] proved the Thom's conjecture: $P_0(n,k;\mathbb{R})$ is a finite set. How about $P_0(n;\mathbb{R})$? It is easy to see that $P_0(1;\mathbb{R})$ contains only three elements. For example, the germs $y = x^2$ and $y = x^4$ are C^0-equivalent (V.I. Arnol'd etc. [2], p. 12). In general, $y = x^{2m}$ and $y = x^{2n}$ belong to be the same class, and $y = x^{2m+1}$ and $y = x^{2n+1}$ belong to be the another class.

2. Homogeneous case

Let $P(x,y)$ be a germ of a real homogeneous polynomial of degree k. Then

$$P(x,y) = a(x - b_1 y) \cdots (x - b_s y)(x - c_1 y) \cdots (x - c_m y)$$

where $a, b_i \in \mathbb{R}, a \neq 0, c_j \in \mathbb{C}$. We have the following.

THEOREM 1. *$P(x,y)$ is C^0-finitely determined if and only if $b_i \neq b_j$ for $i \neq j$. In this case, the degree of C^0-determinacy of P is k.*

THEOREM 2. *Homogeneous polynomial-germs $P(x,y)$ and $Q(x,y)$ are V-equivalent if and only if they have the same number of real factors (do not account the repeated number, if $b_i = b_j$ for some i,j).*

Remark. The degrees of P and Q may be unequal when they are V-equivalent.

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COROLLARY 3. *$P_0(n;\mathbb{R})$ is infinite for $n \geq 2$.*

3. Non-homogeneous case

Let $F(x,y)$ be a germ of the following form

$$F(x,y) = x^n + A_1(y)x^{n-1} + A_2(y)x^{n-2} + \cdots + A_n(y)$$

where $A_i(y)$ is a real polynomial of y. By Newton-Puiseux Theorem,

$$F(x,y) = (x - p_1(y)) \cdots (x - p_s(y))(x - q_1(y)) \cdots (x - q_m(y))$$

where $p_i(y)$ is a real fraction power series in y and $q_j(y)$ has some complex coefficients.

Remark. The coefficients of p_i, q_j can be computed effectively out, so p_i and q_j are called the Puiseux roots.

THEOREM 4. *If $p_1, \ldots, p_s$ are mutually distinct, then $F(x,y)$ is C^0-finitely-determined.*

4. Proofs

LEMMA 1 (Y.C. Lu [3], Theorem 2). *Let $Z(x,y) = Z_1 Z_2 \cdots Z_q$, where $Z_1(x,y)$ is homogeneous of degree a_j and the degree of C^0-determinacy of Z_j is k_j. Moreover, $\{Z_1, Z_2, \ldots, Z_q\}$ is pairwise relatively prime. Then Z is C^0-m-determined, where*

$$m = \max_{1 \leq i \leq q} \{\sum_{j=1}^{q} a_j - a_i + k_i\}.$$

LEMMA 2 (T.C. Kuo [4], Corollary 1). *Let $H(x,y)$ be homogeneous of degree k. If $H(x,y) = 0$ is a non-singular projective variety, i.e. $\operatorname{grad} H(x,y) = 0$ only when $x = y = 0$, then H is C^0-k-determined.*

Proof of Theorem 1. (1) Necessity. If P has a real repeated factor, we have

$$P = (x - ay)^r B(x,y), a \in \mathbb{R}, r \geq 2.$$

Then $x - ay$ is a comon factor of $\frac{\partial P}{\partial x}$ and $\frac{\partial P}{\partial y}$, so the line $x - ay = 0$ is contained in $\left(\frac{\partial P}{\partial x}\right)^{-1}(0)$ and $\left(\frac{\partial P}{\partial y}\right)^{-1}(0)$, and $(0,0)$ is not an isolated critical point of P. By Bochnack and Lojasiewicz [5], P is not C^0-finitely-determined.

(2) Sufficiency. If P has no any real repeated factors, we have

$$\text{(A)} \qquad P = a \prod_{i=0}^{s} (x - b_i y) \prod_{j=0}^{m} (x^2 + c_j xy + d_j y^2)^{w_j}$$
$$= aZ_1 Z_2 \cdots Z_{s+m}. \qquad \text{where } c_j^2 - 4d_j < 0, \quad j = 1, 2, \ldots, m.$$

If $i \neq j$, Z_i and Z_j are relatively prime. Let $g(x,y) = x - by, b \in \mathbb{R}$. Obviously, g is C^0-1-determined. Let $h(x,y) = (x^2 + cxy + dy^2)^t, c^2 - 4d < 0, t \geq 1$, then $x^2 + cxy + dy^2 = 1$ is an elliptic curve, so under new coordinate system, h has the following form:

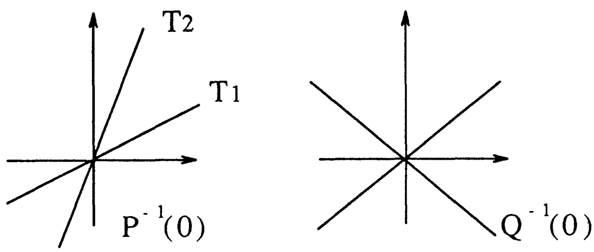
$$h(x,y) = (x^2 + y^2)^t.$$

For $\frac{\partial h}{\partial x} = 2tx(x^2+y^2)^{t-1}$, $\frac{\partial h}{\partial y} = 2ty(x^2+y^2)^{t-1}$, then grad $h = 0$ only when $x = y = 0$. By Lemma 2, h is C^0-k-determined.

