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ON THREEFOLDS WITH $K^3 = 2p_g - 6$

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Abstract

It is known that if X is an n-dimensional normal variety, and D a nef and big Cartier divisor on it such that the associated map φ_D is generically finite then $D^n \geq 2(h^0(X, \mathcal{O}_X(D)) - n)$. We study the case in which the equality holds for $n = 3$ and $D = K_X$ is the canonical divisor.

We also produce a bound for the admissible degree of the canonical map of a threefold, when it is supposed to be generically finite.

1. Introduction

It is well known that if X is a surface of general type then $K_X^2 \geq 2p_g(X) - 4$. Moreover, if the equality $K_X^2 = 2p_g(X) - 4$ holds, then by Castelnuovo-Enriques theorem the canonical map is generically finite (see [2]). These surfaces have been classified by Horikawa in [5]: they are canonical double cover of scrolls in $\boldsymbol{P}^{p_g - 1}$. The purpose of this paper is to study the analogous cases in higher dimensions.

Using Clifford theorem on curves, it has been proved in [7] the following theorem.

THEOREM 1.1. *Let X be a normal n-fold of general type, with at most canonical singularities. Suppose K_X is nef and* $\dim \varphi_{K_X}(X) = n$. *Then $K_X^n \geq 2p_g(X) - 2n$. Moreover, if the equality holds and $n \geq 2$, then K_X is Cartier, the canonical system is base-point-free and φ_{K_X} is a generically finite morphism of degree two on normal varieties of minimal degree in $\boldsymbol{P}^{p_g(X)-1}$.*

In this paper we study the cases $K_X^n = 2p_g - 2n = 2, 4$ for every $n \geq 3$, and the case $n = 3$, that is $K_X^3 = 2p_g - 6$, for every K_X^3. The Castelnuovo-Enriques theorem is no longer true in higher dimension, as the examples in [7] show. Hence, we need to require the hypothesis that the canonical map is generically finite. In particular, we work out the cases in which the image of φ_{K_X} is singular. We also show the nonuniqueness of irreducible components of the

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moduli space to which the families of threefolds with canonical morphism of degree 2 belong.

1.1. Notations and Conventions. Everything works on the complex number field. All varieties are normal and complete unless otherwise stated.

We denote by X a smooth minimal algebraic threefold of general type. We say that X is *pluriregular* if $h^1(\mathcal{O}_X) = h^2(\mathcal{O}_X) = 0$. We denote by ω_X the canonical sheaf of X, by K_X a canonical divisor on X, and by φ_{K_X} or simply by φ the rational map defined by the complete canonical system on X, which by theorem 1.1 will be everywhere defined. We briefly refer to it as the *canonical morphism*.

We recall that the existence of the moduli space of canonically polarized smooth threefolds with Hilbert polynomial $h(t) := \chi(\omega_X^t) = t(2t-1)(t-1)a/12 + (1-2t)(1-b)$ is given in [11]. The subset $\mathcal{M}_{a,b}$ of points representing pluriregular smooth minimal threefolds X with $K_X^3 = a$ and $p_g(X) = b$ is an open inside the *moduli space* and is a coarse moduli space. $\mathcal{M}_{a,b}^2$ will denote the locus in $\mathcal{M}_{a,b}$ of points representing threefolds whose canonical map is a finite morphism of degree 2.

A morphism $\varrho : X \to Y$ between varieties is called a *double cover* if is flat and finite and the function field of X is an extension of degree 2 of the function field of Y. In this case $\varrho_*(\mathcal{O}_X)$ is a locally free $\mathcal{O}_Y$-sheaf of rank 2.

When $\mathcal{G}$ is a locally free sheaf of rank r over a variety Y, $\boldsymbol{P}(\mathcal{G})$ denotes the $\boldsymbol{P}^{r-1}$-bundle over Y given by $\mathrm{Proj}(\mathrm{Sym}\,\mathcal{G})$, $\mathcal{H}$ denotes the tautological bundle $\mathcal{O}_{\boldsymbol{P}(\mathcal{G})}(1)$ over $\boldsymbol{P}(\mathcal{G})$ and $\varphi_{\mathcal{H}}$ the map defined by $\mathcal{H}$.

In particular if $Y = \boldsymbol{P}^1$, $\mathcal{G}$ splits as a direct sum of r linear bundles that can be ordered by increasing degrees. Thus up to tensorize $\mathcal{G}$ by an appropriate line bundle, $\boldsymbol{P}(\mathcal{G})$ is isomorphic to $\boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_r))$ where the integers $a_1 \le a_2 \le \cdots \le a_r$ can be supposed to be nonnegative. The r-tuple $(a_1, a_2, \ldots, a_r)$ is called the *type* of the bundle. In this case $\boldsymbol{P}(\mathcal{G})$ is denoted by $\tilde{W} = \tilde{W}(a_1, a_2, \ldots, a_r)$ and the image of $\tilde{W}$ under $\varphi_{\mathcal{H}}$ is a *rational normal scroll* of type $(a_1, a_2, \ldots, a_r)$, denoted by W. This is a variety of minimal degree in the projective space. If $a_1 = 0$ then W is a cone. The natural projection from $\tilde{W}$ to $\boldsymbol{P}^1$ is denoted by π and the general fiber by F.

We generally use H to denote the hyperplane section.

For a real number α we denote by $(\alpha)^+$ the maximum between α and 0.

1.2. The degree of the canonical map. In analogy with the case of surfaces it is interesting to find the bound of the admissible degree for the canonical map of a threefold, when it is supposed to be generically finite. We prove the following result.

Theorem 1.2. *Let X be a minimal threefold of general type with $h^2(\mathcal{O}_X) - h^1(\mathcal{O}_X) \ge 2$ and with generically finite canonical map φ_{K_X}.*

If $\varphi_{K_X}(X)$ has geometric genus 0 then $\deg \varphi_{K_X} \le 72$. If $\varphi_{K_X}(X)$ has geometric genus $p_g(X)$ then $\deg \varphi_{K_X} \le 24$. No other case occurs.

Proof. Let $\delta = \deg \varphi_{K_X}(X)$ and $d = \deg \varphi_{K_X}$, clearly one has $p_g \geq 4$ and $0 < d\delta \leq K_X^3$. By Miyaoka-Yau inequality (see [8]) one also has

$$0 < K_X^3 \leq -72\chi(\mathcal{O}_X).$$

If the canonical image has geometric genus 0 then $\delta \geq p_g(X) - 3$, thus

$$d(p_g(X) - 3) \leq d\delta \leq 72[p_g(X) - 1 - (h^2(\mathcal{O}_X) - h^1(\mathcal{O}_X))]$$

which implies

$$d \leq 72[p_g(X) - 1 - (h^2(\mathcal{O}_X) - h^1(\mathcal{O}_X))]/(p_g(X) - 3).$$

Since by hypothesis $h^2(\mathcal{O}_X) - h^1(\mathcal{O}_X) \geq 2$ one has $d \leq 72$.

If the canonical image has geometric genus $p_g(X)$ then by [4], section 2, it has to be

$$\delta \geq \dim \varphi_{K_X}(X) \operatorname{codim} \varphi_{K_X}(X) + 2 = 3(p_g(X) - 4) + 2 > 3(p_g(X) - 4).$$

The same kind of computation as above shows that in this case $d \leq 24$.

No other case can occur for the geometric genus of the canonical image, by theorem 3.4 in [2]. □

Remark. One could replace the hypothesis $h^2(\mathcal{O}_X) - h^1(\mathcal{O}_X) \geq 2$ with the inequality $h^2(\mathcal{O}_X) - h^1(\mathcal{O}_X) > 1 - p_g$, which is always verified according to Miyaoka-Yau inequality. Then $-\chi(\mathcal{O}_X) \leq 2(p_g - 1)$. Hence one gets $d \leq 432$ in the first case, and $d \leq 192$ in the second. Therefore under the hypothesis $h^2(\mathcal{O}_X) - h^1(\mathcal{O}_X) \geq 2$ of the theorem a sharper estimate is obtained. Note that the bound for d decreases rapidly when p_g increases.

The same computation shows that if $h^2(\mathcal{O}_X) - h^1(\mathcal{O}_X) \geq 0$ then $d \leq 216$ in the first case, and $d \leq 96$ in the second, achieved respectively for $p_g = 4$ and $p_g = 5$. Again, the bound for d decreases in the two cases to 72 and 24 asymptotically with p_g.

1.3. Double covers. Let X and Y be nonsingular n-dimensional varieties, and $f : X \to Y$ be a finite morphism of degree two. By definition, the ramification locus of f in X is a divisor R such that the so-called Hurwitz formula $K_X = f^*K_Y + [R]$ holds. The divisor $B = f_*(R)$ in Y is called the *branch locus*.

A divisor B of a variety Y is said to be *even* if there exists a linear bundle $\mathcal{L}$ such that $\mathcal{O}(B)$ is linearly equivalent to $\mathcal{L}^{\otimes 2}$. Let $\mathcal{O}(B) = \mathcal{L}^{\otimes 2}$ be an even divisor of Y without multiple components. It is possible to define the *double cover of Y branched along B* as a divisor X' of $\boldsymbol{P}(\mathcal{O}_Y \oplus \mathcal{L})$. When $b = 0$ is the equation of B in the open subset U of Y and z is the coordinate of the fiber $\boldsymbol{C}$ of the natural projection $\boldsymbol{P}(\mathcal{O}_Y \oplus \mathcal{L}) \to Y$, X' is locally given by the equation $b = z^2$ in the open subset $U \times \boldsymbol{C}$ (cf. [6]). Then, denoting by ϱ the finite morphism $X' \to Y$, one has $\varrho_*\mathcal{O}_{X'} = \mathcal{O}_Y \oplus \mathcal{L}^*$. From the fact that B does not contain multiple components, it follows that X' has at most codimension 2 singularities.

Indeed, if X' and Y are nonsingular then B is the branch locus of ϱ according to the above definition. Moreover, one has the following useful projection formula

$$\varrho_*(\Theta_{X'}) = \Theta_Y \otimes \mathscr{L}^* \oplus \Theta_Y(-\log B). \tag{1.1}$$

We recall that if $y_1, \dots, y_n$ are local coordinates for Y and $b = 0$ is the local equation of B, $\Theta_Y(-\log B)$ is the sheaf locally generated by $(b(\partial/\partial b), \partial/\partial y_2, \dots, \partial/\partial y_n)$ (see [9]). The *residue exact sequence*

$$0 \to \Omega^1_Y \to \Omega^1_Y(\log B) -^s \to \mathcal{O}_B \to 0 \tag{1.2}$$

holds, where the map s has the value $g|_B$ on the germ $gdb/b + \theta$.

