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BASE POINTS OF POLAR CURVES ON A SURFACE OF TYPE $z^n = f(x, y)$

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Abstract

We give a method for characterizing base points of polar curves on a normal surface of type $z^n = f(x, y)$, after a resolution of singularities by Jung's method.

1. Introduction

Local polar varieties are collections of subspaces of a reduced equidimensional analytic germ. Roughly speaking a polar variety is the closure of the critical locus of a linear projection restricted to the non-singular locus of the germ. D. T. Lê and B. Teissier have introduced these objects and proved their relation with characteristic classes of singular spaces and different other objects occurring from the local study of singularities [6].

In the case of surfaces, the collection of polar varieties reduces to the one of polar curves. One of the various links between polar curves and the "nature" of the surface singularity occurs while studying the resolution of the surface by normalized Nash modifications.

Nash modification (or Nash blow-up) is a modification of the surface that consists of replacing the singularities by all limits of directions of tangent spaces to the surface at non-singular points.

M. Spivakovsky proved in [11], that any normal surface singularity can be resolved by a finite number of normalized Nash modifications. However, up to now, no method is known for constructing Nash resolution of a general surface singularity from its minimal resolution.

This problem is linked with the one of determining base points of the polar curves on a resolution of the surface; i.e. points of the non-singular surface that belong to the strict transform of almost all the polar curves (see [11] and [3]).

We dedicate this work to the study of these base points on a resolution of a normal surface singularity by the so called Jung's method.

The main idea of this work is to study the image of the polar curves in a projection to $\mathbf{C}^2$ and determine base points of these images after the minimal embedded resolution of the discriminant curve singularity.

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The main interest of this work is to provide a way to construct examples of surfaces for which we can determine base points of polar curves on Jung's resolution or sometimes on the minimal resolution.

We restricted our study to the case of normal surfaces defined by equations of type $z^n = f(x, y)$. The reason is that, in this case, the image of the polar curves in $\mathbf{C}^2$ is mainly a collection of pencils generated by the discriminant locus of the projection and its generic polar curve (with some appropriate powers).

We first characterize the base points of a generic pencil on the minimal embedded resolution of the discriminant locus by valuative criteria. We get more precise results when we apply these criteria to the case of an irreducible discriminant, using for that a theorem by D. T. Lê, F. Michel and C. Weber in [5] on the growth of "Hironaka's quotients".

Finally we give various examples of surfaces of type $z^n = f(x, y)$, that illustrate how useful can be our method for characterizing base points of polar curves.

2. Polar curves in Jung's resolution of a surface

2.1. Polar curves and its base points

Let $(S, 0)$ be a germ of normal surface singularity embedded in some $(\mathbf{C}^N, 0)$. A linear projection $p : \mathbf{C}^N \to \mathbf{C}^2$, induces a morphism $\pi : S \to U$ on a representative of the germ $(S, 0)$ to an open neighborhood of 0 in $\mathbf{C}^2$.

Remark 2.1. *If we choose the kernel of the projection p to be transversal to the tangent cone of the surface S at* 0, *i.e. the intersection of* $\ker p$ *with the tangent cone is* $\{0\}$, *then the morphism π is finite (proper with finite fibers) and its degree is equal to the multiplicity of the surface S at the origin (see for example* [14, I.5.2]). *Such a projection π is called generic.*

Let L be a generic $(N-2)$-plane (in the sense of remark 2.1), and $\pi_L : S \to \mathbf{C}^2$ the restriction of the linear projection whose kernel is L. According to [6, 2.2.2], the polar curve $P_L(S, 0)$, defined by L on S, is the closure in S, of the critical locus of the restriction of π_L to the non-singular locus of S. In other words, it is the closure of

$$\{x \in S \backslash \{0\} / \dim(L \cap T_x S) \geq 1\},$$

where $T_x S$ denotes the direction of the tangent space to S at a (non-singular) point $x \in S \backslash \{0\}$.

When the direction L is generic and the surface normal and singular, the polar curve is non-empty and coincides with the critical locus of the projection that defines it.

A polar curve could be thought of as a generic element of a linear system parameterized by the Grassmannian $G(N-2, N)$ of $(N-2)$-linear subspaces in $\mathbf{C}^N$.

Now, consider a modification $\mu : S' \to S$ over 0. That means a proper map, that induces an analytic isomorphism on the pre-image of $S \backslash \{0\}$. If $\mathscr{C}$ is a curve on S, the closure in S' of $\mu^{-1}(\mathscr{C} \backslash \{0\})$ is called the strict transform of $\mathscr{C}$ by μ and is denoted by $\mathscr{C}'$.

