K. SAWADA
KODAI MATH. J.
16 (1993), 52-59

UNIQUENESS OF FACTORIZATION OF CERTAIN ENTIRE FUNCTIONS

BY KAZUNARI SAWADA

Introduction. For a meremorphic function $F(z)$ in the plane $(|z|<+\infty)$, the representation:

$$F(z)=f\circ g(z)=f(g(z))$$

is called a *factorization* of $F(z)$, where f and g are meromorphic functions (g is entire, if f is transcendental). And then f is called *left-factor* and g is called *right-factor* of F. F is called to be *prime,* if, for every factorization, we can always deduce that either f or g is linear. We state that two factorizations:

$$F(z)=f_1\circ f_2\circ \cdots \circ f_n$$
$$=g_1\circ g_2\circ \cdots \circ g_m$$

are *equivalent,* if $n=m$ and there exist linear functions T_j $(1\leqq j\leqq n-1)$ such that

$$f_1=g_1\circ T_1,\quad f_j=T_{j-1}^{-1}\circ g_j\circ T_j\quad (2\leqq j\leqq n-1),$$

and

$$f_n=T_{n-1}^{-1}\circ g_n.$$

An entire function F is called *uniquely factorizable,* if all the factorizations into non-linear prime entire functions are equivalent to each other.

Urabe [8] proved the following

THEOREM A. *$F(z)=(z+h(e^z))\circ(z+Q(e^z))$ is uniquely factorizable, where h is a non-constant entire function, $h(e^z)$ is of finite order and Q is a non-constant polynomial.*

We have many functions which are uniquely factorizable as its corollaries. Still there are several functions whose unique factorizablity cannot be proved by Theorem A. For example,

$$F(z)=(z+e^z)\circ\left(z+\frac{1}{e^z}\right),$$
$$F(z)=(z+e^z)\circ(z+\sin(-iz))$$

Received November 1, 1991 Revised June 4, 1992.

$$=(z+e^z)\circ\left(z+\frac{e^z-e^{-z}}{2i}\right)$$

and so on.

In this paper we shall prove the following

THEOREM. *Let $R_j(w)$ $(j=1, 2)$ be non-constant rational functions having at most two poles at $w=0$ and $w=\infty$. Then*

$$F(z)=(z+R_1(e^z))\circ(z+R_2(e^z))$$

is uniquely factorizable.

As an easy application of this theorem we have immediately that above functions are uniquely factorizable.

§ 1. Some lemmas. We shall use the following symbols:

$$M_F(r)=M(r, F)=\max_{|z|=r}|F(z)|$$

$$\rho(F)=\limsup_{r\to\infty}\frac{\log\log M_F(r)}{\log r}$$

for an entire function F. And we shall use Nevanlinna's notations such as $T(r, F)$, $m(r, F)$ and $N(r, a, F)$.

LEMMA 1 (Urabe [8]). *Let $J(b)=\{F(z)=cz-H(z)$; $H(z)$ is an entire periodic fnnction with period b $(\neq 0)$ and c is a non-zero constant$\}$. And let $F\in J(b)$ and $F(z)=f(g(z))$ with non-linear entire functions f and g, then $f\in J(b')$ for some $b'\neq 0$ and $g\in J(b)$. Further $b'=c_2\cdot b$, if $g(z)=c_2\cdot z+H_2(z)$.*

LEMMA 2 (Urabe [8]). *Let*

$$F(z)=(z+H_1(z))\circ(z+H_2(z))$$

where H_1, H_2 ($\not\equiv$constant) are periodic entire functions with period $2\pi i$ and $\rho(H_1)<+\infty$ and H_2 is of exponential type. And let $F(z)=f(g(z))$ with non-linear entire functions f and g. Then g is of exponential type.

We recall that g is of exponential type, if $\rho(g)\leq 1$ and

$$\limsup_{r\to+\infty}\frac{\log M_g(r)}{r}<+\infty.$$

LEMMA 3 (Urabe [8]). *Let $H(z)$ ($\not\equiv$constant) be a periodic entire function with period $2\pi i$ and of exponential type. Then there exist a rational function $R(w)$ with at most two poles at $w=0$ and $w=\infty$ such that $H(z)\equiv R(e^z)$.*

LEMMA 4 (Ogawa [4]). *Let $h(w)$ be single-valued and regular in $0<|w|<\infty$.*

// $h(e^z)$ is of finite order, then $h(w)$ is of order zero around $w=0$ and $w=\infty$.

In general, if $h(w)$ is regnlar in $0<|w|<\infty$, there exist two entire fnnctions $h_j(w)$ $(j=1, 2)$ such that

$$h(w)=h_1(w)+h_2\left(\frac{1}{w}\right).$$

The above lemma 4 suggests $\rho(h_j)=0$ for $j=1, 2$.