If one wants to allow X' to be singular and preserve the Hurwitz formula, some details need to be considered.

Following [5], we say that a curve C *has no infinitely near triple points* if

(i) C has no singular points of multiplicity greater than 3;

(ii) after a quadratic transformation centered at a singular point the strict transform has no singular points with multiplicity greater than 2.

In general, we say that a variety B *has no infinitely near triple points* if it has no singular points of multiplicity greater than 3, and after a blow up centered at its singular locus, the strict transform has no singular points with multiplicity greater than 2. We recall theorem 7.2 in Chapter III of [1].

Theorem 1.3. *Let $\varrho : X' \to Y$ be a double cover with X normal and Y a nonsingular surface, branched over the even reduced divisor B. Let $\mathscr{L}$ be the line bundle on Y satisfying $\mathscr{L}^{\otimes 2} = \mathcal{O}_Y(B)$, which determines the covering. Consider $\sigma : \tilde{X}' \to X'$ the resolution of X' by a sequence of blowing up morphisms. Then there is a divisor $Z \geq 0$ on $\tilde{X}'$, with* Supp Z *contained in the union of the exceptional locus for σ, such that*

$$\omega_{\tilde{X}'} = (\pi\sigma)^*(\omega_Y \otimes \mathscr{L}) \otimes \mathcal{O}_{\tilde{X}'}(-Z). \tag{1.3}$$

Moreover, $Z = 0$ if and only is B has no infinitely near triple points.

One can easily generalize this result in any dimension.

Theorem 1.4. *The above theorem holds for Y nonsingular variety of dimension $n \geq 2$.*

Proof. We prove the statement for any dimension by reduction to the sectional surface H^{n-2}, where H is a very ample divisor on Y.

Firstly, note that by applying 1.3 and the adjunction formula to the general intersection of $n-2$ divisors of $|\mathcal{O}_Y(H)|$ one can show that the formula (1.3) holds in any dimension. It remains to be shown the last part of the statement.

Suppose that $Z = 0$, and let P be a point in B, then the intersection of $n-2$ general divisors of $|H|$ passing through P is a smooth surface Σ. By 1.3, $B \cap \Sigma$ has no infinitely near triple points. Therefore P is at most a triple point for B. Suppose that P a triple point for B and that after a blow up centered at P the strict transform of B has a triple point Q in correspondence of P. Consider $n-2$ general divisors in $|H|$ passing through P and tangent to the vector which corresponds to the point Q. Let Σ be the intersection of these divisors. Then Σ is smooth at P but $B \cap \Sigma$ has infinitely near triple points by construction. This contradicts 1.3. Therefore the strict transform of B cannot have a triple point.

Conversely, suppose that B has no infinitely near triple points and consider the resolution of the double cover $\sigma : \tilde{X}' \to X'$ induced by a chain of successive blow-ups which resolves the singularities of B.

If the locus of the triple points of B is nonempty we can suppose that the first blow up of the chain, namely $\mu : Y_1 \to Y$ is done along the locus of triple points of B. Let E be the exceptional locus of μ and B_1 be the strict transform of B. Let $\varrho_1 : X_1' \to Y_1$ be the double cover branched along B_1, and let $\sigma_1 : \tilde{X}_1' \to X_1'$ be a resolution of X_1'. Let Z_1 be such that $\omega_{\tilde{X}_1'} = (\varrho_1\sigma_1)^* (\omega_{Y_1} \otimes \mathscr{L}_1) \otimes \mathcal{O}_Y(-Z_1)$, which exists in consequence of the first part of the theorem.

We want to show that for any triple point in B there exists no component of Z_1 whose image by $\mu\varrho_1\sigma_1$ contains such a point. Let then P be a triple point of B and Σ be the intersection of $n-2$ general divisors in $|\mathcal{O}_Y(H)|$ passing through P. Let Σ_1 be the strict transform of Σ by μ. Then $B_1 \cap \Sigma_1$ has at most double points, because B has no infinitely near triple points by hypothesis. Hence, by 1.3 $Z_1 \cap (\varrho_1\sigma_1)^*(\Sigma_1) = \emptyset$. Thus inside E one has $(\varrho_1\sigma_1)(Z_1) \cap \Sigma_1 = \emptyset$. Therefore $(\mu\varrho_1\sigma_1)(Z_1)$ cannot contain P. This means that no component of Z_1, hence of Z, has as image a triple point of B. That is, triple points of B do not give rise to any contribute to the scheme Z of formula (1.3). Thus it remains to analyze the contribution given by the double points of B.

Let us suppose that B has at most double points. We again blow up Y along the singular locus of B. In the same way as the case of the triple points above it can be shown that for any point P double for B there exists no component of Z whose image by $(\mu\varrho_1\sigma_1)$ in Y contains P. Then $(\varrho\sigma)(Z) \cap \operatorname{Sing} B = \emptyset$, hence Z is zero. $\square$

THEOREM 1.5. *Let X be a nonsingular variety. Suppose that the canonical map φ_{K_X} is a generically finite morphism of degree 2 on a nonsingular variety Y. Let B be the branch locus of φ_{K_X}. Suppose that B is even and has no multiple components. Let $\varrho : X' \to Y$ be the double cover of Y branched along B. Then X' is the canonical model of X and there exists a birational morphism $f : X \to X'$ such that $\varphi_{K_X} = \varrho \circ f$. Moreover B has no infinitely near triple points.*

Proof. Let R be the ramification locus of φ_{K_X}, hence $B = \varphi_{K_X}(R)$. On the open subset $X \backslash R$, a map f such that $\varphi_{K_X} = \varrho \circ f$ is defined up to the involution defined by the double cover. We fix such an f, we have to show that it is everywhere defined. We note that f is an isomorphism on $X \backslash R$, hence it is a birational map between X and X'. Thus

$$p_g(X) = p_g(X') = h^0(\mathcal{O}_Y(1)). \tag{1.4}$$

Let $H \in |\mathcal{O}_Y(1)|$ be a hyperplane section of Y. Since φ_{K_X} is the canonical morphism of X, one has $K_X \sim \varphi_{K_X}^* K_Y + [R] \sim \varphi_{K_X}^*(H)$. Let $\mathscr{L} = \varphi_{K_X *}\mathcal{O}_X(R)$, then $\varphi_{K_X *}(\varphi_{K_X}^* \omega_Y \otimes \mathcal{O}_X(R)) = \varphi_{K_X *}\varphi_{K_X}^*(\mathcal{O}_Y(1))$, that is

$$\omega_Y \otimes \mathscr{L} = \mathcal{O}_Y(1) \tag{1.5}$$

Let $\sigma : \tilde{X}' \to X'$ be a resolution of X'. By theorem 1.4 and (1.5) one has

$$p_g(X') = h^0((\varrho\sigma)^*(\omega_Y \otimes \mathscr{L}) \otimes \mathcal{O}_{\tilde{X}'}(-Z)) = h^0((\varrho\sigma)^*\mathcal{O}_Y(1) \otimes \mathcal{O}_{\tilde{X}'}(-Z)). \tag{1.6}$$

Comparing with (1.4) one deduces that the divisor Z does not impose any condition to the sheaf $(\varrho\sigma)^*\mathcal{O}_Y(1)$. Therefore the singular locus of $X', \sigma(Z)$, does not impose any condition to the canonical sheaf of X'. Thus the singularities of X' are canonical, hence $Z = 0$. Moreover, again by theorem 1.4 B has no infinitely near triple points.

Then X' is a canonical variety. In fact $\omega_{X'} = \varrho^*(\omega_Y \otimes \mathscr{L}) = \varrho^*\mathcal{O}_Y(1)$ is ample, because ϱ is a finite map and X' has canonical singularities (see [10]). Hence (see [10]) X' is the canonical model of X, that is X' is isomorphic to $\mathrm{Proj} \oplus_{m \geq 0} H^0(\mathcal{O}_X(mK_X))$. Therefore the birational map $f : X \to X'$ can be thought to be a birational morphism. $\square$

In order to study threefolds X with $K_X^3 = 2p_g - 6$ having generically finite canonical map, actually generically finite of degree 2 onto a variety of minimal degree in $\boldsymbol{P}^{p_g - 1}$ by theorem 1.1, one can start from the general case, in which the variety of minimal degree is smooth. Hence by theorem 1.5 we can reduce ourselves to study the double covers whose branch locus has no infinitely triple points.

Remark. In [3] a more algebraic approach is produced to study varieties whose canonical map are finite of higher degree.

2. Varieties with $K_X^n = 2$, $p_g = n + 1$ and $K_X^n = 4$, $p_g = n + 2$

The cases $K_X^n = 2$, $p_g = n + 1$ and $K_X^n = 4$, $p_g = n + 2$ can be easily worked out for any dimension n as follows.

LEMMA 2.1. *Let X be an n-fold of general type with $n > 2$, $K_X^n = 2k$ and $p_g = n + k$ for $k = 1, 2$, whose canonical map φ_{K_X} is generically finite. Then the canonical system is base-point-free, and φ_{K_X} gives a double cover of Y, where $Y = \boldsymbol{P}^n$ if $k = 1$, or Y is a quadric if $k = 2$. Suppose that Y is nonsingular if $k = 2$. Then the branch locus is a divisor in the linear system $|\mathcal{O}_Y(2n + 6 - 2k)|$, without infinitely near triple points.*

Proof. Since φ_{K_X} is generically finite, the canonical system is base-point-free and $\varphi_{K_X} : X \to Y$ is actually finite of degree 2 (see [7], proposition (2.2)). Clearly, Y is $\boldsymbol{P}^n$ if $k = 1$, and it is a quadric of $\boldsymbol{P}^{n+1}$ if $k = 2$. In the general case the quadric is smooth, so let us suppose that Y is smooth. Hence one can use the

Hurwitz formula and obtain $K_X = \varphi_{K_X}^*(K_Y) + R = \varphi_{K_X}^*(\mathcal{O}_Y(1))$. Thus $R \in |\varphi_{K_X}^*(\mathcal{O}_Y(n+3-k))|$ and $B = \varphi_{K_X}(R) \in |\mathcal{O}_Y(2n+6-2k)|$. One can suppose that $B = B_1 + D$ where B_1 has no multiple component and $D \in |\mathcal{O}_Y(2d)|$, with $d \geq 0$.

