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KODAI MATH. J.
7 (1984), 1—15

THE DISTRIBUTION OF PICARD DIMENSIONS

Dedicated to Professor Mitsuru Ozawa on the occasion of his 60th birthday

BY MITSURU NAKAI AND TOSHIMASA TADA

The purpose of this paper is to show that Picard dimensions of densities on the punctured unit disk cover all countable cardinal numbers as well as the cardinal number of continuum.

Before stating our result more precisely we first fix terminologies. We denote by Ω the unit punctured disk $0<|z|<1$ which is viewed as an end of the punctured sphere $0<|z|\leqq\infty$ so that the unit circle $|z|=1$ is the relative boundary $\partial\Omega$ of Ω and the origin $z=0$ is the ideal boundary $\delta\Omega$ of Ω. By a *density* P on Ω we mean a nonnegative locally Hölder continuous function $P(z)$ on $\bar{\Omega}=\Omega\cup\partial\Omega$ so that P may or may not have singularity at $z=0$. With a density P on Ω we associate the class $PP(\Omega;\partial\Omega)$ of nonnegative continuous functions u on $\bar{\Omega}$ such that u satisfies the following elliptic equation

$$4\frac{\partial^2}{\partial z\partial\bar{z}}u(z)=P(z)u(z) \tag{1}$$

on Ω and vanishes on $\partial\Omega$. We also denote by $PP_1(\Omega;\partial\Omega)$ the subclass of $PP(\Omega;\partial\Omega)$ consisting of functions u with the following normalization

$$-\int_{\partial\Omega}\frac{\partial}{\partial|z|}u(z)|dz|=2\pi\,. \tag{2}$$

The Choquet theorem (cf. e.g. Phelps [5]) yields that there exists a bijective correspondence $u\longleftrightarrow\mu$ between $PP_1(\Omega;\partial\Omega)$ and the set of probability measures μ on the set $\mathrm{ex}.PP_1(\Omega;\partial\Omega)$ of extremal points of the convex set $PP_1(\Omega;\partial\Omega)$ such that

$$u=\int_{\mathrm{ex}.PP_1(\Omega;\partial\Omega)}v\,d\mu(v)\,.$$

Thus the set $\mathrm{ex}.PP_1(\Omega;\partial\Omega)$ is essential for the class $PP_1(\Omega;\partial\Omega)$, and the cardinal number $\#(\mathrm{ex}.PP_1(\Omega;\partial\Omega))$ of the set $\mathrm{ex}.PP_1(\Omega;\partial\Omega)$ is referred to as the *Picard dimension* of a density P at the ideal boundary $\delta\Omega$ of Ω, $\dim P$ in notation, i.e.

The main result of this paper was announced in [4]. To complete the present work both of the authors were supported in part by Grant-in-Aid for Scientific Research, No. 434005, Japanese Ministry of Education, Science and Culture.

Received February 8, 1983

$$\dim P = \sharp(\mathrm{ex}.\, PP_1(\Omega\,;\,\partial\Omega))\,. \tag{3}$$

We are interested in the range $\dim \mathscr{D}$ of the mapping $P \to \dim P$ from the totality $\mathscr{D}$ of densities P on Ω to the set of cardinal numbers. It is easily seen (cf. e.g. [3]) that $\dim P \geqq 1$ for any density P on Ω. Since $PP_1(\Omega\,;\,\partial\Omega)$ is a subset of the space $C(\Omega)$ of real valued continuous functions on Ω which is separable, the well known fact $\sharp(C(\Omega)) = \mathfrak{c}$ implies that $\dim P \leqq \mathfrak{c}$, where $\mathfrak{c}$ is the cardinal number of continuum. Therefore we have

$$1 \leqq \dim P \leqq \mathfrak{c} \quad (\text{i.e. } \dim \mathscr{D} \subset [1,\ \mathfrak{c}]) \tag{4}$$

for every density P on Ω, where $[1,\ \mathfrak{c}]$ is the interval consisting of cardinal numbers $\mathfrak{m}$ such that $1 \leqq \mathfrak{m} \leqq \mathfrak{c}$. We denote by $\boldsymbol{N}$ the set of positive integers and by $\mathfrak{a}$ the countably infinite cardinal number $\sharp\boldsymbol{N}$ and set $\Xi = \boldsymbol{N} \cup \{\mathfrak{a},\ \mathfrak{c}\}$. The primary purpose of this paper is to prove the following result:

THE MAIN THEOREM. *There exists a density* $P_{\mathfrak{m}}$ *on* Ω *for any cardinal number* $\mathfrak{m}$ *in* Ξ *such that* $\dim P_{\mathfrak{m}} = \mathfrak{m}$.

Therefore we have $\Xi \subset \dim \mathscr{D} \subset [1,\ \mathfrak{c}]$ so that $\dim \mathscr{D} = \Xi$ if we assume the continuum hypothesis $\Xi = [1,\ \mathfrak{c}]$. The proof is divided into two parts: the existence of canonically associated densities discussed in nos. 1-6, and three examples of relative harmonic dimensions considered in nos. 8-14. The deduction of the main theorem from the above two parts is given in no. 7.

§ 1. Canonically associated densities.

1. A sequence $\{K_n\}_1^\infty$ of continua K_n possibly empty in Ω will be referred to as a $\mathscr{K}$*-sequence* in Ω if $K_n \cap K_m = \emptyset$ $(n \neq m)$, $W = \Omega - \bigcup_1^\infty K_n$ is connected, and $\{K_n\}$ converges to $\delta\Omega : z = 0$, i.e. there exist only a finite number of K_n such that $K_n \cap \{\varepsilon \leqq |z| < 1\} \neq \emptyset$ for every $\varepsilon > 0$. We denote by $\mathscr{K}(\Omega)$ the set of $\mathscr{K}$-sequences in Ω. The relative boundary ∂W of the region $W = \Omega - \bigcup_1^\infty K_n$ for a $\mathscr{K}$-sequence $\{K_n\}_1^\infty$ is $\partial W = (\partial\Omega) \cup (\bigcup_1^\infty \partial K_n)$. We then consider the class $HP(W\,;\,\partial W)$ of nonnegative harmonic functions on W with vanishing boundary values on ∂W and the subclass $HP_1(W\,;\,\partial W)$ of $HP(W\,;\,\partial W)$ consisting of those functions u with the normalization (2). Similarly to the Picard dimension we define the *relative harmonic dimension*, $\dim\{K_n\}$ in notation, of a $\mathscr{K}$-sequence $\{K_n\}$ at $\delta\Omega : z = 0$ by

$$\dim\{K_n\} = \sharp(\mathrm{ex}.\, HP_1(W\,;\,\partial W))\,. \tag{5}$$

It is easy to see that, as in the case of Picard dimensions, $1 \leqq \dim\{K_n\} \leqq \mathfrak{c}$ for any $\mathscr{K}$-sequence $\{K_n\}$ in Ω. We will see in § 2 that the range $\dim \mathscr{K}(\Omega)$ of the mapping $\dim : \mathscr{K}(\Omega) \to \{\text{cardinal numbers}\}$ also contains Ξ.

