N. SUITA
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ON SUBADDITIVITY OF ANALYTIC CAPACITY FOR TWO CONTINUA

Dedicated to Professor Mitsuru Ozawa on his 60th birthday

BY NOBUYUKI SUITA

1. Introduction. Let K be a compact set in the complex plane $\boldsymbol{C}$ and let $\Omega(K)$ be the connected component of $\boldsymbol{C}-K$ containing the point at infinity. The analytic capacity $\gamma(K)$ of K is defined by

$$\gamma(K)=\sup_{\|f\|\leqq 1}|f'(\infty)|.$$

Here $f(z)$ is a holomorphic function in $\Omega(K)$ whose expansion at the point at infinity is given by

$$f(z)=f'(\infty)/z+a_2/z^2+\cdots$$

and $\|f\|$ denotes the supnorm of f.

The following problem was first raised by Vituškin [6]. For two disjoint compact sets K_1 and K_2, is there an absolute constant M such that the following semiadditivity holds

$$\gamma(K_1\cup K_2)\leqq M(\gamma(K_1)+\gamma(K_2))\ ? \tag{1}$$

We can find this problem and partial answers in many articles and books [1, 2, 3, 6, 7]. It was shown by Mel'nikov [3] that if K_1 and K_2 are separated by an analytic curve c, then there exists a constant $M(c)$ depending on c for which (1) holds. This shows that the constant M possibly depends on positions of two sets.

In the present paper we shall establish the following subadditive inequality

$$\gamma(K_1\cup K_2)\leqq\gamma(K_1)+\gamma(K_2) \tag{2}$$

for arbitrary disjoint continua K_1 and K_2. To prove (2) we shall solve an auxiliary extremal problem.

2. Extremal problem. Let Ω be a doubly connected domain containing the point at infinity. We suppose that its boundary components are continua K_1 and

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K_2. Let $\mathfrak{F}$ be the family of univalent functions ϕ defined on Ω with a normalization

$$(3) \qquad \phi(z)=z+a_1/z+\cdots \quad \text{at } \infty.$$

We denote by $\phi(K_j)$ the boundary component of the image domain $\phi(\Omega)$ under ϕ which corresponds to K_j $(j=1, 2)$. Our problem is to minimize $\operatorname{Cap}\phi(K_1)+\operatorname{Cap}\phi(K_2)$ within $\mathfrak{F}$, $\operatorname{Cap}\phi(K_j)$ being logarithmic capacity of $\phi(K_j)$. We prove

THEOREM1. *The minimum of the quantity* $\operatorname{Cap}\phi(K_1)+\operatorname{Cap}\phi(K_2)$ *is attained by the conformal mapping* ϕ_0 *which maps* Ω *onto a parallel slit domain* Δ *whose boundary lies on a straight line.*

Proof. Since capacity is invariant under rigid motions, we consider the same problem within a slightly larger class $\tilde{\mathfrak{F}}$ of univalent functions ϕ whose expansion admits

$$(4) \qquad \phi(z)=e^{i\theta}z+a_0+a_1/z+\cdots \quad \text{at } \infty.$$

There exists a unique radial slit disc mapping $\psi(z)$ in $\tilde{\mathfrak{F}}$ such that $\psi(K_1)$ is a disc $|w|\leqq r$, say and $\psi(K_2)$ is a line segment $a\leqq w\leqq b$ $(a>r)$. Then $\operatorname{Cap}\phi(K_1)=r$ and we know that r is equal to the minimum of $\operatorname{Cap}\phi(K_1)$. For these facts see Sario and Oikawa ([5], pp. 148-158). By an elementary mapping $S_r(w)=r(w/r+r/w)$, $\psi(K_1)$ is mapped onto a line segment $[-2r, 2r]$ with the same capacity r. Then $\psi(K_2)$ is mapped onto a new segment $[a', b']$. We show that $\operatorname{Cap}[a', b']=(b'-a')/4$ is the minimum of $\operatorname{Cap}\phi(K_2)$ within $\tilde{\mathfrak{F}}$. In fact by a parallel displacement, $[a', b']$ is moved onto $[-2r', 2r']$, $r'=(b'-a')/4$. The conformal mapping $S_{r'}^{-1}$ maps the last domain onto another radial slit disc, K_2 corresponding to the disc $|w|\leqq r'$. Hence r' is the minimum of $\operatorname{Cap}\phi(K_2)$. Thus $r+r'$ is the desired minimum within $\tilde{\mathfrak{F}}$. $S_r\circ\psi$ is clearly an extremal function. It is easy to see that by composition of elementary mappings an extremal function ϕ_0 can be constructed in $\mathfrak{F}$ as stated in the theorem, which completes the proof.

Remark. The uniqueness of the extremal function in $\mathfrak{F}$ is deduced from that of a radial slit disc mapping. Indeed the radial slit disc mapping ψ with $\lim_{z\to\infty}\psi(z)/z=1$ is a unique extremal mapping minimizing radii within the family of univalent functions ϕ satisfying $\lim_{z\to\infty}\phi(z)/z=1$ and that $\phi(K_1)$ is a disc centered at the origin. It is not difficult to derive the uniqueness directly by means of the method of extremal metrics.

3. Subadditivity. We prove

THEOREM 2. *If* K_1 *and* K_2 *are mutually disjoint continua, then the inequality* (2) *holds.*

Proof. If $\Omega(K_j)\subset\Omega(K_k)$ $(j\neq k)$, the inequality is trivial. Otherwise $\Omega(K_1\cup K_2)$ forms a doubly connected domain. For simplicity's sake we write K_1, K_2 for its boundary components which have the same analytic capacities as the original continua. Let ϕ_0 be the extremal mapping for this domain in Theorem 1. Pommerenke showed that for a compact set on a straight line its analytic capacity is equal to a quater of its total length [4]. Since the analytic capacity is invariant under a conformal mapping with the normalization (4), we have

$$\gamma(K_1\cup K_2)=\gamma(\phi_0(K_1))+\gamma(\phi_0(K_2)) .$$

Since the analytic capacity is equal to the capacity for a continuum [7], we obtain the assertion from the minimum property in Theorem 1.

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DEPARTMENT OF MATHEMATICS
TOKYO INSTITUTE OF TECHNOLOGY