Let now $\varrho_1 : X_1 \to Y$ be the double cover branched on B_1 and $\sigma_1 : \tilde{X}_1 \to X_1$ be the normalization. One has

$$\omega_{\tilde{X}_1} = (\varrho_1\sigma_1)^*(\omega_Y \otimes \mathcal{O}_Y(n+3-k-d)) \otimes \mathcal{O}(-Z) = (\varrho_1\sigma_1)^*(\mathcal{O}_Y(1-d) \otimes \mathcal{O}(-Z))$$

where Z is a subscheme of $\tilde{X}'$ supported on the exceptional locus of σ_1, by theorem 1.4. Thus it turns out that $d = 0$, that is $B = B_1$ has no multiple component, and the canonical morphism of $\tilde{X}_1$ factors through ϱ_1. Hence by theorem 1.5 B has no infinitely near triple points. $\square$

Let $\mathcal{M}^2_{K_X^n, p_g}(n)$ be locus of the coarse moduli space of pluriregular n-folds X with the declared K_X^n and p_g, whose canonical map has degree 2.

THEOREM 2.2. *The locus $\mathcal{M}^2_{K_X^n, p_g}(n)$ is connected, rational and generically smooth of dimension*

$$\binom{3n+4}{n} - (n+1)^2 \quad \textit{if} \quad (K_X^n, p_g) = (2, n+1)$$

or

$$\binom{3n+1}{n+1}\frac{5n+4}{2(2n+1)} - n(n+3)/2 \quad \textit{if} \quad (K_X^n, p_g) = (4, n+2).$$

Proof. Since the canonical map of the general n-fold X is in fact a degree 2 morphism, then by lemma 2.1, and with the same notations, X is a divisor of $\boldsymbol{P}(\mathcal{O}_Y \oplus \mathcal{L})$ living in a linear system which can be parametrized by the linear system of the branch loci.

Therefore, if $k = 1$ the dimension of the family of those n-folds is

$$\binom{3n+4}{n} - 1,$$

while if $k = 2$ one has to consider also the parameter space of the quadrics. Hence

$$\begin{aligned} h^0(\mathcal{O}_Q(2n+2)) &= h^0(\mathcal{O}_{\boldsymbol{P}^{n+1}}(2n+2)) - h^0(\mathcal{O}_{\boldsymbol{P}^{n+1}}(2n)) \\ &= \binom{3n+3}{n+1} - \binom{3n+1}{n+1} \end{aligned}$$

if $k = 2$. The locus $\mathcal{M}^2_{K_X^n, p_g}(n)$ is obtained by quotienting with respect to the action of the group of the automorphisms. Hence one has

$$\dim \mathcal{M}^2_{2,n+1}(n) = h^0(\mathcal{O}_{\boldsymbol{P}^n}(2n+4)) - \dim \boldsymbol{PGL}(n+1) - 1 = \binom{3n+4}{n} - (n+1)^2$$

for $k = 1$ and

$$\dim \mathscr{M}^2_{4,n+2}(n) = (n+3)(n+2)/2 + \left[\binom{3n+3}{n+1} - \binom{3n+1}{n+1}\right] - (n+2)^2 - 1$$

for $k = 2$.

Now it is sufficient to show that the dimension of $\mathscr{M}^2_{K_X^n, p_g}(n)$ coincides with $h^1(\Theta_X)$ and that

$$h^2(\Theta_X) = h^2(\varphi_*(\Theta_X)) = h^2(\Theta_Y(-\log B)) + h^2(\Theta_Y(-n-3+k)) = 0.$$

We compute $h^i(\Theta_X)$ for $i = 0, \dots, 3$ by the projection formula (1.1). We recall that since X is of general type $h^0(\Theta_X) = 0$. We can use the Euler exact sequence and in the case $k = 2$ we also use the restriction sequence

$$0 \to \Theta_{\boldsymbol{P}^{n+1}}(-n-3) \to \Theta_{\boldsymbol{P}^{n+1}}(-n-1) \to \Theta_{\boldsymbol{P}^{n+1}} \otimes \mathscr{O}_Y(-n-1) \to 0$$

where $Y = \varphi_{K_X}(X)$ is a quadric. By Bott theorem we infer that if $k = 1$ the cohomology of $\Theta_Y(-n-2)$ is zero, $k = 2$ one has

$$h^1(\Theta_Y(-n-1)) = h^2(\Theta_Y(-n-1)) = 0. \tag{2.1}$$

Hence

$$h^i(\Theta_X) = h^i(\Theta_Y(-\log B)) = h^{n-i}(\Omega^1_Y(\log B) \otimes \mathscr{O}_Y(-n-2+k))$$

for $k = 1$ and every i or for $k = 2$ and $i = 2$. By projection formula (1.1) one has $h^n(\Omega^1_Y(\log B) \otimes \mathscr{O}_Y(-n-2+k)) \le h^0(\Theta_X) = 0$. Thus the residue exact sequence (1.2) tensored by $\mathscr{O}_Y(-n-2+k)$

$$0 \to \Omega^1_Y(-n-2+k) \to \Omega^1_Y(\log B) \otimes \mathscr{O}_Y(-n-2+k) \to \mathscr{O}_B(-n-2+k) \to 0$$

Together with the restriction sequence on B

$$0 \to \mathscr{O}_Y(-2n-4+2k) \to \mathscr{O}_Y(-n-2+k) \to \mathscr{O}_B(-n-2+k) \to 0$$

imply that for $k = 1$

$$h^2(\Theta_X) = 0,$$

$$h^1(\Theta_X) = \binom{3n+4}{n} - (n+1)^2,$$

and for $k = 2$

$$h^2(\Theta_X) = 0,$$

$$h^1(\Theta_X) = \frac{(n+3)(n+2)}{2} + \binom{3n+3}{n+1} - \binom{3n+1}{n+1} - (n+2)^2 + 1. \qquad \square$$

Remark. Of course specializations of the n-folds with $K_X^n = 4$ and $p_g = n+2$ are given by double covers of quadric cones branched along the intersection with a hypersurface of degree $2n+2$, singular along the vertex of the cone.

3. The case $p_g = 7$, $K_X^3 = 8$ and canonical image is the Veronese cone

If X is a smooth threefold with $p_g = 7$, $K_X^3 = 8$ and φ_{K_X} is generically finite then $W = \varphi_{K_X}(X)$ is a threefold of minimal degree in $\boldsymbol{P}^6$. Therefore W may be either a scroll either a Veronese cone over the surface $\mathscr{V}$ in $\boldsymbol{P}^5$ with vertex a point P. We now study this latter case.

Let ψ be the projection of the cone W from the vertex P, and D' a generator of Pic $\mathscr{V}$. Let $D = \psi^*(D')$, then $\mathscr{O}_{\mathscr{V}}(2D') = \mathscr{O}_{\mathscr{V}}(1)$ and $\mathscr{O}_W(\mathscr{V}) = \mathscr{O}_W(1)$. A resolution $\tilde{W}$ of W is given by

$$\varphi_{\mathscr{H}} : \tilde{W} = \boldsymbol{P}(\mathscr{O}_{\boldsymbol{P}^2} \oplus \mathscr{O}_{\boldsymbol{P}^2}(2)) \to W$$

where $\mathscr{H}$ is the tautological bundle of $\tilde{W}$. Let $E = \varphi_{\mathscr{H}}^{-1}(P)$ be the exceptional locus. If we denote by a tilde the proper transforms of the Weil divisors on W by means of $\varphi_{\mathscr{H}}$ and denote by $\tilde{H}$ a divisor of $|\mathscr{H}|$, then Pic $\tilde{W} = \boldsymbol{Z}\tilde{D} \oplus \boldsymbol{Z}\tilde{H}$. Moreover,

$$\mathscr{O}_{\tilde{W}}(\tilde{H} - E) = \mathscr{O}_{\tilde{W}}(2\tilde{D}). \tag{3.1}$$

By applying the adjunction formula to a general divisor of $|\tilde{H}|$ one can prove that

$$\omega_{\tilde{W}} = \mathscr{O}_{\tilde{W}}(-2\tilde{H} - \tilde{D}).$$

LEMMA 3.1. *Let* $\mathrm{Aut}(\tilde{W})$ *be the group of the automorphisms of* $\tilde{W}$. *Then* $\dim \mathrm{Aut}(\tilde{W}) = 15$.

Proof. Consider the natural projection $\pi : \tilde{W} = \boldsymbol{P}(\mathscr{O}_{\boldsymbol{P}^2} \oplus \mathscr{O}_{\boldsymbol{P}^2}(2)) \to \boldsymbol{P}^2$ and the exact sequence

$$1 \to \mathrm{Aut}_{\boldsymbol{P}^2}(\tilde{W}) \to \mathrm{Aut}(\tilde{W}) \to \mathrm{Aut}\, \boldsymbol{P}^2 \to 1$$

where $\mathrm{Aut}_{\boldsymbol{P}^2}(\tilde{W})$ is the group of the automorphisms of $\tilde{W}$ which fix the fibers on $\boldsymbol{P}^2$. Then $\mathrm{Aut}_{\boldsymbol{P}^2}(\tilde{W})$ is given by the invertible polynomial matrices of type

$$\begin{pmatrix} [0] & [2] \\ 0 & [0] \end{pmatrix}$$

where the numbers inside the square brackets denote the degree of the polynomial entry. Thus

$$\begin{aligned} \dim \mathrm{Aut}(\tilde{W}) &= \dim \mathrm{Aut}_{\boldsymbol{P}^2}(\tilde{W}) + \dim \boldsymbol{P}GL(3) \\ &= (2 + 6 - 1) + 8 = 15. \end{aligned}$$

□

LEMMA 3.2. *Let* $\tilde{W} = \boldsymbol{P}(\mathscr{O}_{\boldsymbol{P}^2} \oplus \mathscr{O}_{\boldsymbol{P}^2}(2))$, *then*

$$h^3(\Omega^1_{\tilde{W}}(-2\tilde{H} - \tilde{D})) = h^0(\Theta_{\tilde{W}}) = 15,$$

$$h^i(\Omega^1_{\tilde{W}}(-2\tilde{H} - \tilde{D})) = h^{3-i}(\Theta_{\tilde{W}}) = 0 \quad \textit{for } i = 0, 1, 2.$$