2. Suppose that each continuum $\overline{Y}_n$ in a $\mathscr{K}$-sequence $\{\overline{Y}_n\}$ in Ω is the closure of a Jordan region Y_n in Ω $(n = 1, 2, \cdots)$. Such a $\mathscr{K}$-sequence will be referred to as a *$\mathcal{Y}$-sequence* in Ω and we denote by $\mathcal{Y}(\Omega)$ the class of $\mathcal{Y}$-sequences in Ω so that $\mathcal{Y}(\Omega)\subset\mathcal{K}(\Omega)$. Consider the region $W=\Omega-\bigcup_1^\infty \overline{Y}_n$ for a $\mathcal{Y}$-sequence $\{\overline{Y}_n\}$ and a density P on Ω such that $\text{supp}.P\subset\bigcup_1^\infty \overline{Y}_n=\Omega-W$. We denote by H_u^W for each u in $PP(\Omega;\partial\Omega)$ the least nonnegative harmonic function on W with boundary values u on ∂W (cf. e.g. Constantinescu-Cornea [1]). It is the lower envelope of the family of superharmonic functions s on W with the lower limit boundary values of s on ∂W being not less than $u|\partial W$. Then the function $T_P u=u-H_u^W$ belongs to the class $HP(W;\partial W)$ for every u in $PP(\Omega;\partial\Omega)$, and $u\mapsto T_P u$ defines a mapping $T_P: PP(\Omega;\partial\Omega)\to HP(W;\partial W)$. It is easy to see that the mapping T_P is order preserving (i.e. $u_1\leqq u_2$ implies $T_P u_1\leqq T_P u_2$), positively homogeneous (i.e. $T_P(\lambda u)=\lambda T_P(u)$ for nonnegative real numbers λ), and additive (i.e. $T_P(u_1+u_2)=T_P u_1+T_P u_2$). In general T_P may or may not be injective and similarly surjective. If the mapping T_P happens to be bijective, then the density P is said to be *canonically associated* with the $\mathcal{Y}$-sequence $\{\overline{Y}_n\}$. If a density P on Ω is canonically associated with a $\mathcal{Y}$-sequence $\{\overline{Y}_n\}$, then we have

$$\dim P=\dim\{\overline{Y}_n\}. \tag{6}$$

To prove this we denote by $l(u)$ the left hand side of (2). Then it is easy to see that $u\mapsto(2\pi/l(T_P u))T_P u$ is a bijective mapping of $PP_1(\Omega;\partial\Omega)$ onto $HP_1(W;\partial W)$ along with T_P. We now prove the following

THEOREM. *There always exists a density P on Ω canonically associated with an arbitrarily given $\mathcal{Y}$-sequence $\{\overline{Y}_n\}$ in Ω.*

The proof of this assertion will be given in nos. 4-6 after establishing an auxiliary result in no. 3.

3. We denote by P_f^U the solution of (1) on the unit disk $U:|z|<1$ with boundary values f on ∂U, where P is a density on $\overline{U}$ and f is in $C(\partial U)$. Give any Jordan region V with $\overline{V}\subset U$ and any positive number ε. We then have the following simple but very useful fact (cf. [2]):

PROPOSITION. *There exists a density $P=P_{V,\varepsilon}$ on $\overline{U}$ with* supp.$P\subset V$ *and satisfying the inequality*

$$\sup_V |P_f^U|\leqq\frac{\varepsilon}{2\pi}\int_0^{2\pi}|f(e^{i\theta})|\,d\theta \tag{7}$$

for any f in $C(\partial U)$.

For a proof of this we fix a disk $X:|z|<r$ $(0<r<1)$ with $\overline{V}\subset X$. Since the Poisson kernel $P(e^{i\theta},z)$ on $\partial U\times U$ is continuous on $\partial U\times\overline{X}$, there exists the maximum c of $P(e^{i\theta},z)$ on $\partial U\times\overline{X}$:

$$c=\max_{\partial U\times\overline{X}} P(e^{i\theta},z).$$

Let Y be an analytic Jordan region with $\bar{Y}\subset V$ and ω be the harmonic function on $X-\bar{Y}$ with boundary values c on ∂X, $\varepsilon/2$ on ∂Y. Taking Y enough close to V we can assume $\omega<\varepsilon$ on ∂V. Fix an analytic Jordan region Z and a conformal mapping $\zeta=\phi(z): \bar{Z}\to\phi(\bar{Z})=\{|\zeta|\leqq 1\}$ such that $\bar{Y}\subset Z$, $\bar{Z}\subset V$, and $\phi(Y)$ is a disk with center at $\zeta=0$. Consider the density ψ_m and the function v_m on $\phi(\bar{Z})$ defined by

$$\psi_m(\zeta)=\frac{4m^2|\zeta|^{2m-2}}{|\zeta|^{2m}+(2m)^{-1}}, \quad v_m(\zeta)=\frac{|\zeta|^{2m}+(2m)^{-1}}{1+(2m)^{-1}} \quad (m=1, 2, \cdots)$$

and observe that

$$4\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}v_m(\zeta)=\psi_m(\zeta)v_m(\zeta) \quad (|\zeta|<1).$$

If we set

$$Q_m(z)=\psi_m(\phi(z))|\phi'(z)|^2, \quad w_m(z)=v_m(\phi(z))$$

the function w_m on Z has boundary values 1 on ∂Z and satisfies

$$4\frac{\partial^2}{\partial z\partial\bar{z}}w_m(z)=Q_m(z)w_m(z)$$

on Z. We take a density P_m on $\bar{U}$ with supp. $P_m\subset V$, $P_m\geqq Q_m$ on $\bar{Z}$ and $P_m\leqq P_{m+1}$ on $\bar{U}$ $(m=1, 2, \cdots)$. Since $P_m\geqq Q_m$ on $\bar{Z}$, the solution u_m of

$$4\frac{\partial^2}{\partial z\partial\bar{z}}u(z)=P_m(z)u(z)$$

on X with boundary values c on ∂X satisfies $u_m\leqq cw_m$ on $\bar{Z}$. Observe that $\{w_m\}$ converges to 0 uniformly on $\bar{Y}$ as $m\to\infty$. Then for some positive integer m_ε we have $u_{m_\varepsilon}<\varepsilon/2$ on ∂Y so that $u_{m_\varepsilon}\leqq\omega$ on $X-\bar{Y}$. Therefore $u_{m_\varepsilon}<\varepsilon$ on ∂V, and hence on $\bar{V}$. Now we set