Definition 2.2. *A point $\eta \in S'$ is a base point of the linear system of polar curves by μ, or simply a base point, if there exists an open dense subset Ω of the Grassmannian $G(N-2, N)$, such that for every $L \in \Omega$, $\eta \in P'_L(S, 0)$.*

When the modification is the blow-up of the singularity, a full characterization of the base points of polar curves is given in [10, 6.3 and 5.8].

Remark 2.3. *Nash modification, or Nash blow-up, of a normal surface singularity has the property of being the "minimal" modification of a normal surface singularity for which the polar curves have no base point* (*see* [3, 1.2]).

Let $\rho : X \to S$, be a resolution of singularities of S, i.e. a modification with X non-singular.

Characterizing base points of polar curves on a resolution of a normal surface singularity is an open problem. The answer is known only in some particular cases: [3] for the case of rational double points and [11] in the case of minimal singularities.

There are different ways to remove a surface singularity (see [15], [11], [1] ...). We will study base points of polar curves under a resolution of singularities by Jung's method.

2.2. Jung's method

Consider a finite projection $\pi : S \to U \subset \mathbf{C}^2$, and call Δ its discriminant locus; Δ is defined as the reduced image of the critical locus. Let $\rho : Z \to U$ be an embedded resolution of Δ, i.e. a modification of U over 0 such that the strict transform of Δ is non-singular and the total pre-image of Δ has only normal crossing singularities. Let $\tilde{S}$ be the normalization of the pull-back of S by ρ. Denote by $\pi' : \tilde{S} \to Z$ the pull-back of π by ρ and ρ' the pull-back of ρ by π. The discriminant locus of π' is contained in the total pre-image of Δ. The surface $\tilde{S}$ has only normal quasi-ordinary singularities, whose resolution $r : X \to \tilde{S}$ is given by combinatoric data of the ramified covering (see for example [1, III.5]).

$$
\begin{array}{ccccc}
X & \xrightarrow{r} & \tilde{S} & \xrightarrow{\rho'} & S \\
 & & \Big\downarrow{\scriptstyle \pi'} & & \Big\downarrow{\scriptstyle \pi} \\
 & & Z & \xrightarrow{\rho} & U
\end{array}
\tag{1}
$$

Definition 2.4. *We call Jung's resolution of a germ of surface any resolution obtained by the method described in the diagram* 1.

Our goal in this work, is to follow the linear system of polar curves all along this diagram in order to be able to get some information about the base points on the surface $\tilde{S}$.

Denote by $\mathscr{C}_L$ the image of a polar curve $P_L(S,0)$ on $U \subset \mathbf{C}^2$ by the finite map π. Let us call the family of plane curves $\{\mathscr{C}_L, L \in G(N-2,N)\}$ the "image-system" of the linear system of polar curves on the open set U, or simply the image-system. We will study the base points of the image-system by the resolution ρ.

The notations introduced in this section will be used all along this work.

2.3. Case of surfaces of the form $z^n = f(x,y)$

Suppose the surface S is defined by an equation of the form $z^n = f(x,y)$ in $\mathbf{C}^3$ with coordinates x, y and z. Assume the function f is reduced in order to ensure the normality of S at the origin.

The linear system of polar curves on S is given by:

$$\alpha \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y} + \gamma z^{n-1} = 0, \quad (\alpha : \beta : \gamma) \in \mathbf{P}^2.$$

Consider the projection $\pi : S \to U \subset \mathbf{C}^2$ induced by $(x,y,z) \mapsto (x,y)$. The discriminant locus Δ of π is defined by the equation $f = 0$. The image-system on U is the family of plane curves defined by the equations:

$$\left(\alpha \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y}\right)^n + (-1)^{n-1}\gamma^n f^{n-1} = 0; \quad (\alpha : \beta : \gamma) \in \mathbf{P}^2.$$

Note that the curve Δ'_t defined by $\alpha \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y} = 0$ is the polar curve in the direction $t = (\alpha : \beta) \in \mathbf{P}^1$ associated to the curve defined by $f = 0$ (see [5]). We will denote by f'_t any representative of $\alpha \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y}$ modulo multiplication by a complex number.

We have then:

Proposition 2.5. *An open dense part of the image-system in $U \subset \mathbf{C}^2$ is given by the linear systems of curves defined by*

$$\lambda (f'_t)^n + \mu f^{n-1} = 0, \quad ((\lambda : \mu), t) \in \mathbf{P}^1 \times \mathbf{P}^1.$$

In other words, it is the collection parameterized by $t \in \mathbf{P}^1$, of pencils generated by the $(n-1)^{th}$ power of the reduced discriminant Δ and the n^{th} power of its polar curve Δ'_t.