Proof. Since $\Theta_{\tilde{W}/\boldsymbol{P}^2} = \mathcal{O}_{\tilde{W}}(2) \otimes \pi^* \mathcal{O}_{\boldsymbol{P}^2}(-2) = \mathcal{O}_{\tilde{W}}(2H - 2D)$ the exact sequence of the relative tangent bundle

$$0 \to \Theta_{\tilde{W}/\boldsymbol{P}^2} \to \Theta_{\tilde{W}} \to \pi^* \Theta_{\boldsymbol{P}^2} \to 0$$

is equivalent to

$$0 \to \mathcal{O}_{\tilde{W}}(-2\tilde{H} - \tilde{D}) \to \Theta_{\tilde{W}} \to \pi^* \Theta_{\boldsymbol{P}^2} \to 0. \tag{3.2}$$

By projecting on $\boldsymbol{P}^2$ one obtains that

$$h^0(\tilde{W}, \mathcal{O}_{\tilde{W}}(-2\tilde{H} - \tilde{D})) = 7,$$
$$h^i(\tilde{W}, \mathcal{O}_{\tilde{W}}(-2\tilde{H} - \tilde{D})) = 0 \quad \text{for } i = 1, 2, 3.$$

Moreover, since

$$h^0(\tilde{W}, \pi^*(\Theta_{\boldsymbol{P}^2})) = h^0(\Theta_{\boldsymbol{P}^2}) = 8,$$
$$h^i(\tilde{W}, \pi^*(\Theta_{\boldsymbol{P}^2})) = h^i(\Theta_{\boldsymbol{P}^2}) = 0 \quad \text{for } i = 1, 2, 3,$$

then

$$\begin{aligned} h^0(\Theta_{\tilde{W}}) &= h^0(\Theta_{\tilde{W}/\boldsymbol{P}^2}) + h^0(\pi^* \Theta_{\boldsymbol{P}^2}) = 7 + 8 = 15, \\ h^i(\Theta_{\tilde{W}}) &= 0 \quad \text{for } i = 1, 2, 3. \end{aligned} \tag{3.3}$$

□

THEOREM 3.3. *There exists a unirational component $\mathcal{M}^2_{8,7}$ of the moduli space of smooth minimal threefolds with $K^3 = 8$ and $p_g = 7$. For such a general threefold X the canonical system is base-point-free and the canonical morphism is of degree 2 on the cone W over the Veronese surface. The branch locus is a general divisor of the linear system $|\varphi_{\mathcal{H}*} \mathcal{O}_{\tilde{W}}(6\tilde{H} + 2\tilde{D})|$. The double cover of W branched on the same divisor is the canonical model of X. The component $\mathcal{M}^2_{8,7}$ is generically smooth of dimension 355.*

Proof. By Bertini's theorem a general divisor of the linear system

$$|\varphi_{\mathcal{H}*} \mathcal{O}_{\tilde{W}}(6\tilde{H} + 2\tilde{D})|$$

is smooth and irreducible. Thus we can consider a double cover $\varrho : X' \to \tilde{W}$ as in the statement. By theorem 1.4, denoting by $\sigma : \tilde{X}' \to X'$ a resolution of X', one has

$$\omega_{\tilde{X}'} = \sigma^* \circ \varrho^*(\mathcal{O}_{\tilde{X}'}(\tilde{H})).$$

Therefore $\varphi_{\mathcal{H}} \circ \varrho : X' \to W$ is the canonical morphism of X'. Moreover by contracting $\varrho^{-1}(E)$ we obtain the canonical model of X', which has an isolated node in the image of $\varrho^{-1}(E)$.

Since the intersection form on $\tilde{W}$ gives $\tilde{H}^3 = 4$, $\tilde{H}^2 E = \tilde{H} E^2 = 0$ and $E^3 = 4$, we have $K^3_{\tilde{X}'} = 8$ and $p_g(X') = h^0(\mathcal{O}_{\tilde{W}}(\tilde{H})) = 7$.

For a fixed cone W one has a family of threefolds parametrized by $|\mathcal{O}_{\tilde{W}}(6\tilde{H} + 2\tilde{D})|$, on which the group $\mathrm{Aut}(\tilde{W})$ of the automorphisms of $\tilde{W}$ acts. One can compute $h^0(\tilde{W}, \mathcal{O}_{\tilde{W}}(6\tilde{H} + 2\tilde{D}))$ by projecting on $\boldsymbol{P}^2$, then

$$\begin{aligned} &h^0(\tilde{W}, \mathcal{O}_{\tilde{W}}(6\tilde{H} + 2\tilde{D})) \\ &\qquad = h^0(\tilde{W}, \mathcal{O}_{\tilde{W}}(6\tilde{H}) \otimes \varphi_{\mathscr{H}}^* \mathcal{O}_{\boldsymbol{P}^2}(2)) = h^0(\boldsymbol{P}^2, \pi_* \mathcal{O}_{\tilde{W}}(6\tilde{H}) \otimes \mathcal{O}_{\boldsymbol{P}^2}(2)) \\ &\qquad = h^0(\boldsymbol{P}^2, \mathrm{Sym}^6(\mathcal{O}_{\boldsymbol{P}^2} \oplus \mathcal{O}_{\boldsymbol{P}^2}(2)) \otimes \mathcal{O}_{\boldsymbol{P}^2}(2)) = 371. \end{aligned}$$

The quotient of an open subset of $\boldsymbol{P}H^0(\tilde{W}, \mathcal{O}_{\tilde{W}}(6\tilde{H} + 2\tilde{D}))$ by the automorphism group of $\tilde{W}$ immerges itself as a subscheme, which is clearly unirational, in the moduli space $\mathscr{M}^2_{8,7}$ of threefolds with the predicted invariants. Thus by lemma 3.1 it has dimension $(371 - 1) - 15 = 355$. The unirationality and the nonsingularity of the component, say $\mathscr{M}^2_{8,7(\mathscr{V})}$, that has been selected in $\mathscr{M}^2_{8,7}$ follow once that the constructed family is proved to be an open subset of $\mathscr{M}^2_{8,7(\mathscr{V})}$. It is sufficient to show that $\dim \mathscr{M}^2_{8,7(\mathscr{V})} = h^1(\Theta_X) = 355$. The projection formula (1.1) says that

$$h^1(\Theta_X) = h^1(\Theta_{\tilde{W}}(-\log B)) + h^1(\Theta_{\tilde{W}}(-B'))$$

where $2B'$ is linearly equivalent to B, that is $B' \sim 3\tilde{H} + \tilde{D}$. Consider the exact sequence (3.2) tensored by $\mathcal{O}(-B')$

$$0 \to \mathcal{O}_{\tilde{W}}(-\tilde{H} - 3\tilde{D}) \to \Theta_{\tilde{W}}(-3\tilde{H} - \tilde{D}) \to \pi^* \Theta_{\boldsymbol{P}^2}(-3\tilde{H} - \tilde{D}) \to 0$$

projecting on $\boldsymbol{P}^2$ one can compute that

$$h^i(\tilde{W}, \mathcal{O}_{\tilde{W}}(-\tilde{H} - 3\tilde{D})) = 0 \quad \text{for } i = 0, \ldots, 3$$

hence

$$h^1(\Theta_{\tilde{W}}(-B')) = h^1(\Theta_{\tilde{W}}(-3\tilde{H} - \tilde{D})) = 0. \tag{3.4}$$

To compute $h^1(\Theta_{\tilde{W}}(-\log B)) = h^2(\Omega^1_{\tilde{W}}(\log B - 2\tilde{H} - \tilde{D}))$ one can use

$$0 \to \Omega^1_{\tilde{W}}(-2\tilde{H} - \tilde{D}) \to \Omega^1_{\tilde{W}}(\log B - 2\tilde{H} - \tilde{D}) \to \mathcal{O}_B(-2\tilde{H} - \tilde{D}) \to 0. \tag{3.5}$$

One has

$$\begin{aligned} &h^2(\mathcal{O}_B(-2\tilde{H} - \tilde{D})) \\ &\qquad = h^2(\mathcal{O}_B(K_{\tilde{W}})) = h^0(\mathcal{O}_B(B)) \\ &\qquad = h^0(\mathcal{O}_{\tilde{W}}(B)) - 1 = 370. \end{aligned} \tag{3.6}$$

By projection formula (1.1) one also has

$$h^3(\Omega^1_{\tilde{W}}(\log B - 2\tilde{H} - \tilde{D})) = h^0(\Theta_{\tilde{W}}(-\log B)) \le h^0(\Theta_X)$$

but since X is of general type $h^0(\Theta_X) = 0$, hence

$$h^3(\Omega^1_{\tilde{W}}(\log B - 2\tilde{H} - \tilde{D})) = 0.$$

Thus from lemma 3.2, from (3.6) and from the exact sequence (3.5), one gets

$$\begin{aligned} h^1(\Theta_{\tilde{W}}(-\log B)) &= h^2(\Omega^1_{\tilde{W}}(-2\tilde{H}-\tilde{D})) \\ &= h^2(\mathcal{O}_B(-2\tilde{H}-\tilde{D})) - h^3(\Omega^1_{\tilde{W}}(-2\tilde{H}-\tilde{D})) \\ &= 370 - 15 = 355. \end{aligned}$$

Then

$$\begin{aligned} h^1(\Theta_X) &= h^1(\Theta_{\tilde{W}}(-\log B)) + h^1(\Theta_{\tilde{W}}(-B')) \\ &= h^1(\Theta_{\tilde{W}}(-\log B)) = 355. \end{aligned}$$

□

Remark. The moduli space of the threefolds with $p_g = 7$ and $K^3 = 8$ has another component, whose generic point represents a threefold with base-point-free canonical system and canonical morphism of degree 2 over a scroll. This one falls in the general case which is studied in the next section. These two irreducible components have no intersection, because there exists no deformation of a scroll to a Veronese cone.