$$P=P_{V,\varepsilon}=P_{m_\varepsilon}, \quad v=u_{m_\varepsilon}.$$

We denote by $g(\zeta, z)$, $G(\zeta, z)$ the harmonic Green's function, P-Green's function on U, respectively. We remark that $G(\zeta, z)\leqq g(\zeta, z)$ and

$$P(e^{i\theta}, z)=\left[-\frac{\partial}{\partial|\zeta|}g(\zeta, z)\right]_{\zeta=e^{i\theta}}$$

for $e^{i\theta}$ in ∂U and z in U. If we set

$$K(e^{i\theta}, z)=\left[-\frac{\partial}{\partial|\zeta|}G(\zeta, z)\right]_{\zeta=e^{i\theta}},$$

then we have $0<K(e^{i\theta}, z)\leqq P(e^{i\theta}, z)$. Since $P(e^{i\theta}, z)\leqq c$ on $\partial U\times\bar{X}$ $v(z)=c$ on ∂X we have $K(e^{i\theta}, z)\leqq v(z)$ on X so that $K(e^{i\theta}, z)<\varepsilon$ on $\partial U\times\bar{V}$. Thus we have for f in $C(\partial U)$ and z in $\bar{V}$

$$
\begin{aligned}
|P_f^U(z)| &= \left|\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})K(e^{i\theta}, z)d\theta\right| \\
&\leqq \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|K(e^{i\theta}, z)d\theta \\
&\leqq \frac{\varepsilon}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|d\theta . \qquad \square
\end{aligned}
$$

4. We proceed to the proof of Theorem in no. 2, i.e. the existence of canonically associated densities. Let $\{\overline{Y}_n\}$ be any $\mathcal{Y}$-sequence on Ω and U_n be a slightly larger analytic Jordan region in Ω than Y_n containing $\overline{Y}_n$. We may assume $\overline{U}_n \cap \overline{U}_m = \emptyset$ for $n \neq m$. We fix a point z_1 in $\Omega - \bigcup_1^\infty \overline{U}_n$ and denote by F the set of nonnegative harmonic functions u on $W = \Omega - \bigcup_1^\infty \overline{Y}_n$ with $u(z_1)=1$. Then the Harnack principle yields for every nonnegative integer n

$$b_n \equiv \sup_F \max_{\partial U_n} h < +\infty .$$

Using a conformal mapping ϕ_n of $\overline{U}_n$ to $\phi_n(\overline{U}_n) = \{|z| \leqq 1\}$ we define a density $P = P_W$ on Ω by

$$P(\zeta) = \begin{cases} P_n(\phi_n(\zeta)) & (\zeta \in \overline{U}_n\,;\ n=1, 2, \cdots) \\ 0 & (\zeta \in \Omega - \bigcup_1^\infty \overline{U}_n) \end{cases}$$

where P_n is a density on $\overline{U} = \{|z|<1\}$ which satisfies (7) for $V = \phi_n(Y_n)$, $\varepsilon = 1/n(b_n+1)$. We will show that P is a canonically associated density with $\{\overline{Y}_n\}$.

5. First we prove that the mapping T_P is injective. Let u_1, u_2 be functions in $PP(\Omega\,;\partial\Omega)$ with $T_P u_1 = T_P u_2$. Then we have $u_1 - u_2 = H_{u_1-u_2}^W$ on W. Assume that $u_1 \not\equiv u_2$. Then we may assume $\sup_W(u_1-u_2)>0$ if necessary by exchanging indices of u_1, u_2. Since $H_{u_1-u_2}^W$ is quasibounded we have $\sup_W H_{u_1-u_2}^W = \sup_{\partial W} H_{u_1-u_2}^W$. We set $v_k = u_k/(u_k(z_1)+1)$ $(k=1, 2)$. Then v_k satisfies $v_k \leqq b_n$ on ∂U_n so that we have for z in $\overline{Y}_n$

$$v_k(z) \leqq \frac{1}{2\pi n(b_n+1)}\int_0^{2\pi} v_k(\phi_n^{-1}(e^{i\theta}))d\theta \leqq \frac{1}{n}.$$

Then we have $H_{u_1-u_2}^W = u_1 - u_2 \leqq (u_1(z_1)+u_2(z_1)+2)/n$ on ∂Y_n, and hence the harmonic function $H_{u_1-u_2}^W$ on W attains its supremum at a point in some ∂Y_n. Therefore the function $u_1 - u_2$ in $PP(\Omega\,;\partial\Omega)$ attains its maximum at the same point as that of $H_{u_1-u_2}^W$ so that u_1-u_2 is identically a positive constant. This contradicts the fact that $u_1 - u_2 \leqq (u_1(z_1)+u_2(z_1)+2)/n$ on $\overline{Y}_n$ for every positive integer n. Thus $u_1 \equiv u_2$.

6. Next we prove that T_P is surjective. Let h be any function in $HP(W\,;\partial W)$. Since T_P is positively homogeneous we may assume that h is in F. We denote by w the harmonic measure of $\partial W - \partial\Omega$ considered on W and we set $h_1 = h + w$

on W. Observe that for z in ∂Y_n

$$P_{h_1}^{U_n}(z) \leqq \frac{1}{2\pi n(b_n+1)} \int_0^{2\pi} h_1(\phi_n^{-1}(e^{i\theta}))d\theta$$

$$\leqq \frac{b_n+1}{n(b_n+1)} \leqq 1 = h_1(z).$$

Then we have $P_{h_1}^{U_n} \leqq h_1$ on $U_n - \overline{Y}_n$ by the maximum principle. Therefore the function s on Ω defined by $s = h_1$ on $\Omega - \bigcup_1^\infty U_n$, $s = P_{h_1}^{U_n}$ on U_n ($n=1, 2, \cdots$) is a supersolution of (1) (i.e. superharmonic with respect to (1)) on Ω with $h \leqq s \leqq h_1$ on W. On the other hand h is a subsolution of (1) (i.e. subharmonic with respect to (1)) on Ω if we define $h=0$ on $\bigcup_1^\infty Y_n$. Now the lower envelope u of the family of supersolutions of (1) on Ω which dominate h on Ω is a solution of (1) with $h \leqq u$ on Ω, $u \leqq h_1$ on W so that u is in $PP(\Omega; \partial\Omega)$. If we set $v = -h + u - H_u^W$ on W, v satisfies $v \leqq -h + u \leqq -h + h_1 = w \leqq 1$, $v \geqq -H_u^W \geqq -H_{h_1}^W = -w \geqq -1$. On the other hand v vanishes on ∂W and therefore $v \equiv 0$, i.e. $T_P u = h$. This completes the proof of Theorem in no. 2. □

§ 2. Relative harmonic dimension.