Remark 2.6. *The discriminant curve of the projection π can be given a scheme structure* [13, §§1 *and* 2]. *In this case it is the hypersurface defined by $f^{n-1} = 0$, that is one of the generators of the pencils of the image-system.*

In order to look for base points of the linear system of polar curves, our idea is first to look for base points of the pencil of curves generated by the discriminant and its general polar curve.

3. Base points of pencils of curves

We will now look for base points of a pencil of curves of the form $\lambda(f_t')^n + \mu f^{n-1}$ under a modification of the origin, f being a reduced holomorphic function, f_t' a derivative of f in a generic direction, and n a positive integer.

In a general setting, let f and g be reduced holomorphic functions around the origin of $\mathbf{C}^2$, without common branches, a and b two positive integers. Base points of the pencil generated by the functions f^a and g^b after a modification μ are defined as in definition 2.2; they are also called indeterminacy points of the "function" f^a/g^b in the modification μ. D. T. Lê and C. Weber, studied this indeterminacy for two reduced plane curves without common branches in [7]. Following their work, we are going to characterize base points under a modification by valuative criteria:

Let $\nu : \tilde{U} \to U$ be a normalized modification of $0 \in \mathbf{C}^2$ (i.e. a modification composed with a normalization). We will call $D_1, \dots, D_n$ the irreducible components of the total transform $((f^a g^b) \circ \nu)^{-1}(0)$; it includes the irreducible components of the exceptional divisor and the strict transforms. We recall that the valuation $v_p(f)$ of f at a point $p \in D_i$ is the smallest degree of a Taylor expansion of $f \circ \nu$ at p. We define the valuation $v_i(f)$ (resp. $v_i(g)$) of f (resp. of g) on a component D_i as the valuation $v_p(f)$ (resp. $v_p(g)$) at a generic point of D_i.

Let $\rho : Z \to U$ be the minimal embedded resolution of the curve Δ defined by $f = 0$, as in §2.2.

THEOREM 3.1. *The base points of the pencil generated by f^a and g^b under the resolution ρ are the intersection points of two components D_i and D_j such that*

$$v_i(f)/v_i(g) < b/a < v_j(f)/v_j(b)$$

where we allow the quotient of valuations to be the infinity.

Proof. Let $\eta \in Z$ be a point of the exceptional divisor of ρ. Locally around η, the total transform of f^a and g^b can be written respectively on the form $uh_1^{a_1} \cdots h_n^{a_n}$ and $vh_1^{b_1} \cdots h_n^{b_n}$, where u and v are units and h_i is a reduced local equation of the irreducible component D_i; with a_i and b_i non-negative integers. We note that the integers a_i and b_i are the respective valuations of f^a and g^b on D_i.

Let $(\lambda : \mu) \in \mathbf{P}^1$. The total transform of $\lambda f^a + \mu g^b$ around η is

$$\lambda uh_1^{a_1} \cdots h_n^{a_n} + \mu vh_1^{b_1} \cdots h_n^{b_n}. \tag{2}$$

If η is on D_1, but not at the intersections with the other components, then in (2) we have $a_1 = v_1(f^a)$, $b_1 = v_1(g^b)$ and $a_i = b_i = 0$ for $2 \le i \le n$.

We distinguish two cases:

If $a_1 \neq b_1$ then the total transform of $\lambda f^a + \mu g^b$ is of the form $h_1^{|a_1 - b_1|} \cdot unit$. Hence η is not in the strict transform of any member of the pencil generated by f^a and g^b.

If $a_1 = b_1$, then the total transform of $\lambda f^a + \mu g^b$ is of the form $h_1^{a_1}(\lambda u' + \mu v')$. For a generic $(\lambda : \mu) \in \mathbf{P}^1$, $\lambda u' + \mu v'$ is a unit, and hence η is not a base point by ρ of the pencil generated by f^a and g^b.

Let η be an intersection point of some components, let us say for simplicity $D_1 \cdots D_k$, and suppose $v_i(f^a) \leq v_i(g^b)$ for $1 \leq i \leq k$. Then, for a generic $(\lambda : \mu) \in \mathbf{P}^1$, the total transform of $\lambda f^a + \mu g^b$ is of the form $h_1^{a_1} \cdots h_k^{a_k} \cdot unit$. Hence, η is not a base point.

If there exist two components, let us say D_1 and D_2 such that $\eta \in D_1 \cap D_2$ with $v_1(f^a) < v_1(g^b)$ and $v_2(g^b) < v_2(f^a)$ then the total transform of $\lambda f^a + \mu g^b$ is of the form $h_1^{a_1} h_2^{b_2}(\lambda u \varphi + \mu v \psi)$, with $\varphi(\eta) = \psi(\eta) = 0$ for all $(\lambda : \mu) \in \mathbf{P}^1$. Hence η is a base point.