4. The general case

Let us suppose from now on that $p_g > 5$ and $W = \varphi(X) = \varphi_{K_X}(X)$ is a rational normal scroll, let $K_X^3 = 2\mu$ and $\mu = p_g - 3 = \deg W$. Denote by (a_1, a_2, a_3) the type of the scroll W (see [4]), then $0 \le a_1 \le a_2 \le a_3$ and $a_1 + a_2 + a_3 = \mu = p_g - 3$. W is the image of the birational morphism

$$\varphi_{\mathscr{H}} : \tilde{W} = \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1}(a_1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_2) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_3)) \to W$$

defined by the tautological bundle $\mathscr{H}$ on the abstract scroll $\tilde{W}$. The morphism $\varphi_{\mathscr{H}}$ is an isomorphism if and only if $a_1 \ge 0$. We recall that if L is the general 2-dimensional projective space which is the fiber of the projection

$$\pi : W \to \boldsymbol{P}^1$$

then

$$\operatorname{Pic} \tilde{W} = \boldsymbol{Z}L \oplus \boldsymbol{Z}H$$

$$\omega_{\tilde{W}} = \mathcal{O}_{\tilde{W}}(-3H + (\mu - 2)L). \tag{4.1}$$

We also recall that the intersection form on the scrolls is given by

$$H^3 = \mu, \quad H^2L = 1, \quad HL^2 = L^3 = 0. \tag{4.2}$$

By theorem 1.5 we can reduce ourselves to recover the threefold X as a double cover of $\tilde{W}$ with branch locus $\tilde{B}$ with no infinitely near triple points. On the other hand by theorem 1.4, since the double cover has to be defined by the linear system $|\omega_X|$, by comparing (1.3) with (4.1) we deduce that $\tilde{B}$ has to be a general divisor in $|\mathcal{O}_{\tilde{W}}(8H - 2(\mu - 2)L)|$. Therefore in what follows we study the singularities of the divisors in $|\mathcal{O}_{\tilde{W}}(8H - 2(\mu - 2)L)|$. For this purpose it is useful to distinguish the cases in which two consecutive a_i are equal or not.

It is known that the linear system $|\mathcal{O}_{\tilde{W}}(H - a_1L)|$ of the abstract projective bundle $\tilde{W}$ defines a birational morphism to a cone whose vertex has dimension 1 if $a_1 < a_2$, dimension 2 if $a_1 = a_2$.

Let S be a general surface in $|\mathcal{O}_{\tilde{W}}(H - a_1L)|$, we study the singularities of the divisors $|\mathcal{O}_{\tilde{W}}(8H - 2(\mu - 2)L)|$ restricted to S.

LEMMA 4.1. *Let $\tilde{W} = \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1}(a_1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_2) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_3))$ and $\mu = a_1 + a_2 + a_3$. The linear system $|\mathcal{O}_{\tilde{W}}(8H - 2(\mu - 2)L)|$ has no surface $\bar{S}$ belonging to $|\mathcal{O}_{\tilde{W}}(H - a_1L)|$ as fixed component.*

Proof. Let $\bar{S}$ be a given surface in $|\mathcal{O}_{\tilde{W}}(H - a_1L)|$. Let x_i be a fixed section of $\mathcal{O}_{\tilde{W}}(H - a_iL)$ in $\tilde{W}$ for $i = 1, 2, 3$, such that $\bar{S}$ has equation $x_1 = 0$.

One can write the equation of the general element of $|\mathcal{O}_{\tilde{W}}(8H - 2(\mu - 2)L)|$ as

$$G = \sum_{i,j \geq 0,\, i+j \leq 8} g_{ij} x_1^{8-i-j} x_2^i x_3^j \tag{4.3}$$

$$= g_{00}x_1^8 + x_1^7(g_{10}x_2 + g_{01}x_3) + \cdots + (g_{80}x_2^8 + \cdots + g_{08}x_3^8) \tag{4.4}$$

where g_{ij} are homogeneous forms of degree $(8 - i - j)a_1 + ia_2 + ja_3 - 2(\mu - 2)$ on $\boldsymbol{P}^1$. Note that if $i + j = 8$ then

$$\begin{aligned} \deg g_{80} &= 8a_2 - 2(\mu - 2) \\ &\leq \deg g_{ij} = ia_2 + ja_3 - 2(\mu - 2) \\ &\leq \deg g_{08} = 8a_3 - 2(\mu - 2). \end{aligned}$$

The monomial x_1 divides G if and only if $g_{ij} = 0$ for $i + j = 8$, hence x_1 divides the general G in $|\mathcal{O}_{\tilde{W}}(8H - 2(\mu - 2)L)|$ if and only if $8a_3 - 2(\mu - 2) < 0$, that is $4a_3 < \mu - 2$ hence $3a_3 < a_1 + a_2 - 2$, but this inequality never holds. $\square$

4.1. The case $a_2 = a_3$.

THEOREM 4.2. *Let $\tilde{W} = \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1}(a_1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_2) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_3))$.*

(a) *When $a_1 = a_2 = a_3$ the linear system $|\mathcal{O}_{\tilde{W}}(8H - 2(\mu - 2)L)|$ is base-point-free and the general section is nonsingular.*

(b) *When $a_1 < a_2 = a_3$ the general element of the linear system $|\mathcal{O}_{\tilde{W}}(8H - 2(\mu - 2)L)|$ has no infinitely near triple points if and only if*

$$5a_1 \leq 2a_2 + 2 \quad \text{and} \quad a_2 \leq 3a_1 + 4. \tag{4.5}$$

In particular, if $a_1 = 0$ the condition (4.5) is true only for the following types (a_1, a_2, a_3):

$$\begin{matrix} (0,1,1) & (0,2,2) \\ (0,3,3) & (0,4,4). \end{matrix}$$

Proof. (a) If $a_1 = a_2 = a_3 = 1$ then $\mu = 3$, $\tilde{W} = W$ is a projective scroll on the rational normal cubic curve and the branch locus $\tilde{B}$ belongs to the linear system $|\mathcal{O}_{\tilde{W}}(2(4H - L))|$, whose general element is irreducible and nonsingular, since $|\mathcal{O}_{\tilde{W}}(4H - L)|$ has dimension greater than 1 and its general element is irreducible and nonsingular.

If $a_1 = a_2 = a_3 = a > 1$, $\mu = 3a$, then the system $|\mathcal{O}_{\tilde{W}}(8H - 2(\mu - 2)L)|$ may be thought as twice $|\mathcal{O}_{\tilde{W}}(4H - (3a - 2)L)|$. This is the sum of divisors of type

$$4(H - (a-1))L \quad \text{and} \quad (a-2)L.$$

The former is very ample, this can be seen by considering equivalently 4 times the tautological bundle of the scroll $\boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1}(a-1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a-1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a-1))$, the latter is a pencil, so the conclusion follows from Bertini's theorem.

(b) Let $a_1 < a_2 = a_3$, and $\mu = a_1 + 2a_2$. We recall that the general element S of $|H - a_1 L|$ is a 2-dimensional scroll of type $(a_2 - a_1, a_2 - a_1)$.

Denote by l_1 and l_2 two generators of the pencils of lines of a nonsingular quadric Q in $\boldsymbol{P}^3$, then $\varphi_H(S)$ is the image of Q via the morphism f defined by the linear system $|l_1 + (a_2 - a_1)l_2|$. Moreover, in $\tilde{W}$ one has $(H - a_2 L)^3 = -(a_2 - a_1)$, while the linear system $|\mathcal{O}_{\tilde{W}}(H - a_2 L)|$ is a pencil having a base curve γ. One has

$$S\gamma = (H - a_1 L)(H - a_2 L)^2 = 0$$

and the birational morphism defined by $|\mathcal{O}_{\tilde{W}}(H - a_1 L)|$ contracts γ to the vertex of the cone.

Since

$$f^*(H|_S) = l_1 + (a_2 - a_1)l_2 \quad \text{and} \quad f^*(L|_S) = l_2$$

then

$$\begin{aligned} f^*(\tilde{B}) &= f^*((8H - 2(\mu - 2)L)_S) \\ &= 8(l_1 + (a_2 - a_1)l_2) - 2(\mu - 2)l_2) \\ &= 8l_1 + (-10a_1 + 4a_2 + 4)l_2. \end{aligned}$$

For $h^0(S, \mathcal{O}_S(\tilde{B})) \neq 0$ it is necessary that $5a_1 \leq 2a_2 + 2$. Consider the equation (4.3) of the general element of $|\mathcal{O}_{\tilde{W}}(8H - 2(\mu - 2)L)|$

$$G = g_{00}x_1^8 + x_1^7(g_{10}x_2 + g_{01}x_3) + \cdots + (g_{80}x_2^8 + \cdots + g_{08}x_3^8). \tag{4.6}$$

The homogeneous polynomials g_{ij} such that $i + j = k$ have degree $(8 - k)a_1 + ka_2 - 2(\mu - 2) = (6 - k)a_1 + (k - 4)a_2 + 4$ on $\boldsymbol{P}^1$. If $3a_1 - a_2 + 4 < 0$ then the g_{ij} terms with $i + j = k \leq 3$ are zero and the curve γ of equation $x_2 = x_3 = 0$, base locus of the linear system $|\mathcal{O}_{\tilde{W}}(H - a_2 L)|$, would be contained in the base locus of $|\mathcal{O}_{\tilde{W}}(8H - 2(\mu - 2)L)|$ with multiplicity at least 4.

Conversely, suppose that the conditions (4.5) are true. We distinguish 4 cases, by the type of the scroll.

(b1) When $6a_1 - 4a_2 + 4 \geq 0$ the linear system $|\mathcal{O}_{\tilde{W}}(8H - 2(\mu - 2)L)|$ may be thought as the sum of $|\mathcal{O}_{\tilde{W}}(8(H - a_1 L))|$ and $|\mathcal{O}_{\tilde{W}}((6a_1 - 4a_2 + 4)L)|$. The first one is of positive dimension and is not composed with a pencil, the second one is base-point-free. By Bertini's theorem the general element of $|\mathcal{O}_{\tilde{W}}(8H - 2(\mu - 2)L)|$ is smooth.

Suppose now that $6a_1 - 4a_2 + 4 < 0$ and let P be a point of a general $\tilde{B}$ in $|\mathcal{O}_{\tilde{W}}(8H - 2(\mu - 2)L)|$. If it does not lie on γ then one can suppose that $x_3(P) \neq 0$ and take affine coordinates for (4.6) near the point P. For $w_j = x_j/x_3$ one has

$$\begin{aligned} G &= \sum_{i,j \geq 0,\, i+j \leq 8} g_{ij} w_1^{8-i-j} w_2^i \\ &= g_{08} + (g_{07} w_1 + g_{17} w_2) + \cdots + (g_{00} w_1^8 + g_{10} w_1^7 w_2 + \cdots + g_{80} w_2^8) \end{aligned}$$

and $\deg g_{08} = 8a_3 - 2(\mu - 2) = 2(2a_2 - a_1 + 2)$. By (4.5) $\deg g_{08} \geq 0$, therefore P is a smooth point for $\tilde{B}$. It remains to study the multiplicity of the points in γ.