7. In view of (6) and the theorem in no. 2, the proof of the main theorem is reduced to showing that $\dim \mathcal{Y}(\Omega)$ contains Ξ. In passing we remark that $\mathcal{Y}(\Omega) \subset \mathcal{K}(\Omega)$ implies that $\dim \mathcal{K}(\Omega)$ contains Ξ along with $\dim \mathcal{Y}(\Omega)$. Thus the proof of the main theorem will be complete if we show the following fact which may have an independent interest in its own right:

THEOREM. *There exists a $\mathcal{Y}$-sequence $\{\overline{Y}_n\}$ in Ω for any cardinal number $\mathfrak{m}$ in Ξ such that* $\dim\{\overline{Y}_n\} = \mathfrak{m}$.

Therefore we have $\Xi \subset \dim \mathcal{Y}(\Omega) \subset \dim \mathcal{K}(\Omega) \subset [1, \mathfrak{c}]$ so that $\dim \mathcal{Y}(\Omega) = \dim \mathcal{K}(\Omega) = \Xi$ if we assume the continuum hypothesis $\Xi = [1, \mathfrak{c}]$. The proof will be given in nos. 8-14 by exhibiting three examples in nos. 9, 12, and 14.

8. Before proceeding to our three examples we remark the following simple fact (cf. e.g. Constantinescu-Cornea [1]) which plays an important role in verifying that the examples in nos. 9, 12, and 14 are required ones. Let $\{U_n\}_1^\infty$ be a sequence of relatively compact subregion U_n in Ω such that $\overline{U}_n \cap \overline{U}_m = \emptyset$ ($n \neq m$) and $\{\overline{U}_n\}$ converges to $\delta\Omega: z=0$. Consider a region $W = \Omega - \bigcup_1^\infty K_n$ for a $\mathcal{K}$-sequence $\{K_n\}$ in Ω. Set $V_n = W \cap U_n$ and $V = \bigcup_1^\infty V_n$. Let δW, $\delta_1 W$ be the Martin boundary of W over $\delta\Omega: z=0$, the set of minimal points in δW, respectively and $(W-V)^a$ the closure of $W-V$ considered in the Martin compactification of W. Then

$$\delta W - (W-V)^a \subset \delta W - \delta_1 W. \tag{8}$$

9. EXAMPLE 1. First we exhibit an example of a $\mathcal{Y}$-sequence $\{Y_n\}_1^\infty$ with $\dim\{\overline{Y}_n\}=m$ for any given positive integer m. Fix a sequence $\{a_n\}_1^\infty$ in (0, 1) with $a_{n+1}<a_n$ $(n=1, 2, \cdots)$ and $\lim a_n=0$. Fix a positive numbers $\theta_1, \cdots, \theta_m$; $\eta_1, \cdots, \eta_m$ with $\theta_\mu+2\eta_\mu<-2\eta_{\mu+1}+\theta_{\mu+1}$ $(\mu=0, 1, \cdots, m)$, where $\theta_0+2\eta_0=0$, $-2\eta_{m+1}+\theta_{m+1}=2\pi$. We choose a sequence $\{b_n\}_1^\infty$ in (0, 1) with $a_{n+1}<b_n<a_n$. Let (see Fig. 1)

$$S_{n\mu}=\{b_n<|z|<a_n,\ |\arg z-\theta_\mu|<\eta_\mu\}\quad (\mu=1, \cdots, m;\ n=1, 2, \cdots).$$

Observe that any positive integer k has a unique expression $k=(n-1)m+\mu$ with positive integers n and μ with $1\leqq\mu\leqq m$. We set

$$Y_k=S_{n\mu}\quad (k=(n-1)m+\mu).$$

Then the sequence $\{\overline{Y}_k\}_1^\infty=\{\overline{S}_{n\mu}\}$ is clearly a $\mathcal{Y}$-sequence. If we choose the sequence $\{b_n\}$ so as to make the sequence $\{b_n-a_{n+1}\}_1^\infty$ converges to zero enough rapidly, i.e. satisfying (9), (11), and (14) below then we can show that $\dim\{\overline{S}_{n\mu}\}$ $=m$ in the following way.

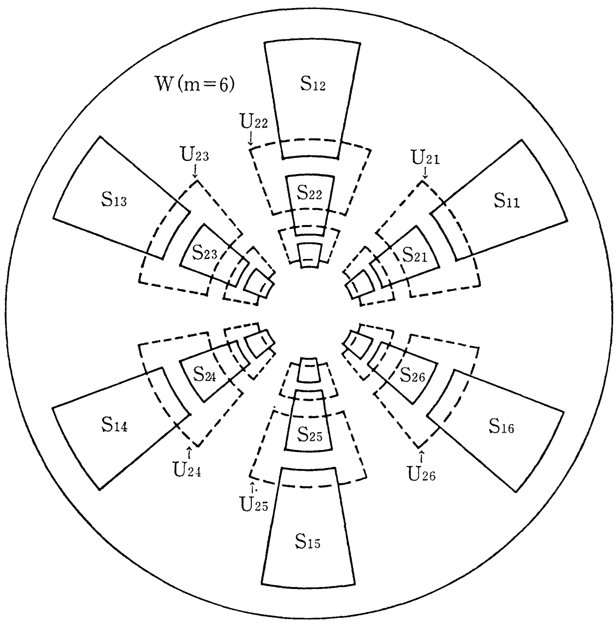

Fig. 1.

10. Fix a sequence $\{\delta_n\}_2^\infty$ in (0, 1) with $a_{n+1}+\delta_{n+1}<a_n-\delta_n$ $(n=1, 2, \cdots)$, where $\delta_1=0$. We set (see Fig. 1)

$$U_{n\mu}=\{||z|-a_n|<\delta_n,\ |\arg z-\theta_\mu|<2\eta_\mu\}\quad (\mu=1, \cdots, m;\ n=2, 3, \cdots)$$

and $U=\bigcup_{n=2}^{\infty}\bigcup_{\mu=1}^{m}U_{n\mu}$. Then we have $\bar{U}_{n\mu}\cap\bar{U}_{k\nu}=\emptyset$ $((n, \mu)\neq(k, \nu))$. The first property which $\{b_n\}$ has to satisfy is

$$b_n<a_{n+1}+\delta_{n+1} \quad (n=1, 2, \cdots). \tag{9}$$

We consider subregions W and W_n $(n=1, 2, \cdots)$ of Ω given by

$$W=\Omega-\bigcup_{n=1}^{\infty}\bigcup_{\mu=1}^{m}\bar{S}_{n\mu},$$

$$W_n=W-\bigcup_{\mu=1}^{m}\{0<|z|\leqq a_n, \ |\arg z-\theta_\mu|\leqq\eta_\mu\}$$

and denote by $g(\zeta, z)$, $g_n(\zeta, z)$ the Green's functions on W, W_n, respectively. Fix a reference point a of the Martin kernels $k(\zeta, z)=g(\zeta, z)/g(\zeta, a)$, $k_n(\zeta, z)=g_n(\zeta, z)/g_n(\zeta, a)$ on W, W_n, respectively and a neighbourhood D of a with $\bar{D}\subset W-\bar{U}$. Finally we fix a sequence $\{\varepsilon_n\}_1^{\infty}$ in $(0, 1)$ with

$$\sum_{n=1}^{\infty}\varepsilon_n<1, \quad \prod_{n=1}^{\infty}(1-\varepsilon_n)\geqq\frac{1}{2}. \tag{10}$$