Denote the degree of Z_j by a_j, and the degree of C^0-determinacy of Z_j by k_j, then $a_j = k_j$ for all j from the above argument and $\sum_{j=1}^{s+m} a_j = k$. Hence by Lemma 1, P is C^0-k-determined.

Example. Let $P(x,y) = x^5 + y^5$. Since $z^5 + 1 = 0$ has only one real root, so P has only one real factor of its decomposition. By Theorem 1, P is C^0-5-determined. From D. Siersma [6] (p. 26), P is C^∞-6-determined. By Y.C. Lu [7] (p. 59), P is not C^∞-5-determined. This example and Theorem 1 show that for germs of homogeneous polynomials, the degree of C^0-determinacy is exactly the degree of polynomial if it is finite-determined, but it is not ture for the smooth case.

Proof of Theorem 2. In the express (A), $P(x,y) = 0$ if and only if either $x - b_i y = 0$ or $x^2 + c_1xy + d_jy^2 = 0$ is satisfied. The curve $x - b_i y = 0$ is a straight line T_i passing through the origin, and the curve $x^2 + c_1xy + d_jy^2 = 0$ contains only one point $(0,0)$, because $c_j^2 - 4d_j < 0$, then $P^{-1}(0)$ consists of lines T_i.

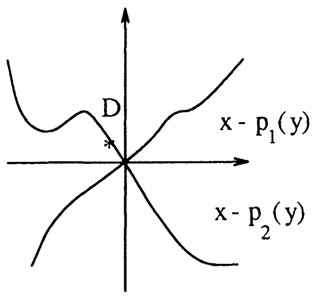


It is easy to see that $P^{-1}(0)$ and $Q^{-1}(0)$ have the same topological type if and only if they have the same number of lines T_i.

Proof of Corollary 3. For $n = 2$, let $f_k(x,y) = (x-y)(x-2y)\cdots(x-ky)$. By Theorem 2, f_i is not C^0-equivalent to f_j for $i \neq j$. For $n > 2$, let $\bar{f}_k(x_1,\ldots,x_n) = (x_1 - x_2)\cdots(x_1 - kx_2)(x_3^2 + \cdots + x_n^2)$, then $f_k(x_1,x_2) = \bar{f}_k(x_1,x_2,0,\ldots,0)$. Since $f_i(x_1,x_2)$ is not C^0-equivalent to $f_j(x_1,x_2)$ for $i \neq j$, hence $\bar{f}_i$ is not C^0-equivalent to $\bar{f}_j$.

Proof of Theorem 4. For any point D near $(0,0)$,

$$\begin{aligned}\frac{\partial F}{\partial x} &= \sum_{i=1}^{s}[x - p_1(y)]\cdots(x - \hat{p}_i(y))\cdots(x - p_s(y))(x - q_1(y))\cdots(x - q_m(y))\\ &+ \sum_{j=1}^{m}(x - p_1(y))\cdots(x - p_s(y))(x - q_1(y))\cdots(x - \hat{q}_j(y))\cdots(x - q_m(y)).\end{aligned}$$



Therefore, $\frac{\partial F}{\partial x} = 0$ at D if and only if $D = (0,0)$, and so $(0,0)$ is the isolated critical point of F. By Kuo-Bochnack-Lojasiewicz Theorem, F is C^0-finitely-determined.

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REFERENCE

[1] T. Fukuda, *Topologiques Des polynomes*, I. H. E. S. Publ. Math. **46**, (1976), 87–106.
[2] V.I. Arnol'd, S.M. Gusein-Zade and A.N. Varchenko, *Singularities of Differentiable Maps, Vol. I*, Birkhauser, Boston Inc., 1985.
[3] Yung-Chen Lu, *Sufficiency of jets in $J^r(2,1)$ via Decomposition*, Invent. Math. **10** (1970), 119–127.
[4] T.C. Kuo, *On C^0-sufficiency of jets of potential functions*, Topology **8** (1969), 167–171.
[5] J. Bochnack and S. Lojasiewicz, *A converse of the Kuiper-Kuo Theorem*, LNM-Springer **192** (1971), 254–261.
[6] D. Siersma, *Classification and deformation of singularities*, Ph. D. thesis, Academis Service, Vinkveen (1974).
[7] Y.C. Lu, *Singularity Theory and Introduction to Catastrophe*, Springer-Verlag, New York, 1976.
[8] R.J. Walker, *Algebraic Curves*, Princeton Univ. press, 1950.

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