So, η is a base point by ρ of the considered pencil if and only if, η is an intersection point of at least two components D_i and D_j such that

$$(v_i(f^a) - v_i(g^b))(v_j(f^a) - v_j(g^b)) < 0. \tag{3}$$

By the definition we gave for a valuation of a function along an irreducible curve, we have that $v_i(h^k) = k v_i(h)$.

If $v_i(g) v_j(g) \neq 0$, then we divide (3) by $v_i(g) v_j(g)$ and get that η is a base point if and only if

$$v_i(f)/v_i(g) < b/a < v_j(f)/v_j(g)$$

or the same formula inverting the inequalities.

If $v_i(g) = 0$ then, both $v_j(g)$ and $v_i(f)$ are non zero, and hence we get the same inequality with $v_i(f)/v_i(g) = \infty$. □

Remark 3.2. i) *The affirmation of theorem* 3.1 *is mainly the same as the one in the proof of* [7, 2.1]. *In our case we do not need to suppose that we have resolved the singularities of the curve* $(fg)^{-1}(0)$.

ii) *The case we are interested in is when* $g = f_t'$, *where* f_t' *is an equation of a generic polar curve associated to* f, $b = n$ *and* $a = n - 1$. *Since the function* f *is supposed to be reduced, the curves* $f = 0$ *and* $f_t' = 0$ *do not have any common branch.*

An obvious base point would have been an intersection point of the strict transforms of $f = 0$ *and* $f_t' = 0$. *However, such a point does not exist in any embedded resolution of* $f = 0$, *see* [5, 2.1].

4. Hironaka's quotients

The quotients of valuation used in the previous section appear in the literature as "Hironaka's quotients" of the functions f and f_t'. In [5] the authors study the growth of these quotients in the resolution diagram of a curve. Using their results, we will express some conditions on the base points of pencils generated by the functions f^{n-1} and $(f_t')^n$.

First we need to introduce some vocabulary on resolution diagram of plane curves.

4.1. Vocabulary on resolution diagrams

Let $f \in \mathbf{C}\{x, y\}$ be a reduced holomorphic function, and let $r : Z \to (\mathbf{C}^2, 0)$ be the minimal embedded resolution of the curve $f = 0$. One knows it can be obtained as a composition of blow-ups of points [2, III]. Denote by $E_1, \dots, E_r$ the irreducible components of the exceptional divisor $r^{-1}(0)$.

The dual graph Γ of the resolution r is the graph obtained by associating to each component E_i a vertex that we still call E_i, and to each intersection $E_i \cap E_j$ an edge connecting the two corresponding vertices. We will represent the irreducible components of the strict transform of the curve $(f = 0)$ by arrows supported on the vertices representing the irreducible exceptional components intersecting it. We may also represent by some other arrows, the strict transform of any other curve C. In this case it may happen that an arrow has support on an edge and the intersections need not be transversal.

Following the definitions of [5, §2], we call #1 the vertex corresponding to the irreducible component that appears in the first blow-up. An *extremity* of the graph is a vertex other than #1 that is attached to only one other vertex and not attached to any arrow of the curve $(f = 0)$. A *rupture vertex* is a vertex attached to at least 3 other vertices or arrows of the curve $(f = 0)$. A *dead branch* of Γ is a geodesic of Γ, linking a rupture vertex to an extremity without containing any other rupture vertex.

4.2. The growth of Hironaka's quotients

Let f and $g : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ be two reduced holomorphic functions, and $r : Z \to (\mathbf{C}^2, 0)$ the minimal embedded resolution of the curve $f^{-1}(0)$. Denote by $E_1, \dots, E_r$ the irreducible components of the exceptional divisor $r^{-1}(0)$ and by Γ the dual graph of the resolution r.

DEFINITION 4.1. *Hironaka's quotient of the component E_i is the rational number $q_i(f, g) = v_i(f)/v_i(g)$ (or simply q_i), where $v_i(f)$ (resp. $v_i(g)$) is the valuation of f (resp. g) along the irreducible exceptional component E_i.*

To each vertex of the dual graph we associate Hironaka's quotient of the corresponding irreducible component. The following theorem on the growth of Hironaka's quotient is proved by D. T. Lê, F. Michel and C. Weber in [5, 3.2].