Then the second property which $\{b_n\}$ has to satisfy is

$$g_{n+1}(\zeta, z)-g_n(\zeta, z)\leqq\varepsilon_n g_n(\zeta, z) \tag{11}$$

for any z in D and ζ in $W-U$. Assume (11) is valid for every positive integer n. Then we have

$$g_{n+j}(\zeta, z)-g_n(\zeta, z)\leqq\alpha_n g_{n+j}(\zeta, z)$$

so that

$$\begin{cases} g_n(\zeta, z)\leqq g_{n+j}(\zeta, z)\leqq(1-\alpha_n)^{-1}g_n(\zeta, z) \\ g_n(\zeta, z)\leqq g(\zeta, z)\leqq(1-\alpha_n)^{-1}g_n(\zeta, z) \end{cases}$$

for every positive integer j, where $\alpha_n=\sum_{k=n}^{\infty}\varepsilon_k$. Therefore Martin kernels k, k_n satisfy on $(W-U)\times D$

$$(1-\alpha_n)k_n(\zeta, z)\leqq k_{n+j}(\zeta, z)\leqq(1-\alpha_n)^{-1}k_n(\zeta, z), \tag{12}$$

$$(1-\alpha_n)k_n(\zeta, z)\leqq k(\zeta, z)\leqq(1-\alpha_n)^{-1}k_n(\zeta, z). \tag{13}$$

Let δW_n and $\delta_1 W_n$ be the Martin boundary of W_n over $\delta\Omega$: $z=0$ and the set of minimal points in δW_n, respectively. Then we have $\delta W_n=\delta_1 W_n=\delta W_1=\delta_1 W_1=\{p_1, \cdots, p_m\}$ and we may assume for every positive integer n, μ, ν with $1\leqq\mu$, $\nu\leqq m$

$$\lim_{\Gamma_\mu\ni z\to 0}k_n(p_\nu, z)=\begin{cases} +\infty & (\mu=\nu) \\ 0 & (\mu\neq\nu) \end{cases},$$

where $\Gamma_\mu=\{0<|z|<1, \ \arg z=(\theta_\mu+2\eta_\mu+\theta_{\mu+1}-2\eta_{\mu+1})/2\}$ and $\theta_{m+1}-2\eta_{m+1}=\theta_1-2\eta_1$. Making $\zeta\to p_\mu$ in (12) we have

$$(1-\alpha_n)k_n(p_\mu, z) \leqq k_{n+j}(p_\mu, z) \leqq (1-\alpha_n)^{-1}k_n(p_\mu, z)$$

for every positive integer μ with $1 \leqq \mu \leqq m$. Then the following limit exists for any z in D: $A_\mu(z) = \lim_{n\to\infty} k_n(p_\mu, z)$, and hence $\{k_n(p_\mu, z)\}_{n=1}^\infty$ converges to a nonnegative harmonic function A_μ on W uniformly on every compact subset of W. On the other hand from (13) it follows that harmonic functions

$$\overline{A}_\mu(z) = \overline{\lim_{\zeta\to p_\mu}}\, k(\zeta, z), \quad \underline{A}_\mu(z) = \underline{\lim}_{\zeta\to p_\mu}\, k(\zeta, z)$$

on W satisfy

$$(1-\alpha_n)k_n(p_\mu, z) \leqq \underline{A}_\mu(z) \leqq \overline{A}_\mu(z) \leqq (1-\alpha_n)^{-1}k_n(p_\mu, z)$$

on D. By making $n\to\infty$ in the above inequalities we have $A_\mu(z) = \underline{A}_\mu(z) = \overline{A}_\mu(z)$ on D, and hence on W. Then every p_μ defines unique Martin boundary point q_μ of W over $\partial\Omega$: $z=0$ such that

$$\lim_{n\to\infty} k_n(p_\mu, z) = \lim_{\zeta\to p_\mu} k(\zeta, z) = k(q_\mu, z)$$

on W. We remark that it may happen $q_\mu = q_\nu$ for some μ, ν with $\mu \neq \nu$. Let q be any point in $(W-U)^a \cap \partial W$. Then there exists a sequence $\{\zeta_n\}_1^\infty$ in $W-U$ with $\zeta_n \to q$. Since a subsequence $\{\zeta_n'\}_1^\infty$ of $\{\zeta_n\}$ converges to a point p_μ in $\partial\Omega_1$ we have $q = q_\mu$ so that $\partial W - (W-U)^a$ contains $\partial W - \{q_\mu\}_1^m$. Therefore by (8) $\partial W - \delta_1 W$ contains $\partial W - \{q_\mu\}_1^m$, and hence we have $\delta_1 W \subset \{q_\mu\}_1^m$. Thus we conclude that $\dim\{\bar{S}_{n\mu}\} \leqq m$.

11. Now we give the last property which $\{b_n\}$ has to satisfy. Consider the harmonic function $u_{n\mu}$ on W_{n-1} with boundary values $k_n(p_\mu, z)$ on ∂W_{n-1} ($\mu = 1, \cdots, m$; $n = 2, 3, \cdots$). We require $\{b_n\}$ to satisfy

$$u_{n+1,\mu}(z) \leqq \varepsilon_n \tag{14}$$

on $W-U$ for every μ. Since the nonnegative harmonic function $k_{n+1}(p_\mu, z) - u_{n+1,\mu}(z)$ vanishes on ∂W_n it is represented by $k_n(p_\mu, z)$:

$$k_{n+1}(p_\mu, z) - u_{n+1,\mu}(z) = (1-u_{n+1,\mu}(a))k_n(p_\mu, z).$$

Assume (14) is valid for every positive integer n. Then if we set $v_{1\mu}(z) = k_1(p_\mu, z)$ and