(b2) When $6a_1 - 4a_2 + 4 < 0$ and $5a_1 - 3a_2 + 4 \geq 0$ then $g_{00} = 0$ and the curve γ is in the base locus of $|8H - 2(\mu - 2)L|$ with multiplicity 1.

(b3) When $5a_1 - 3a_2 + 4 < 0$ and $4a_1 - 2a_2 + 4 \geq 0$, $g_{00} = g_{10} = g_{01} = 0$ then the curve γ is the base locus for $|\mathcal{O}_{\tilde{W}}(8H - 2(\mu - 2)L)|$ with multiplicity 2: the general element of $|\mathcal{O}_{\tilde{W}}(8H - 2(\mu - 2)L)|$ has γ as locus of double points.

(b4) Finally, when $4a_1 - 2a_2 + 4 < 0$, since by hypothesis also $3a_1 - a_2 + 4 \geq 0$, the g_{ij} are zero if $i + j = k \leq 2$ and are nonzero if $i + j = 3$. Hence γ is a base locus for $|\mathcal{O}_{\tilde{W}}(8H - 2(\mu - 2)L)|$ with multiplicity exactly 3, that is, the general element $\tilde{B}$ of $|\mathcal{O}_{\tilde{W}}(8H - 2(\mu - 2)L)|$ has γ as locus of triple points and it is smooth outside γ.

We need to know the kind of singularity of $\tilde{B}$ at a point P of γ. By the genericity of $\tilde{B}$ one can suppose that the polynomials g_{ij} such that $i + j = 3$ do not have common zeroes, then any point of γ has multiplicity exactly 3 for $\tilde{B}$. We look at an affine neighbour of $\tilde{B}$: if $z_j = x_j/x_1$ the local equation of $\tilde{B}$ is

$$G = (g_{30} z_2^3 + g_{03} z_3^3 + g_{12} z_2 z_3^2 + g_{21} z_2^2 z_3) + g_{40} z_2^4 + \cdots$$

The degree 3 part is the tangent cone to $\tilde{B}$ at the point P.

If $3a_1 - a_2 + 4 = 0$ then the polynomials g_{ij} are constant if $i + j = 3$. Therefore for a general $\tilde{B}$ the tangent cone is a nonsingular cubic and P is an ordinary triple point. If $3a_1 - a_2 + 4 \geq 0$ then the polynomials g_{ij} are nonconstant along γ for $i + j = 3$ (γ is a nonsingular rational curve). Again, one can suppose that for the general $\tilde{B}$ the tangent cone to $\tilde{B}$ in P is an irreducible cubic, showing that P is an ordinary triple point. $\square$

Corollary 4.3. *Let $a_1 \leq a_2 = a_3$, there exists a double cover $\varrho : X' \to W(a_1, a_2, a_3)$ branched on a divisor $\tilde{B}$ in $|8H - 2(\mu - 2)L|$ such that X' has at most canonical singularities. Such a threefold is of general type and its minimal desingularization $\sigma : X \to X'$ is such that $K_X^3 = 2p_g - 6$ and that $\varrho \circ \sigma$ is its canonical morphism.*

Proof. The statement follows from theorems 1.4 and 4.2, from (4.1), and from the intersection form on the scrolls (4.2) as soon as $h^0(\mathcal{O}_{\tilde{W}}(8H-2(\mu-2)L)) > 0$, which will be proved in the next section. $\square$

4.2. The case $a_2 < a_3$. Suppose now that $a_2 < a_3$. A smooth surface S in $|H - a_1 L|$ is a scroll of type $(a_2 - a_1, a_3 - a_1)$, having a section of minimal self-intersection $\delta_0 = H - a_3 L|_S$, the self-intersection being $-(a_3 - a_2)$. We recall that the section of minimal self-intersection δ_0 is a fixed curve in S. Denote by δ the linear system $H - a_2 L|_S$, then δ is base-point-free, and has self-intersection $a_3 - a_2$ and dimension $a_2 + a_3 + 1$.

LEMMA 4.4. *Let $a_2 < a_3$ and $\mu = a_1 + a_2 + a_3$, suppose $\varrho : X \to \tilde{W}(a_1, a_2, a_3)$ is a double cover branched on a divisor of $|\mathcal{O}_{\tilde{W}}(8H - 2(\mu-2)L)|$. Then X has at most canonical singularities if and only if*

$$a_2 \geq (a_3 + 2a_1 - 4)/5.$$

In particular, if $a_1 = 0$ then $5a_2 \geq a_3 - 4$, if $a_1 = a_2 = 0$ then $\mu = 3$ or 4.

Proof. Suppose that $\tilde{B}$ is nonreduced. Since $|\mathcal{O}_{\tilde{W}}(H - a_1 L)|$ is base-point-free a general surface S of $|\mathcal{O}_{\tilde{W}}(H - a_1 L)|$ intersects $\tilde{B}$ along a nonreduced curve $\tilde{B}_S = S \cap \tilde{B}$, which is branch locus for a degree 2 morphism. Therefore $\tilde{B}_S$ does not contain δ_0 with multiplicity greater than 1, thus $(\tilde{B}_S - \delta_0) \cdot \delta_0 \geq 0$ that is

$$(H - a_1 L)(H - a_3 L)(8H - 2(\mu - 2)L) \geq \delta_0^2 = -(a_3 - a_2)$$

The conclusion follows from the values of the intersection form. $\square$

Let $a_2 < a_3$ and $\tilde{W} = \tilde{W}(a_1, a_2, a_3)$. Denote by Δ_0 the unique divisor of the linear system $|\mathcal{O}_{\tilde{W}}(H - a_3 L)|$. Δ_0 is a 2-dimensional scroll of type (a_1, a_2). Let $\delta_{0,0}$ be the curve of minimal self-intersection of Δ_0, it is cut out on Δ_0 by the linear system $|\mathcal{O}_{\tilde{W}}(H - a_2 L)|$. One has

THEOREM 4.5. *Suppose that $a_2 < a_3$, $\tilde{W} = \tilde{W}(a_1, a_2, a_3)$, $\mu = a_1 + a_2 + a_3$. Denote by Bs the base locus of $|\mathcal{O}_{\tilde{W}}(8H - 2(\mu - 2)L)|$. Then*

(a) *$Bs = \emptyset$ if and only if $8a_1 - 2\mu + 4 \geq 0$.*

In this case the general element of $|\mathcal{O}_{\tilde{W}}(8H - 2(\mu-2)L)|$ is irreducible and nonsingular.

(b) *When $8a_1 - 2\mu + 4 < 0$ and $8a_2 - 2\mu + 4 \geq 0$ the support of Bs is the curve $\delta_{0,0}$ and the multiplicity of $\delta_{0,0}$ in Bs is less than or equal 3. The general $\tilde{B}$ in $|\mathcal{O}_{\tilde{W}}(8H - 2(\mu-2)L)|$ has no infinitely near triple points if and only if $3a_1 - 2a_2 + a_3 + 4 \geq 0$.*

(c) *When $8a_2 - 2\mu + 4 < 0$ and $7a_2 - 2\mu + 4 + a_3 \geq 0$ the support of Bs is the surface Δ_0; any element in $|\mathcal{O}_{\tilde{W}}(8H - 2(\mu-2)L)|$ is reduced in $\overline{B} + \Delta_0$.*

The base locus of $|\mathcal{O}_{\tilde{W}}(\overline{B})|$ is empty if and only if $7a_1 + a_3 - 2\mu + 4 = 5a_1 - 2a_2 - a_3 + 4 \geq 0$.

If $7a_1 + a_3 - 2\mu + 4 < 0$ *then the base locus of* $|\mathcal{O}_{\tilde{W}}(\bar{B})|$ *is the curve* $\delta_{0,0}$.

In this case, a surface $\bar{B}$ *has at most double point along* $\delta_{0,0}$ *if and only if* $3a_1 - 2a_2 + a_3 + 4 \geq 0$. *Then* $\bar{B}$ *has no infinitely near triple points if and only if* $3a_1 - 2a_2 + a_3 + 4 \geq 0$.

Proof. (a) The system $|\mathcal{O}_{\tilde{W}}(H - a_1L)|$ on $\tilde{W}$ is base-point-free and not composed with a pencil. Let us decompose

$$8H - 2(\mu - 2)L = 8(H - a_1L) + (-2\mu + 4 + 8a_1)L.$$

Since $8a_1 - 2\mu + 4 \geq 0$ the linear system $|\mathcal{O}_{\tilde{W}}(8H - 2(\mu - 2)L)|$ is a sum of two linear systems which are base-point-free and such that one of them is not composed with a pencil. Hence by Bertini's theorem its general element is smooth and irreducible.

Conversely, let $\tilde{B}$ be a divisor in $|\mathcal{O}_{\tilde{W}}(8H - 2(\mu - 2)L)|$. Then

$$\tilde{B}\delta_{0,0} = (8H - 2(\mu - 2)L)(H - a_2L)(H - a_3L) \tag{4.7}$$

$$= 8a_1 - 2\mu + 4. \tag{4.8}$$

Therefore if $8a_1 < 2\mu + 4$ the curve $\delta_{0,0}$ is a fixed locus for the linear system $|\tilde{B}|$.

(b) Recall the equation (4.3) of the general surface $\tilde{B}$ in $|\mathcal{O}_{\tilde{W}}(8H - 2(\mu - 2)L)|$, and note that the degree of the homogeneous form g_{ij} on $\boldsymbol{P}^1$ is $(8 - i - j)a_1 + ia_2 + ja_3 - 2(\mu - 2)$. In particular when $i + j = k$

$$\begin{aligned}(8 - k)a_1 + ka_2 - 2(\mu - 2) &= \deg g_{k0} \leq \deg g_{ij} \leq \deg g_{0k} \\ &= (8 - k)a_1 + ka_3 - 2(\mu - 2).\end{aligned}$$

Note that $\delta_{0,0}$ has equations $x_2 = x_3 = 0$.