LEMMA 5. *Let $F(z)$ be the same function in the theorem. And let $F(z)=f(g(z))$ with an entire function f and $g(z)=z+Q(e^z)$, where $Q(w)$ is a rational function with at most two poles at $w=0, \infty$. Then $\rho(f)<+\infty$.*

Proof. By Pólya's result,

$$M_F(r)\geqq M_f\left(d\cdot M_g\left(\frac{r}{2}\right)\right) \qquad (r\geqq r_0)$$

for some positive constant d. And by the form of F, there exists a positive constant K such that

$$M_F(r)\leqq e^{e^{K\cdot r}}$$

for any $r\geqq r_1$. Further for any $\varepsilon>0$, there exist $r_2(>0)$ and some natural number c such that

$$e^{c/2\cdot r-\varepsilon}\leqq M_g\left(\frac{r}{2}\right)\leqq e^{c/2\cdot r+\varepsilon}$$

for $r\geqq r_2$.

Therefore, there exists R_0 (>0) such that

$$M_f(R)\leqq \exp\left[\left(e^{\varepsilon}\cdot\frac{R}{d}\right)^{2K/c}\right]$$

for $R\geqq R_0$. It means that $p(f)<+\infty$. q. e. d.

§ 3. **Proof of theorem.** By the assumption of theorem,

$$F(z)=z+R_2(e^z)+R_1[e^{z+R_2(e^z)}].$$

Here the function $R_2(e^z)+R_1[e^{z+R_2(e^z)}]$ is a periodic function with period $2\pi i$. By lemma 1, if

$$F(z)=f(g(z)) \tag{1}$$

with non-linear entire functions / and g, then

$$f(z)=c_1\cdot z+H_1(z), \qquad g(z)=c_2\cdot z+H_2(z)$$

where H_1, H_2 are periodic with period $2\pi c_2 i$, $2\pi i$ respectively. Substituting these into (1), we have $c_1\cdot c_2=1$ and hence

$$f(c_2\cdot z)=c_1c_2z+H_1(c_2\cdot z)=z+H_1(c_2\cdot z)$$

and

$$\frac{1}{c_2}\cdot g(z)=z+\frac{1}{c_2}\cdot H_2(z)$$

belong to $J(2\pi i)$. Therefore, without loss of generality, we may assume that

$$\begin{cases} f(z)=z+H_1(z) \\ g(z)=z+H_2(z) \end{cases}$$

where H_j $(j=1,2)$ are periodic entire functions with period $2\pi i$. Further, in general, a periodic entire function with period $2\pi i$ is represented as $h(e^z)$ with some regular function $h(w)$ in $0< w|<+\infty$. Hence

$$\begin{cases} f(z)=z+h_1(e^z) \\ g(z)=z+h_2(e^z) \end{cases} \tag{2}$$

where $h_j(w)$ are regular in $0<|w|<+\infty$ $(j=1, 2)$.

Since $\rho(R_1(e^z))=1<+\infty$ and $R_2(e^z)$ is of exponential type, g must be of exponential type by lemma 2. And then h_2 must be a rational function by lemma 3. By (1) and (2), we have

$$h_2(e^z)+h_1[e^z\cdot e^{h_2(e^z)}]=R_2(e^z)+R_1[e^z\cdot e^{R_2(e^z)}].$$

Now we put $w=e^z$. Then

$$h_2(w)-R_2(w)=-h_1[w\cdot e^{h_2(w)}]+R_1[w\ e^{R_2(w)}]. \tag{3}$$

This gives a key of our proof of this theorem. By the above investigation, we assume that

$$\begin{aligned} R_j(w)&=(a_{N_j}{}^j\cdot w^{N_j}+\cdots+a_0{}^j)+\left(a_{-1}{}^j\cdot\frac{1}{w}+\cdots+a_{-M_j}{}^j\cdot\frac{1}{w^{M_j}}\right) \\ &=R_j{}^+(w)+R_j{}^-(w) \qquad (j=1, 2), \\ h_2(w)&=(b_{n_2}{}^2\cdot w^{n_2}+\cdots+b_0{}^2)+\left(b_{-1}{}^2\cdot\frac{1}{w}+\cdots+b_{-m_2}{}^2\cdot\frac{1}{w^{m_2}}\right) \\ &=h_2{}^+(w)+h_2{}^-(w). \end{aligned}$$

Similarly we write

$$h_1(w)=h_1{}^+(w)+h_1{}^-(w),$$

where in this case both $h_1{}^+(w)$ and $h_1{}^-(1/w)$ are entire functions. By lemma 5, $\rho(f-z)=\rho(f)<+\infty$. And by lemma 4, $\rho(h_1{}^+)=\rho(h_1{}^-(1/w)=0$.