THEOREM 4.2. a) *If E_0 carries an arrow of $g = 0$ then, there exists a vertex E_1 carrying an arrow of $f = 0$ such that q_i is strictly increasing along the geodesic on the dual graph from E_0 to E_1.*

b) *q_i is constant on the closure of each connected component of the complement of all the geodesics linking the strict transforms of f with the ones of g.*

By closure we mean that, to each connected component we glue the vertex that attaches it to the rest of the dual graph.

We are actually stating the theorem in the way of [12, II.1], restricted to the case of functions defined on an open set of $\mathbf{C}^2$.

Remark 4.3. *In* [12, II.1], *the morphism r is supposed to be a resolution of the curve* $(fg)^{-1}(0)$. *However, the same proof is valid in the situation of theorem* 4.2.

4.3. The case of an irreducible curve

In this part we will restrict to the case where the function f is analytically irreducible and the function g is a derivative f_t' of f in a generic direction.

The dual graph Γ of the minimal embedded resolution r of the curve $f = 0$ can be drown as in fig. 1.

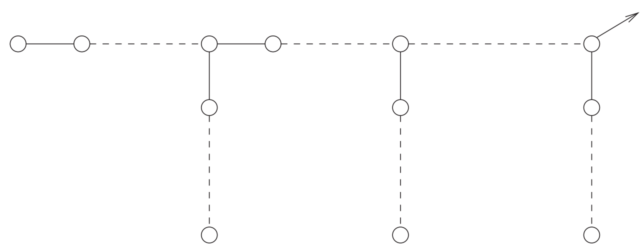

FIGURE 1.

There exists a unique way to write the graph Γ as a disjoint union of connected linear subgraphs Γ_i such that each Γ_i contains one rupture vertex or the arrow. Such a subgraph Γ_i will be called an "L-square" or simply an "L".

The L's of a dual graph can be ordered from the first one that contains the vertex #1 to the last one that contains the vertex carrying the arrow. There is a one-to-one correspondence between the ordered L's of the graph Γ and the ordered Puiseux pairs associated to the irreducible function f.

Call m the multiplicity at the origin of the curve $f = 0$ and let n be a positive integer.

THEOREM 4.4. *Consider the pencil generated by* f^{n-1} *and* $(f_t')^n$. *The base points of this pencil in the minimal embedded resolution of* $f = 0$ *satisfy the following conditions*:

– *If* $m < n$, *the only possible base points are on the dead branches of the graph* Γ.

– *If* $m = n$, *the only possible base points are on the dead branches of* Γ, *but not on the first L.*

– *If* $n < m$, *and the graph* Γ *contains two L's or more, there is no base point on the first L and they are possible on all the others.*

We can have a more precise result if we require that the irreducible curve $f = 0$ has only one Puiseux pair. In this case the dual graph Γ of the minimal embedded resolution of $f = 0$ has only one L, and at most one rupture vertex and one dead branch, as in fig. 2 (see [2, III]).

Corollary 4.5. – *If $n = m$, there is no base point.*

– *If $n < m$, the only base point is the intersection of the strict transform of $f^{-1}(0)$ with the exceptional divisor.*

– *If $n > m$, the only possible base points belong to the exceptional components corresponding to vertices of the dead branch of Γ.*

Before proving the theorem 4.4, let us recall that in [5], D. T. Lê, F. Michel and C. Weber studied the "position" of the strict transform of a generic polar

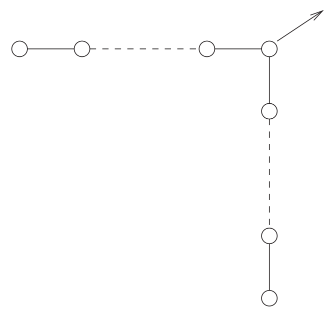

Figure 2.

curve (associated to a plane curve) in the dual graph of the minimal embedded resolution of the given curve. In the case where the curve $f = 0$ has only one branch, we can summarize their result saying that: the strict transforms of a generic polar curve intersect only the components of the dead branches. We also refer to [9] for the irreducible case.

Proof of theorem 4.4 *and corollary* 4.5. By theorem 4.2, Hironaka's quotient $v_i(f)/v_i(f_t')$ is constant over the vertices of the first L that are not on the dead branch (including the rupture vertex). Furthermore, it is equal to $m/(m-1)$ since the multiplicity of a general polar curve at the origin is $m-1$. On the geodesic from the first rupture vertex to the vertex carrying the arrow of $f = 0$, $v_i(f)/v_i(f_t') > m/(m-1)$. Along the dead branch of the first L $v_i(f)/v_i(f_t') < m/(m-1)$.

Now, apply theorem 3.1, to determine base points of the pencil.

– If $m \leq n$, then $n/(n-1) \leq m/(m-1)$. Hence, outside the dead branches, including the arrow of $f = 0$, $v_i(f)/v_i(f_t') \geq n/(n-1)$. Hence there are no base points outside the dead branches.