$$v_{n\mu}(z) = \Big(\prod_{k=2}^n (1-u_{k\mu}(a))\Big)^{-1} k_n(p_\mu, z) \quad (\mu = 1, \cdots, m;\ n = 2, 3, \cdots),$$

by (10) and (14) we have

$$|v_{n+1,\mu}(z) - v_{n\mu}(z)| = \Big(\prod_{k=2}^{n+1}(1-u_{k\mu}(a))\Big)^{-1} |k_{n+1}(p_\mu, z) - (1-u_{n+1,\mu}(a))k_n(p_\mu, z)|$$

$$\leqq \Big(\prod_{k=1}^\infty (1-\varepsilon_k)\Big)^{-1} \varepsilon_n \leqq 2\varepsilon_n$$

on $W-U$. Therefore $\{v_{n\mu}\}_{n=1}^{\infty}$ converges to a nonnegative harmonic function v_μ in $HP(W;\partial W)$ uniformly on every compact subset of W. Since we have

$$|v_{n+j,\mu}-v_{n\mu}|\leqq 2(\varepsilon_n+\cdots+\varepsilon_{n+j-1})$$

for every positive integer j, every function v_μ satisfies $v_{n\mu}-2\alpha_n\leqq v_\mu\leqq v_{n\mu}+2\alpha_n$ on $W-U$. Then v_μ has the same limit as that of $v_{n\mu}$ at $\delta\Omega$: $z=0$ along Γ_ν:

$$\lim_{\Gamma_\nu\ni z\to 0} v_\mu(z)=\begin{cases} +\infty & (\mu=\nu) \\ 0 & (\mu\neq\nu) \end{cases}.$$

Thus we have $\dim\{\bar{S}_{n\mu}\}\geqq m$. □

12. EXAMPLE 2. Next we exhibit an example of a $\mathcal{Y}$-sequence $\{\bar{Y}_n\}_1^\infty$ with $\dim\{\bar{Y}_n\}=\mathfrak{a}$. Fix a sequence $\{a_n\}_1^\infty$ in $(0, 1)$ with $a_{n+1}<a_n$ $(n=1, 2, \cdots)$ and $\lim a_n=0$. Fix sequences $\{\theta_\mu\}_1^\infty$, $\{\eta_\mu\}_1^\infty$ of positive numbers θ_μ, η_μ with $0<\theta_1-2\eta_1$, $\theta_\mu+2\eta_\mu<-2\eta_{\mu+1}+\theta_{\mu+1}<2\pi$ for every positive integer μ. We choose a sequence $\{b_n\}_1^\infty$ in $(0, 1)$ with $a_{n+1}<b_n<a_n$ $(n=1, 2, \cdots)$. Let

$$S_{n\mu}=\{b_n<|z|<a_n,\ |\arg z-\theta_\mu|<\eta_\mu\}\quad (\mu=1, \cdots, n;\ n=1, 2, \cdots).$$

Observe that any positive integer k has a unique expression $k=n(n-1)/2+\mu$ with positive integers n and μ satisfying $n\geqq\mu$. We set (see Fig. 2)

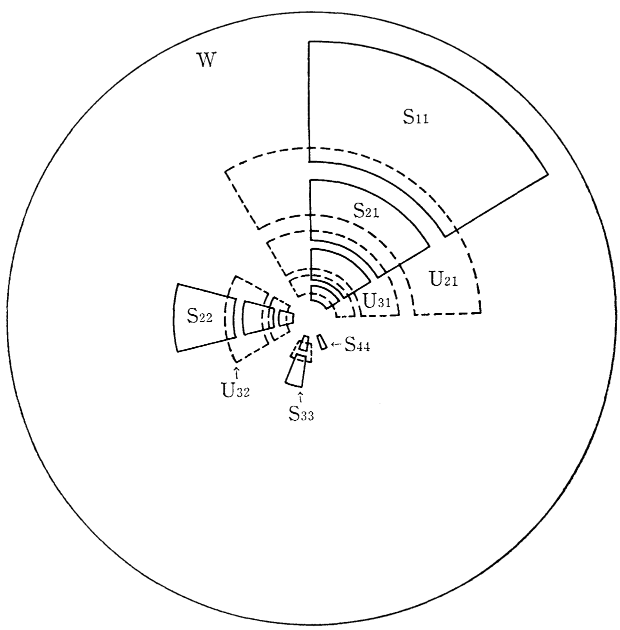

Fig. 2.

$$Y_k = S_{n\mu} \quad (k = n(n-1)/2 + \mu).$$

Then the sequence $\{\bar{Y}_k\}_1^\infty = \{\bar{S}_{n\mu}\}$ ($\mu = 1, \cdots, n$; $n = 1, 2, \cdots$) is a $\mathcal{Y}$-sequence. If we choose the sequence $\{b_n\}$ so as to make the sequence $\{b_n - a_{n+1}\}_1^\infty$ converges to zero enough rapidly, i.e. satisfying (15), (16), and (17) below then we can show that $\dim\{\bar{S}_{n\mu}\} = \mathfrak{a}$ in the similar way as in nos. 10 and 11.

13. Fix a sequence $\{\delta_n\}_2^\infty$ in $(0, 1)$ with $a_{n+1} + \delta_{n+1} < a_n - \delta_n$ ($n = 1, 2, \cdots$), where $\delta_1 = 0$. We set (see Fig. 2)

$$U_{n\mu} = \{||z| - a_n| < \delta_n,\ |\arg z - \theta_\mu| < 2\eta_\mu\} \quad (\mu = 1, \cdots, n-1;\ n = 2, 3, \cdots)$$

and $U = \bigcup_{n=2}^\infty \bigcup_{\mu=1}^{n-1} U_{n\mu}$. Then we have $\bar{U}_{n\mu} \cap \bar{U}_{k\nu} = \emptyset$ ($(n, \mu) \neq (k, \nu)$). The first property which $\{b_n\}$ has to satisfy is

$$(15) \qquad b_n < a_{n+1} + \delta_{n+1} \quad (n = 1, 2, \cdots).$$

We consider subregions W and W_n ($n = 1, 2, \cdots$) of Ω given by

$$W = \Omega - \bigcup_{n=1}^\infty \bigcup_{\mu=1}^n \bar{S}_{n\mu},$$

$$W_n = W - \bigcup_{\mu=1}^n \{0 < |z| \leqq a_n,\ |\arg z - \theta_\mu| \leqq \eta_\mu\}$$

$$- \bigcup_{\mu=n+1}^\infty \{0 < |z| \leqq a_\mu,\ |\arg z - \theta_\mu| \leqq \eta_\mu\}$$