In the following we shall prove that h_1 must be a rational function. Now we assume that $h_1{}^+$ is a transcendental function. Then we will show that tt$_2$^Λ/2 as follows. As noted above, $\rho(h_1{}^+)=0$, and hence by $\cos\pi\rho$—theorem, for any $\varepsilon(>0)$, there exists an unbounded sequence of positive real numbers $\{r_n\}$ such that

$$m_{h_1^+}(r_n) \geqq M_{h_1^+}(r_n)^{1-\varepsilon} \qquad (n=1,\ 2,\ \cdots), \tag{4}$$

where $m_{h_1^+}(r)$ is the minimum modulus of h_1^+, that is,

$$\min_{|w|=r} |h_1^+(w)|.$$

Here assuming $n_2 < N_2$, we consider the following equation:

$$R_2^+(w) = 2\pi ti \qquad (t \in R,\ t \neq 0). \tag{5}$$

As is well-known, the set of roots of equation (5) tend to $w - \infty$ as $|t| \to \infty$ and possesses $2N_2$ lines:

$$\arg w = \frac{1}{N_2} \operatorname{Arg}\left(\frac{i}{a_{N_2}^{\ 2}}\right) + \frac{1}{N_2} 2j\pi \qquad (j=,\ 0,\ 1,\ \cdots,\ N_2-1;\ \text{as } t \to +\infty)$$

$$\arg w = \frac{1}{N_2} \operatorname{Arg}\left(\frac{i}{a_{N_2}^{\ 2}}\right) + \frac{1}{N_2} \cdot (2j+1)\pi \qquad (j=0,\ 1,\ \cdots,\ N_2-1;\ \text{as } t \to -\infty)$$

as asymptotic lines. If $n_2 > 0$, then (because of $n_2 < N_2$), among these $2N_2$ lines, we have a line, say l, on which

$$Re[b_{n_2}^{\ 2} \cdot e^{n_2 \theta i}] > 0 \qquad (z = r \cdot e^{i\theta} \in l).$$

And there exists a subset (continuity) $\{w(t)\}$ of roots of (5) such that

$$R_2^+(w(t)) = 2\pi it$$

and further $\{w(t)\}$ possesses the line l as asymptotic line. Therefore by $R_2(w(t)) = R_2^+(w(t)) + o(1)$,

$$|e^{R_2(w(t))}| \longrightarrow 1 \qquad (\text{as } t \to +\infty, \text{ or as } t \to -\infty) \tag{6}$$

and further, there exists some constant $L(>0)$ such that

$$|e^{h_2(w(t))} > e^{L \cdot |w(t)|^{n_2}} \qquad \text{as } |t| \to +\infty \tag{7}$$

by the assumption of $\{w(t)\}$. Here, consider a sequence $\{t_n\}$ of real numbers such that

$$|w(t_n)\ e^{h_2(w(t_n))}| = r_n.$$

Then by (3), (4), (6), (7) and maximum modulus principle, we have

$$M_{h_1^+}(|w(t_n)|e^{L|w(t_n)|^{n_2}})^{1-\varepsilon} \leqq O(|w(t_n)|^K) \qquad (n=1,\ 2,\ \cdots) \tag{8}$$

for some constant K. Since h_1^+ is assumed to be transcendental, this leads us to a contradiction. Hence $n_2 \geqq N_2$. Now let us note that, even if $n_2 = 0$, the above inequality (8) can be shown to be valid without using the special line l

and hence we get to the same conclusion.

Similarly, if h_1^- is transcendental, then we can prove $m_2 \geqq M_2$.

Since h_2 is non-constant, $n_2>0$ or ra$_2>0$. Without loss of generality, we may assume $n_2>0$.

Let, for a sufficiently small $\delta>0$ $(|b_{n_2}{}^2|/2>\delta)$,

$$O_1=\{z=r\cdot e^{i\theta}\ ;\ Re(b_{n_2}{}^2\cdot e^{in_2\theta})>\delta\}\ ,$$

$$O_2=\{z=r\cdot e^{i\theta}\ ;\ Re(b_{n_2}{}^2\cdot e^{in_2\theta})<-\delta\}\ .$$

By $h_2(w)=b_{n_2}{}^2\cdot w^{n_2}\cdot(1+o(1))(|w|\to+\infty)$, it is noted that the function $h_1^+(we^{h_2(w)})$ is bounded in $O_2\cap\{|w|>R_0\}$ and $h_1^-(we^{h_2(w)})$ is bounded in $O_1\cap\{|w|>R_0\}$. By (3),

$$M(r, R_1[we^{R_2(w)}]-h_2(w)+R_2(w))\geqq h_1[we^{h_2(w)}]\ . \tag{9}$$

Also we have

$$|we^{h_2(w)}|>r\cdot e^{Kr^{n_2}} \qquad (w\in O_1,\ |w|=r>R_0) \tag{10}$$

and

$$|we^{h_2(w)}|<r\cdot e^{-Kr^{n_2}} \qquad (w\in O_2,\ |w|=r>R_0)$$

for some positive constant K.