– If $m = n$, then $n/(n-1) = m/(m-1)$ and $v_i(f)/v_i(f_t') < n/(n-1)$ on the vertices of the dead branch of the first L. Hence it contains no base point. In particular, if $f = 0$ has one Puiseux pair, then there are no base points.

– If $n < m$, then $m/(m-1) < n/(n-1)$. For all the vertices of the first L we have $v_i(f)/v_i(f_t') < n/(n-1)$. So, if $f = 0$ has more than one Puiseux pair, there is no base point on the first L. If it has only one Puiseux pair then, the intersection point of the arrow of $f = 0$ with the exceptional divisor is a base point. □

5. Relation with the polar curves on the surface

Our goal is to determine or characterize base points of polar curves on a normal surface S of the form $z^n = f(x, y)$, after Jung's resolution of the surface singularities.

We have already seen in proposition 2.5 that, under the projection $\pi : (x, y, z) \mapsto (x, y)$, the collection of pencils

$$\lambda(f_t')^n + \mu f^{n-1} = 0$$

$(\lambda : \mu) \in \mathbf{P}^1$, $t \in \mathbf{P}^1$ is a dense part of the image-system.

It is clear that a base point of the image-system is a base point of the pencil generated by $(f_t')^n$ and f^{n-1} for a generic direction t. The converse is not necessarily true. In fact, it may happen that each pencil $(f^{n-1}, (f_t')^n)$ has a base point p_t that depends on the parameter t. Such a point will not be a base point of the image-system.

In order to state the exact correspondence between base points of both families of curves, we need to introduce some additional vocabulary.

Consider a pencil of curves generated by the functions g and h supposed to be reduced without common factors. Call $E_1, \dots, E_n$ the irreducible components of the exceptional divisor of sequence of point blow-ups. Following [7, §2], we will call a *dicritical component* a component E_i, such that the function g/h is well defined on E_i and is not constant.

Consider now the family of curves $f_t' = 0$ parameterized by $t \in \mathbf{P}^1$. It is the pencil generated by $\partial f/\partial x$ and $\partial f/\partial y$. Let η be a base point of the pencil generated by f^{n-1} and $(f_t')^n$ in the minimal embedded resolution ρ of the curve $f = 0$. By analogy with the previous definition, we will say that η is a *mobile base point*, if it is a point of some dicritical component for the family of curves $f_t' = 0$, $t \in \mathbf{P}^1$.

Let $\tilde{S}$ be the normalized pull-back of S by ρ and π' the pullback of π by ρ, as in the diagram 1.

Proposition 5.1. *The base points of the polar curves of S on the surface $\tilde{S}$ are the inverse image by π' of the non-mobile base points of the linear system generated by f^{n-1} and $(f_t')^n$, for a generic direction $t \in \mathbf{P}^1$.*

Proof. The image by π' of a base point of the polar curves in $\tilde{S}$ is a base point of the image system. It is then a non-mobile base point of the pencil generated by f^{n-1} and $(f_t')^n$ for a generic direction t.

Conversely, a non-mobile base point of the pencils generated by f^{n-1} and $(f_t')^n$ is the same base point for all these pencils. Since the morphism π' is finite it is an image of a base point of polar curves on $\tilde{S}$.

Since the surface S is a cyclic covering ramified over $f = 0$, the surface $\tilde{S}$ is locally the normalization of a cyclic covering ramified over the total pre-image of $f = 0$ by ρ. Hence, if ξ_1 and ξ_2 are two points of $\tilde{S}$ such that $\pi'(\xi_1) = \pi'(\xi_2)$, then the germs $(\tilde{S}, \xi_1)$ and $(\tilde{S}, \xi_2)$ are analytically isomorphic. So any point of $\tilde{S}$ whose image by π' is a non-mobile base point of a pencil $(f^{n-1}, (f_t')^n)$ for a generic direction t is a base point of the polar curves of the surface S. $\square$

This proposition, combined with theorems 3.1 and 4.4 and corollary 4.5 allow us to check in many cases whether a resolution of a surface has base points for its polar curves.

6. Examples

In the examples of this section, we will use the process of desingularization explained in [8]. The process is manly Jung's resolution, with some extra-techniques for constructing the weighted dual graph of the desingularization of the surface. See also [4].

6.1. D_n-Surfaces

Consider the germ at the origin of the surface S defined by $z^2 = x^2y + y^{n-1}$, $n \geq 4$; the so-called D_n-singularity. The projection $S \to \mathbf{C}^2$ that maps (x, y, z) to (x, y) has its discriminant locus defined by $f(x, y) = x^2y + y^{n-1} = 0$.