The proof goes as in that of theorem 4.2. Since by hypothesis $8a_1 - 2\mu + 4 < 0$ and $8a_2 - 2\mu + 4 \geq 0$, the term g_{00} is zero and $\delta_{0,0}$ is in the base locus for the linear system $|\mathcal{O}_{\tilde{W}}(8H - 2(\mu - 2)L)|$. By passing to the affine open subset $\{x_3 \neq 0\}$, one can see that if P is a point of $\tilde{B}$ not lying on $\delta_{0,0}$ then it cannot be a base point for $|\mathcal{O}_{\tilde{W}}(8H - 2(\mu - 2)L)|$, thus it is a smooth point for $\tilde{B}$.

If $3a_1 - 2a_2 + a_3 + 4 < 0$ then g_{ij} is zero for $i + j \leq 3$ and $\delta_{0,0}$ lies with multiplicity at least 4 in $\tilde{B}$.

Conversely, if $3a_1 - 2a_2 + a_3 + 4 \geq 0$ then g_{03} is nonzero, thus the multiplicity of $\delta_{0,0}$ in $\tilde{B}$ is less or equal to 4. In this case by the genericity of the forms g_{ij} one can deduce that there are no infinitely near triple points for $\tilde{B}$ along $\delta_{0,0}$.

(c) When $8a_2 - 2\mu + 4 < 0$ and $7a_2 - 2\mu + 4 + a_3 \geq 0$, g_{i0} is zero for any i hence x_3 divides G and (4.3) becomes

$$G = x_3(g_{01}x_1^7 + x_1^6(g_{11}x_2 + g_{02}x_3) + \cdots + g_{08}x_3^7)$$

Δ_0 is a fixed component of the linear system $|8H - 2(\mu - 2)L|$. Let us consider the linear system

$$|\mathcal{O}_{\tilde W}(\bar B)| := |\mathcal{O}_{\tilde W}(8H - 2(\mu-2)L - \Delta_0)| \\ = |\mathcal{O}_{\tilde W}(7H + (2\mu + 4 + a_3)L)|.$$

Proceeding as in the previous case one deduces that the base locus of $|\mathcal{O}_{\tilde W}(\bar B)|$ is empty if and only if $7a_1 + a_3 - 2\mu + 4 = 5a_1 - 2a_2 - a_3 + 4 \geq 0$.

Moreover, if $7a_1 + a_3 - 2\mu + 4 = 5a_1 - 2a_2 - a_3 + 4 < 0$ the base locus of $|\mathcal{O}_{\tilde W}(\bar B)|$ is the curve $\delta_{0,0}$. In this case a surface $\bar B$ has at most $\delta_{0,0}$ as locus of ordinary double points if and only if the term g_{03} is nonzero, hence if and only if $3a_1 - 2a_2 + a_3 + 4 \geq 0$. Therefore if g_{ij} are general the tangent locus to $\bar B$ in a point of $\delta_{0,0}$ is different from the tangent locus to Δ_0 in the same point. □

As in the previous case we can conclude with the following

Corollary 4.6. *Let $a_1 \leq a_2 < a_3$. There exists a double cover $\varrho : X' \to W(a_1, a_2, a_3)$ branched on a surface $\tilde B$ in $|8H - 2(\mu-2)L|$, X' having at most canonical singularities. Such a threefold is of general type, and its minimal desingularization $\sigma : X \to X'$ is such that $K_X^3 = 2p_g - 6$ and $\varrho \circ \sigma$ is its canonical morphism.*

Remark. By theorems 4.2 and 4.5, for fixed values of K_X^3 and p_g not all the scrolls of degree $K_X^3/2$ in $\boldsymbol{P}^{p_g-1}$ occur. Moreover even if $\tilde B$ is reducible (see case (c) in theorem 4.5) it is always connected, while this is not always the case for the surfaces (see [5]).

4.3. The dimension of the moduli space.

Lemma 4.7. *Let $W \subset \boldsymbol{P}^{\mu+2}$ be nonsingular. Then*

$$h^0(W, \mathcal{O}_W(8H - 2(\mu-2)L)) = 30\mu + 225 + h^1(W, \mathcal{O}_W(8H - 2(\mu-2)L)),$$
$$h^2(W, \mathcal{O}_W(8H - 2(\mu-2)L)) = h^3(W, \mathcal{O}_W(8H - 2(\mu-2)L)) = 0.$$

Proof. Consider the exact sequence

$$(4.9) \qquad 0 \to \mathcal{O}_W(8H - 2(\mu-2)L) \to \mathcal{O}_W(8H) \xrightarrow{\rho} \mathcal{O}_{2(\mu-2)L}(8H) \to 0$$

Since W is nonsingular an element of $|\mathcal{O}_W(2(\mu-2)L)|$ is formed by $2(\mu-2)$ disjoint planes. Since the scrolls are projectively of Cohen-Macaulay the dimension of the linear system cut by the hypersurfaces of the projective space depend only on the degree. Therefore we choose a particular scroll to compute $h^0(W, \mathcal{O}_W(8H))$ and make use of the projection on $\boldsymbol{P}^1$

$$\pi : \tilde W = \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1}(a_1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_2) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_3)) \to \boldsymbol{P}^1.$$

In fact

$$\begin{aligned} h^0(W, \mathcal{O}_W(8H)) &= h^0(\tilde{W}, \mathcal{O}_{\tilde{W}}(8H)) \\ &= h^0(\boldsymbol{P}^1, \pi_*(\mathcal{O}_{\tilde{W}}(8H)) \\ &= h^0(\boldsymbol{P}^1, [\mathrm{Sym}^8(\mathcal{O}_{\boldsymbol{P}^1}(a_1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_2) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a_3))]). \end{aligned}$$

One can do the computation in the 3 cases (a, a, a), $(a, a, a+1)$, $(a, a+1, a+1)$, depending on the class of μ modulo 3, by projecting on $\boldsymbol{P}^1$. In all cases one has $h^0(\tilde{W}, \mathcal{O}_{\tilde{W}}(8H)) = 120\mu + 45$. Moreover by the projective normality of W $h^1(\mathcal{O}_W(8H)) = 0$ thus by the exact sequence of restriction (4.9) one has

$$\begin{aligned} &h^0(W, \mathcal{O}_W(8H - 2(\mu-2)L)) \\ &\quad = h^0(W, \mathcal{O}_W(8H)) - h^0(\mathcal{O}_{2(\mu-2)L}(8H)) + h^1(W, \mathcal{O}_W(8H - 2(\mu-2)L)) \\ &\quad = 30\mu + 225 + h^1(W, \mathcal{O}_W(8H - 2(\mu-2)L)). \qquad \square \end{aligned}$$

Remark. One can also compute $h^1(W, \mathcal{O}_W(8H - 2(\mu-2)L)) = h^1(\tilde{W}, \mathcal{O}_{\tilde{W}}(8H - 2(\mu-2)L))$ by projection. Hence

$$h^1(\tilde{W}, \mathcal{O}_{\tilde{W}}(8H - 2(\mu-2)L)) = (-8a_1 + 2\mu + 3)^+ + \cdots + (-8a_3 + 2\mu + 3)^+.$$

Thus

$$(4.10) \quad h^1(\tilde{W}, \mathcal{O}_{\tilde{W}}(8H - 2(\mu-2)L)) = 0 \quad \text{if and only if} \quad 8a_1 - 2(\mu-2) \geq -2$$

(note that the condition $8a_1 - 2(\mu-2) \geq -2$ is equivalent to $6a_1 - 2a_2 - 2a_3 + 6 \geq 0$). Thus the dimension of $|\mathcal{O}_{\tilde{W}}(8H - 2(\mu-2)L)|$ jumps if $8a_1 - 2(\mu-2) < -2$.

Let $g : \mathscr{W} \to \mathscr{B}$ be a family of scrolls in $\boldsymbol{P}^{\mu+2}$ of degree μ such that any element is of type (a_1, a_2, a_3), where $8a_1 - 2(\mu-2) \geq -2$.

By (4.10), the sheaf $\mathscr{Y}$ defined on $\mathscr{W}$ such that when restricted to any fiber W of g is $\varphi_{\mathscr{H}*}\mathcal{O}_{\tilde{W}}(8H - 2(\mu-2)L)$ is flat and gives rise to a unique family of canonical double covers of the scrolls $g : \mathscr{X} \to \mathscr{B}$.

Theorem 4.8. *Let $\mu = a_1 + a_2 + a_3$ and $8a_1 - 2(\mu-2) \geq -2$, let $W = W(a_1, a_2, a_3)$ be a rational nonsingular normal scroll of $\boldsymbol{P}^{\mu+2}$. Then there exists a family of isomorphism classes of nonsingular threefolds with $p_g = \mu + 3$ and $K_X^3 = 2p_g - 6$ and canonical map of degree 2 onto W. Such a family is unirational and has dimension $30\mu + 213$.*

Proof. For a fixed μ and for any scroll W compatible with the conditions given in theorems 4.2 and 4.5 there exists a family of double covers of W with $K^3 = 2\mu$ and $p_g = \mu + 3$ parametrized by an open subset of $\boldsymbol{P}H^0(\tilde{W}, \mathcal{O}_{\tilde{W}}(8H - 2(\mu-2)L))$. One obtains the required family by moding out the automorphisms group of W. When W is nonsingular this is a unirational family whose parameter space is generically smooth, by (4.10). By lemma 4.7 the dimension of the family is $\mathrm{fdim} = 30\mu + 224 - \dim \mathrm{Aut}(W)$. We note that the maximum value for fdim is achieved when W is of type (a, a, a), $(a, a, a+1)$, or $(a, a+1, a+1)$. In fact we can compute $\dim \mathrm{Aut}(W)$ by the exact sequence

$$1 \to \mathrm{Aut}_{\boldsymbol{P}^1}(\tilde{W}) \to \mathrm{Aut}(\tilde{W}) \to \mathrm{Aut}\, \boldsymbol{P}^1 \to 1 \tag{4.11}$$

where $\mathrm{Aut}_{\boldsymbol{P}^1}(\tilde{W})$ is the group of the automorphisms of $\tilde{W}$ fixing the fibers of p, as in the Veronese case.