Now assuming that h_1^+ is transcendental, we use (4) with $\varepsilon=1/2$. Then there exists $\{r_n\}$ such that

$$m(r_n,\ h_1^+)\geqq M(r_n, h_1^+)^{1/2}\,.$$

Then we can find an unbounded sequence $\{t_n\}$ of real numbers such that $|w\cdot e^{h_2(w)}|=r_n$ for some w ($w\in O_1$ and $|w|=t_n$). In this case,

$$\begin{aligned}|h_1(w\cdot e^{h_2(w)})| &\geqq m(r_n,\ h_1^+)+O(1)\\ &\geqq M(r_n,\ h_1^+)^{1/2}\end{aligned} \tag{11}$$

On the other hand, for any natural number N, there exists $R_0=R_0(N)$ such that

$$M(R,\ h_1^+)>R^N \qquad (\text{for } R\geqq R_0)$$

because of transcendency of h_1^+. Therefore (11) becomes

$$|h_1(w\cdot e^{h_2(w)})|\geqq(r_n)^{N/2}\,.$$

Now by (10), $r_n>t_n\cdot e^{K\cdot t_n{}^{n_2}}$. Hence (noting (9)), we have the inequality

$$c\cdot t_n{}^{N_1}\cdot e^{c' N_1 t_n{}^{N_2}}>t_n{}^{N/2}\cdot e^{(1/2)NKt_n{}^{n_2}} \qquad (n\geqq n_0)$$

for some constants c and c' (>0). This contradicts $n_2\geqq N_2$ and the arbitrariness of N. And hence h_1^+ must be a polynomial.

We can prove that $h_1^-(1/w)$ must be a polynomial in the similar way.

Hence we deduce that h_1 is a rational function, as is to be proved.

Finally we prove that both sides of (3) are constants. Putting $h_2(w)-R_2(w)$ $-G(w)$ and assuming that $G(w)$ is a non-constant rational function, then we have

$$G(w)=-h_1[we^{R_2(w)+G(w)}]+R_1[we^{R_2(w)}].$$

Furthermore let us substitute w by e^z, then

$$G(e^z)=-h_1[e^{z+R_2(e^z)+G(e^z)}]+R_1[e^{z+R_2(e^z)}] \tag{12}$$

Assuming that $R_2(w)+G(w)\not\equiv$constant, then we can easily show that

$$T(r, G(e^z))=o\{T(r, e^{z+R_2(e^z)})\},$$
$$T(r, G(e^z))=o\{T(r, e^{z+R_2(e^z)+G(e^z)})\}$$

as $r\to+\infty$. By Borel's unicity theorem [3], (12) is immpossible, because that $h_1(u)$ and $R_1(w)$ are rational functions in w whose coefficients are constants.

Next if $R_2(w)+G(w)\equiv$constant, say c, then

$$h_1[e^{z+R_2(e^z)+G(e^z)}]=h_1[e^{z+c}].$$

Hence (12) is immpossible in the similar way.

Therefore $G(w)$ is a constant, say K.

Then by (3),

$$\begin{cases} h_2(w)=R_2(w)+K \\ h_1[w\cdot e^{h_2(w)}]=R_1[w\cdot e^{R_2(w)}]-K. \end{cases} \tag{13}$$

Hence

$$h_1[w\cdot e^K\cdot e^{R_2(w)}]=R_1[w\cdot e^{R_2(w)}]-K.$$

Let x be $w\ e^K\ e^{R_2(w)}$, then we have

$$h_1(x)=R_1(e^{-K}\cdot x)-K. \tag{14}$$

By (2), (13) and (14),

$$\begin{cases} f(z)=z-K+R_1(e^{z-K}) \\ g(z)=K+z+R_2(e^z). \end{cases}$$

Then

$$\begin{cases} f\circ T(w)=w+R_1(e^w) \\ T^{-1}\circ g(z)=z+R_2(e^z) \end{cases}$$

with $z=T(w)=w+K$. This completes the proof of our theorem. q. e. d.

Acknowledgement. The auther wishes to thank the referee for many valuable comments and suggestions.

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DEPARTMENT OF MATHEMATICS,
SCIENCE UNIVERSITY OF TOKYO,
NODA, CHIBA, JAPAN.