We need to distinguish two cases:

n is even: Let us say $n = 2k$.

In this case, the discriminant has three smooth branches and two tangents. The dual graph of its minimal embedded resolution is given in fig. 3, where

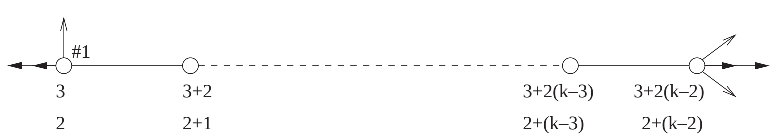

FIGURE 3.

the upper row of integers refers to the valuation of f and the lower row to the one of f_t'. The simple arrows represent the strict transform of $f = 0$ and the double ones, represent the strict transform of $f_t' = 0$ for a generic $t \in \mathbf{P}^1$.

By [5, 2.1], the general polar curve associated to the discriminant has two branches whose strict transform will intersect #1 and the rupture vertex.

By theorem 3.1, the base points of the linear system generated by f and $(f_t')^2$ are the intersection points of the strict transform of the discriminant with the exceptional divisor.

If we apply the process explained in [8], we can blow-up once these intersection points to ensure that the normalized pull-back is non-singular.

We get the new dual graph given in fig. 4.

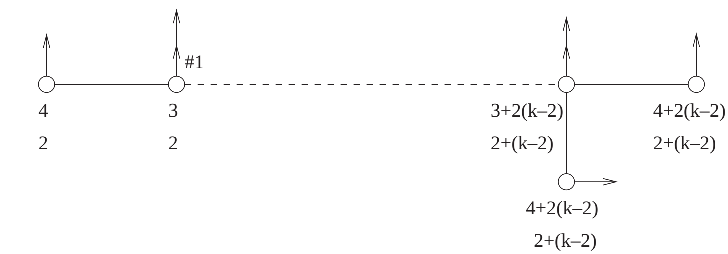

FIGURE 4.

by theorem 3.1, there are no more base points.

We pull-back this embedded resolution and get a resolution of the surface that coincides in this case with the minimal resolution (See [8]). Hence there is no base point on the minimal resolution of a D_{2k}-singularity.

n is odd: Let us say $n = 2k + 1$.

The discriminant has two branches, a smooth one $(y = 0)$ and a cuspidal singularity $(x^2 + y^{2k-1} = 0)$. The dual graph of its minimal resolution is given in fig. 5, the notations being the same as the previous case.

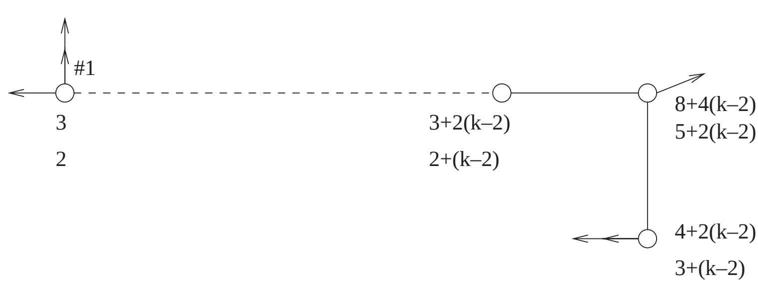

FIGURE 5.

In this case also the base points of the linear system generated by f and $(f_t')^2$ are the intersection points of the strict transforms of the discriminant with the exceptional divisor.

In order to ensure that the normalized pull-back is non-singular, we need to blow-up once the intersection of the strict transform with the irreducible component #1 of the exceptional divisor. We get fig. 6.

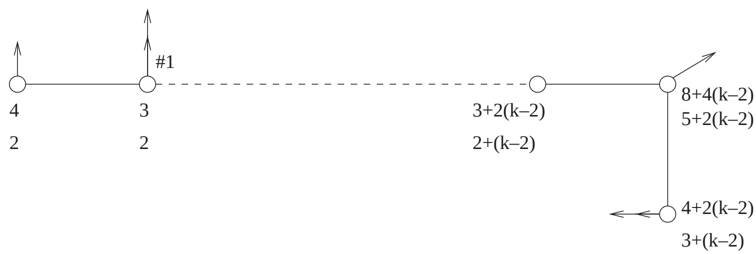

FIGURE 6.

By theorem 3.1, there is only one base point on the new configuration: the intersection point of the strict transform with the rupture vertex. We apply [8] and get the minimal resolution, with the dual graph of fig. 7.

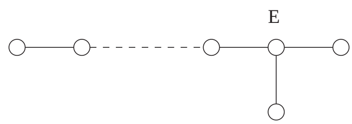

FIGURE 7.