and denote by $g(\zeta, z)$, $g_n(\zeta, z)$ the Green's functions on W, W_n, respectively. Fix a reference point a of the Martin kernels $k(\zeta, z) = g(\zeta, z)/g(\zeta, a)$, $k_n(\zeta, z) = g_n(\zeta, z)/g_n(\zeta, a)$ on W, W_n, respectively and a neighbourhood D of a with $\bar{D} \subset W - \bar{U}$. Finally we fix a sequence $\{\varepsilon_n\}_1^\infty$ in $(0, 1)$ with $\sum_1^\infty \varepsilon_n < 1$, $\prod_1^\infty (1 - \varepsilon_n) \geqq 1/2$. Then the second property which $\{b_n\}$ has to satisfy is

$$(16) \qquad g_{n+1}(\zeta, z) - g_n(\zeta, z) \leqq \varepsilon_n g_n(\zeta, z)$$

for any z in D and ζ in $W - U$. Let δW_n and $\delta_1 W_n$ be the Martin boundary over $\delta\Omega : z = 0$ and the set of minimal points in δW_n, respectively. Then we have $\delta W_n = \delta_1 W_n = \delta W_1 = \delta_1 W_1 = \{p_1, p_2, \cdots;$ and $p_\infty\}$ and we may choose a family $\{\Gamma_\mu : \mu = 1, 2, \cdots;$ and $\infty\}$ of pairwize disjoint curves Γ_μ in $W - \bar{U}$ converging to $\delta\Omega : z = 0$ such that

$$\lim_{\Gamma_\mu \ni z \to 0} k_n(p_\nu, z) = \begin{cases} +\infty & (\mu = \nu) \\ 0 & (\mu \neq \nu) \end{cases}$$

for every positive integer n. Assume (16) is valid for every n. Then $k_n(p_\mu, z)$ and $k(\zeta, z)$ converge to a same function in $HP(W; \partial W)$ uniformly on every compact subset of W as $n \to \infty$ and $\zeta \to p_\mu$, respectively for every $\mu = 1, 2, \cdots;$ and ∞. Therefore every p_μ defines unique Martin boundary point q_μ of W over $\delta\Omega$:

$z=0$ such that

$$\lim_{n\to\infty} k_n(p_\mu, z) = \lim_{\zeta\to p_\mu} k(\zeta, z) = k(q_\mu, z)$$

on W. We remark that it may happen $q_\mu = q_\nu$ for some μ, ν with $\mu \neq \nu$. We denote by ∂W and $\partial_1 W$ the Martin boundary of W over $\partial\Omega: z=0$ and the set of minimal points in ∂W, respectively. Observe that $\partial W-(W-U)^a$ contains $\partial W-\{q_1, q_2, \cdots; \text{ and } q_\infty\}$. Then by (8) $\partial W-\partial_1 W$ contains $\partial W-\{q_1, q_2, \cdots; \text{ and } q_\infty\}$ and hence $\partial_1 W \subset \{q_1, q_2, \cdots; \text{ and } q_\infty\}$. Thus we conclude that $\dim\{\bar{S}_{n\mu}\} \leqq \mathfrak{a}$.

We give the last property which $\{b_n\}$ has to satisfy. Consider the harmonic function $u_{n\mu}$ on W_{n-1} with boundary values $k_n(p_\mu, z)$ on ∂W_{n-1} for $n=2, 3, \cdots$ and $\mu=1, 2, \cdots$; and ∞. We requir $\{b_n\}$ to satisfy

$$u_{n+1,\mu}(z) \leqq \varepsilon_n \tag{17}$$

on $W-U$ for every μ. We set $v_{1\mu}(z)=k_1(p_\mu, z)$ and

$$v_{n\mu}(z)=\Big(\prod_{k=2}^{n}(1-u_{k\mu}(a))\Big)^{-1} k_n(p_\mu, z) \quad (n=2, 3, \cdots; \ \mu=1, 2, \cdots; \text{ and } \infty).$$

Assume (17) is valid. Then $v_{n\mu}$ converges to a function v_μ in $HP(W; \partial W)$ uniformly on every compact subset of W and v_μ has the same limit as that of $k_n(p_\mu, z)$ at $\partial\Omega: z=0$ along Γ_ν:

$$\lim_{\Gamma_\nu \ni z \to 0} v_\mu(z) = \begin{cases} +\infty & (\mu=\nu) \\ 0 & (\mu \neq \nu) \end{cases}$$

for μ, $\nu=1, 2, \cdots$; and ∞. Thus we have $\dim\{\bar{S}_{n\mu}\} \geqq \mathfrak{a}$. □

14. EXAMPLE 3. Finally we exhibit an example of a $\mathcal{Y}$-sequence $\{\bar{Y}_n\}_1^\infty$ with $\dim\{\bar{Y}_n\}=\mathfrak{c}$. Fix a sequence $\{a_n\}_1^\infty$ in $(0, 1)$ with $a_{n+1}<a_n$ $(n=1, 2, \cdots)$ and $\lim a_n=0$. Fix a sequence $\{\eta_\mu\}_0^\infty$ of positive numbers η_μ with

$$4\Big(\eta_0+\sum_{\mu=1}^{\infty} 2^{\mu-1}\eta_\mu\Big)<2\pi\,.$$

For every positive integer n we set $\eta_{n0}=\eta_0$ and for every positive integer μ with $\mu \leqq 2^n-1$ we set $\eta_{n\mu}=\eta_{n-j}$ if $\mu=k2^j$ for positive odd integer k. Then we define a sequence $\{\theta_{n\mu}\}$ $(0 \leqq \mu \leqq 2^n-1, \ n \geqq 1)$ in $[0, 2\pi)$ by induction: $\theta_{10}=0$, $\theta_{11}=\pi$; and in the case $n \geqq 2$ we set

$$\theta_{n\mu}=\theta_{n-1,\mu/2}\,,$$

if μ is a nonnegative even integer,

$$\theta_{n\mu}=\frac{1}{2}(\theta_{n-1,(\mu-1)/2}+2\eta_{n-1,(\mu-1)/2}$$

$$-2\eta_{n-1,(\mu+1)/2}+\theta_{n-1,(\mu+1)/2})\,,$$

if μ is a positive odd integer, where $\theta_{n\nu}=2\pi$, $\eta_{n\nu}=\eta_0$ if $\nu=2^n$. We choose a sequence $\{b_n\}_1^\infty$ in (0, 1) with $a_{n+1}<b_n<a_n$ $(n=1, 2, \cdots)$. Let (see Fig. 3)

$$S_{n\mu}=\{b_n<|z|<a_n,\ |\arg z-\theta_{n\mu}|<\eta_{n\mu}\}\quad (0\leqq\mu\leqq 2^n-1,\ n\geqq 1).$$

Observe that any positive integer k has a unique expression $k=2^n+\mu-1$ with positive integer n and nonnegative integer μ satisfying $\mu\leqq 2^n-1$. We set

$$Y_k=S_{n\mu}\quad (k=2^n+\mu-1).$$

Then the sequence $\{\bar{Y}_n\}_1^\infty=\{\bar{S}_{n\mu}\}$ $(0\leqq\mu\leqq 2^n-1,\ n\geqq 1)$ is a $\mathcal{Y}$-sequence. If we choose the sequence $\{b_n\}$ so as to make the sequence $\{b_n-a_{n+1}\}_1^\infty$ converges to zero enough rapidly, i.e. satisfying (18) and (19) below then we can show that $\dim\{\bar{S}_{n\mu}\}=\mathfrak{c}$ in the similar way as in nos. 10 and 11.