The only base point is on the irreducible component E of the exceptional divisor.

Remark 6.1. *G. Gonzalez-Sprinberg studied the Nash resolution of rational double points of surfaces in* [3]. *The result of our example coincides with his result concerning base points of the polar curves on D_n-singularities.*

6.2. $z^k = x^2y + y^{n-1}$

The discriminant curve is the same one as in the previous example. However, base points depend also on the integer k.

In fact, we can see on the dual graph of the resolution of the discriminant that, if $k \geq 4$ and $n \geq 6$ then the quotient of the valuations of the discriminant and its polar curve will be strictly bigger than $k/(k-1)$. Hence, by theorem 3.1, the base points of the linear system generated by f^{k-1} and $(f_t')^k$ on the minimal embedded resolution of the discriminant are the intersection points of the strict transform of $f_t' = 0$ with the exceptional divisor.

By proposition 5.1, we need then to check if these points are base points for the pencil $f_t' = 0$, $t \in \mathbf{P}^1$.

For simplicity, we will make the computations for a special value of n. Let us assume $n = 7$.

We get the dual graph in fig. 8, in which the upper row of integers are the valuations of f_x and the lower one are the valuations of f_y.

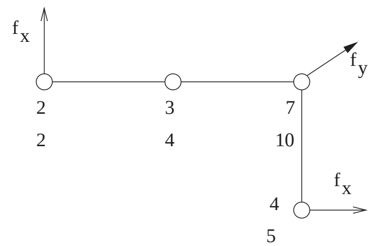

FIGURE 8.

Applying the valuative criterion of theorem 3.1, to the functions f_x and f_y, we find that the family of polar curves f_t', $t \in \mathbf{P}^1$ has a base point corresponding to the intersection of the strict transform of $f_x = 0$ with the exceptional component with valuations 4 and 5.

Hence the surface $z^k = x^2y + y^6$ has a base point on its Jung's resolution. We leave to the reader the case $k = 3$.

6.3. $z^k = x^4 + y^5$

The discriminant curve is defined by $f(x, y) = x^4 + y^5$. It is an irreducible curve with one Puiseux pair. The dual graph of its minimal resolution is given in fig. 9.

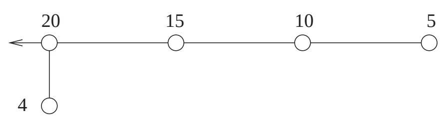

FIGURE 9.

For $k = 4$, by corollary 4.5, the pencil generated by f^3 and $(f_t')^4$ has no base point. Hence Jung's resolution of the surface $z^4 = x^4 + y^5$ has no base point for the polar curves of the surface.

For $2 \le k \le 3$ the pencil generated by f^{k-1} and $(f_t')^k$ has a unique base point on the intersection of the strict transform of $f = 0$ with the exceptional divisor.

We apply the process of [8], and we find that Jung's resolution of $z^k = x^4 + y^5$ has one base point for the polar curves, $k = 2$ or $k = 3$.

For $k > 4$, the pencil generated by f^{k-1} and $(f_t')^k$ may have some base points on the dead branch (by corollary 4.5). In order to localize them, we need to fill the dual graph with the valuations of the generic polar curve $f_t' = 0$. We get the filled graph of fig. 10, where the integer between parenthesis refers to the valuation of the generic polar curve along the exceptional divisor, and the double arrow refers to the strict transform of the generic polar curve.

We find out that the only base point of the pencil is the intersection of the strict transform $f_t' = 0$ with the dual graph.

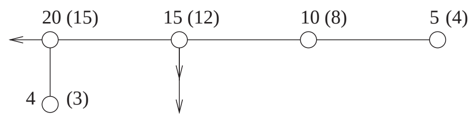

FIGURE 10.

In order to apply proposition 5.1, we need to check wether this intersection point is a mobile point or not. If we follow, on the dual graph of the resolution of the discriminant, the valuations of $\partial f/\partial x$ and $\partial f/\partial y$, we find that the intersection point is a mobile point of the polar curves associated to $f = 0$. Hence, by proposition 5.1, Jung's resolution of $z^k = x^4 + y^5$ has no base point for $k > 4$.

Remark 6.2. *Jung's resolution of a surface, is not in general the minimal resolution. In order to get the minimal resolution from Jung's resolution, one needs to blow-down all the smooth rational irreducible components of the exceptional divisor that have a self-intersection equal to* -1. *This contraction may create base points of the linear system of polar curves on the minimal resolution of the surface. In order to determine these possibly new base points, one needs to characterize the dicritical components of the linear system on Jung's resolution.*

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