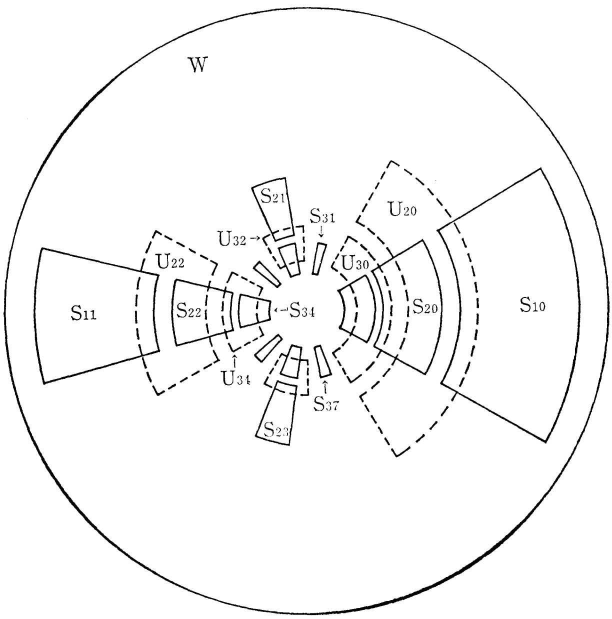

Fig. 3.

Fix a sequence $\{\delta_n\}_2^\infty$ in (0, 1) with $a_{n+1}+\delta_{n+1}<a_n-\delta_n$ $(n=1, 2, \cdots)$, where $\delta_1=0$. We set for every integer n with $n\geqq 2$ and every even integer μ with $0\leqq\mu\leqq 2^n-1$

$$U_{n\mu}=\{||z|-a_n|<\delta_n,\ |\arg z-\theta_{n\mu}|<2\eta_{n\mu}\}$$

and (see Fig. 3)

$$U=\bigcup_{n=2}^{\infty}\bigcup_{\nu=0}^{2^{n-1}-1}U_{n,2\nu}.$$

Then we have $\bar{U}_{n\mu}\cap\bar{U}_{k\nu}=\emptyset$ $((n,\mu)\neq(k,\nu))$. The first property which $\{b_n\}$ has to satisfy is

$$b_n < a_{n+1} + \delta_{n+1} \quad (n=1, 2, \cdots). \tag{18}$$

We consider subregions W and W_n $(n=1, 2, \cdots)$ of Ω given by

$$W = \Omega - \bigcup_{n=1}^{\infty} \bigcup_{\mu=0}^{2n-1} \bar{S}_{n\mu},$$

$$W_n = W - \bigcup_{k=1}^{n} \bigcup_{\mu=0}^{2k-1} \{0 < |z| \leqq a_n,\ |\arg z - \theta_{k\mu}| \leqq \eta_{k\mu}\}$$

$$- \bigcup_{k=n+1}^{\infty} \bigcup_{\mu=0}^{2k-1} \{0 < |z| \leqq a_k,\ |\arg z - \theta_{k\mu}| \leqq \eta_{k\mu}\}$$

and denote by $g(\zeta, z)$, $g_n(\zeta, z)$ the Green's functions on W, W_n, respectively. Fix a reference point a in $W - \bar{U}$ of the Martin kernels $k(\zeta, z) = g(\zeta, z)/g(\zeta, a)$, $k_n(\zeta, z) = g_n(\zeta, z)/g_n(\zeta, a)$ on W, W_n, respectively and let δW_n, $\delta_1 W_n$ be the Martin boundary of W_n over $\delta\Omega: z=0$, the set of minimal points in δW_n, respectively. If we set $\Lambda = \{0, 1\}^N$ then the cardinal number of Λ is $\mathfrak{c}$ and every point x in Λ defines unique minimal point p_x in $\delta_1 W_n = \delta_1 W_1$: we have $\delta W_n = \delta_1 W_n = \delta W_1 = \delta_1 W_1 = \{p_x : x \in \Lambda\}$ and we may choose a family $\{\Gamma_x : x \in \Lambda\}$ of pairwise disjoint curves Γ_x in $W - \bar{U}$ converging to $\delta\Omega: z=0$ such that

$$\lim_{\Gamma_x \ni z \to 0} k_n(p_y, z) = \begin{cases} +\infty & (x=y) \\ 0 & (x \neq y) \end{cases}$$

for every x, y in Λ and positive integer n. Consider the harmonic function u_{nx} on W_{n-1} with boundary values $k_n(p_x, z)$ on ∂W_{n-1} for every x in Λ and integer n with $n \geqq 2$. Finally we fix a sequence $\{\varepsilon_n\}_1^\infty$ in $(0, 1)$ with $\sum_1^\infty \varepsilon_n < 1$, $\prod_1^\infty (1-\varepsilon_n) \geqq 1/2$. Then the second property which $\{b_n\}$ has to satisfy is

$$u_{n+1, x}(z) \leqq \varepsilon_n \tag{19}$$

on $W - U$ for every x in Λ. We set $v_{1x} = k_1(p_x, z)$ and

$$v_{nx}(z) = \left(\prod_{k=2}^{n} (1 - u_{kx}(a)) \right)^{-1} k_n(p_x, z) \quad (x \in \Lambda,\ n \geqq 2).$$

Assume (19) is valid for every n. Then $\{v_{nx}\}$ converges to a function v_x in $HP(W; \partial W)$ uniformly on every compact subset of W and v_x has the same limit as that of $k_n(p_x, z)$ at $\delta\Omega: z=0$ along Γ_y:

$$\lim_{\Gamma_y \ni z \to 0} v_x(z) = \begin{cases} +\infty & (x=y) \\ 0 & (x \neq y) \end{cases}$$

for every y in Λ. Then we have $\dim\{\bar{S}_{n\mu}\} \geqq \mathfrak{c}$. Since the dimension of any $\mathcal{Y}$-sequence is at most $\mathfrak{c}$, we conclude that $\dim\{\bar{S}_{n\mu}\} = \mathfrak{c}$. □

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