In any of the three cases one has $\dim \mathrm{Aut}_{\boldsymbol{P}^1}(\tilde{W}) = 8$, hence

$$\dim \mathrm{Aut}(\tilde{W}) = \dim \mathrm{Aut}_{\boldsymbol{P}^1}(\tilde{W}) + \dim \mathrm{Aut}(\boldsymbol{P}^1) = 11.$$

Therefore one has $\mathrm{fdim} = 30\mu + 224 - \dim \mathrm{Aut}(W) = 30\mu + 213$. □

With an additional hypothesis on the a_i's, it can be shown that the above family fills an open subset of a reduced component $\mathscr{M}$ of the moduli space $\mathscr{M}^2_{2\mu,\mu+3}$. It is sufficient to show that

$$\dim \mathscr{M} = h^1(\Theta_X) = 30\mu + 213. \tag{4.12}$$

Again, one can use the projection formula (see (1.1))

$$h^i(\Theta_X) = h^i(\Theta_{\tilde{W}}(-\log \tilde{B})) + h^i(\Theta_{\tilde{W}}(-\tilde{B}')) \tag{4.13}$$

where $2\tilde{B}' \sim \tilde{B}$.

Lemma 4.9. Let $\tilde{B}' \in |\mathscr{O}_{\tilde{W}}(4H - 2(\mu-2)L)|$ then

$$h^i(\Theta_{\tilde{W}}(-\tilde{B}')) = 0 \quad \textit{for } i = 1, 2, 3.$$

Proof. One has

$$h^i(\mathscr{O}_{\tilde{W}}(-\tilde{B}')) = h^{3-i}(\mathscr{O}_{\tilde{W}}(K_{\tilde{W}} + \tilde{B}')) \tag{4.14}$$

$$= h^{3-i}(\mathscr{O}_{\tilde{W}}(H)) = \begin{cases} \mu + 3 & \text{for } i = 3 \\ 0 & \text{for } i = 0, 1, 2 \end{cases} \tag{4.15}$$

$$h^i(\mathscr{O}_{\tilde{W}}(H - a_k L - \tilde{B}')) = h^{3-i}(\mathscr{O}_{\tilde{W}}(H - (H - a_k L))) \tag{4.16}$$

$$= h^{3-i}(\mathscr{O}_{\tilde{W}}(a_k L)) = \begin{cases} a_k + 1 & \text{for } i = 3 \\ 0 & \text{for } i = 0, 1, 2. \end{cases} \tag{4.17}$$

It is known that if $\tilde{W}$ is a scroll of type (a_1, a_2, a_3) and π is the projection on $\boldsymbol{P}^1$, then

$$0 \to \Theta_{\tilde{W}|\boldsymbol{P}^1} \to \Theta_{\tilde{W}} \to \pi^* \Theta_{\boldsymbol{P}^1} \to 0, \tag{4.18}$$

$$0 \to \mathscr{O}_{\tilde{W}} \to \mathscr{O}_{\tilde{W}}(H - a_1 L) \oplus \mathscr{O}_{\tilde{W}}(H - a_2 L) \oplus \mathscr{O}_{\tilde{W}}(H - a_3 L) \tag{4.19}$$

$$\to \Theta_{\tilde{W}|\boldsymbol{P}^1} \to 0. \tag{4.20}$$

From (4.14), (4.16) and (4.19) tensored by $\mathscr{O}(-\tilde{B}')$ one has

$$h^i(\Theta_{\tilde{W}/\boldsymbol{P}^1}(-\tilde{B}')) = 0 \quad \text{for } i = 0, 1. \tag{4.21}$$

Since one has

$$(4.22)\quad h^i(\tilde{W}, (\pi^*\Theta_{\boldsymbol{P}^1})(-\tilde{B}')) = h^i(\boldsymbol{P}^1, \Theta_{\boldsymbol{P}^1} \otimes \pi_*\mathcal{O}_{\tilde{W}}(-\tilde{B}')) = 0 \quad \text{for } i = 2, 3$$

and by duality one also has

$$(4.23)\quad \begin{aligned} & h^1(\tilde{W}, (\pi^*\Theta_{\boldsymbol{P}^1})\mathcal{O}_{\tilde{W}}(-\tilde{B}')) \\ & \quad = h^1(\boldsymbol{P}^1, \Theta_{\boldsymbol{P}^1} \otimes \pi_*(-\tilde{B}')) = h^0(\boldsymbol{P}^1, \mathcal{O}_{\boldsymbol{P}^1}(2H - \pi_*(\tilde{B}')) = 0. \end{aligned}$$

The conclusion follows comparing (4.22), (4.23), (4.21) with (4.18) tensored by $\mathcal{O}_{\tilde{W}}(-\tilde{B}')$. □

From (4.18) and (4.19) it also follows that (with the same notation as above)

(4.24)

$$\begin{aligned} & h^0(\Theta_{\tilde{W}}) = 2(a_3 - a_1)^+ + 8 + (a_1 - a_2 + 1)^+ + (a_1 - a_3 + 1)^+ + (a_2 - a_3 + 1)^+, \\ & h^1(\Theta_{\tilde{W}}) = (a_2 - a_1 - 1)^+ + (a_3 - a_1 - 1)^+ + (a_3 - a_2 - 1)^+, \\ & h^2(\Theta_{\tilde{W}}) = 2(a_3 - a_1) = h^3(\Theta_{\tilde{W}}) = 2(a_3 - a_1)^+ = 0. \end{aligned}$$

In particular one has

$$(4.25)\qquad h^1(\Theta_{\tilde{W}}) = 0 \quad \text{if and only if } a_3 - a_1 \leq 2.$$

Lemma 4.10. *If $a_3 - a_1 \leq 2$ then*

$$(4.26)\qquad h^i(\Theta_{\tilde{W}}(-\log \tilde{B})) = \begin{cases} 30\mu + 213 & \textit{for } i = 1 \\ 0 & \textit{for } i = 0, 2, 3. \end{cases}$$

Proof. Computations can be carried out by using the residue exact sequence

$$(4.27)\qquad 0 \to \Theta_{\tilde{W}}(-\log \tilde{B}) \to \Theta_{\tilde{W}} \to \mathcal{O}_{\tilde{B}}(\tilde{B}) \to 0$$

and the restriction sequence

$$0 \to \mathcal{O}_{\tilde{W}} \to \mathcal{O}_{\tilde{W}}(\tilde{B}) \to \mathcal{O}_{\tilde{B}}(\tilde{B}) \to 0.$$

By lemma 4.7 one has

$$h^i(\mathcal{O}_{\tilde{W}}(\tilde{B})) = \begin{cases} 30\mu + 225 & \text{for } i = 0 \\ 0 & \text{for } i = 1, 2, 3 \end{cases}$$

hence, since $\tilde{W}$ is rational,

$$(4.28)\qquad h^i(\mathcal{O}_{\tilde{B}}(\tilde{B})) = \begin{cases} 30\mu + 224 & \text{for } i = 0 \\ 0 & \text{for } i = 1, 2. \end{cases} \quad .$$

Moreover since the double cover is of general type one has $h^0(\Theta_{\tilde{W}}(-\log \tilde{B})) = 0$. Thus the conclusion follows from (4.27), (4.25) and (4.28). □

We note that if $a_3 - a_1 \leq 2$ the conditions required by theorems 4.2 and 4.5 are satisfied. By (4.13), lemma 4.9 and lemma 4.10 it can be deduced the following theorem.

THEOREM 4.11. *There exists a unirational component of the moduli space* $\mathcal{M}^2_{(K^3, p_g)} = \mathcal{M}^2_{2\mu, \mu+3}$ *of the threefolds with* $K^3 = 2p_g - 6$ *whose general element has a smooth minimal model, a base-point-free canonical system and a canonical morphism* φ *of degree* 2 *on a nonsingular scroll* W *of* $\boldsymbol{P}^{p_g - 1}$. *The class of the branch locus of* φ *is* $|\mathcal{O}_{\tilde{W}}(8H - 2(\mu - 2)L))|$.

Such a component is reduced of dimension $30\mu + 213$.

Remark. In the theorem only smooth scrolls with $a_3 - a_1 \leq 2$ are used. But by theorems 4.2 and 4.5 also cones without the condition $a_3 - a_1 \leq 2$ are allowed. In this case the generically finite canonical morphism $\varphi : X \to W$ factors through $\tilde{W}$.

Whenever one has a flat family $\mathcal{W}$ of deformations of W inside $\boldsymbol{P}^{\mu+2}$ such that the sheaf $\mathcal{O}_W(8H - 2(\mu - 2)L)$ extends to a flat sheaf on $\mathcal{W}$ one also has a flat family of deformation of double covers, which belongs to the same irreducible component of the moduli space $\mathcal{M}^2_{2\mu, \mu+3}$. This happens for instance if the type of W is such that $h^1(W, \mathcal{O}_W(8H - 2(\mu - 2)L)) = 0$.

On the other hand, in general, there are many different irreducible components of $\mathcal{M}^2_{2\mu, \mu+3}$, even of dimension greater than $30\mu + 213$. In fact different components come out when one considers double covers of cones. An example is proposed in the following theorem.

THEOREM 4.12. *Let* $W = W(0, a, a)$ *(or* $W_1 = W(0, a, a+1)$*) be a cone with vertex a point* P, *let* $\mu = 2a$ *(or* $\mu = 2a + 1$*) and suppose* $a > 4$. *Then* $h^0(W, \mathcal{O}_W(8H - 2(\mu - 2)L)) = h^0(W_1, \mathcal{O}_{W_1}(8H - 2(\mu - 2)L)) = 40\mu + 175$.

There exists a family of isomorphism classes of nonsingular threefolds X *such that* $K_X^3 = 2\mu = 2p_g - 6$ *and canonical morphism of degree* 2 *on* W *(or* W_1*). This family is unirational of dimension* $39\mu + 167$.

Proof. Let us suppose that $\mu = 2a$, then

$$\begin{aligned}
&h^0(\tilde{W}, \mathcal{O}_{\tilde{W}}(8H - 2(\mu - 2)L)) \\
&\quad = h^0(\boldsymbol{P}^1, \mathrm{Sym}^8[\mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(a) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a))] \otimes \mathcal{O}_{\boldsymbol{P}^1}(-2(\mu - 2))) \\
&\quad = h^0(\boldsymbol{P}^1, [\oplus_{k=0}^8 \mathcal{O}_{\boldsymbol{P}^1}(ka)^{\oplus(k+1)}] \otimes \mathcal{O}_{\boldsymbol{P}^1}(-4a + 4) \\
&\quad = h^0(\boldsymbol{P}^1, \oplus_{k=4}^8 \mathcal{O}_{\boldsymbol{P}^1}((k - 4)a + 4))^{\oplus(k+1)}) \\
&\quad = \sum_{k=4}^{8}((k - 4)a + 5)(k + 1)) = 40\mu + 175.
\end{aligned}$$

The hypothesis of theorem 4.2 are satisfied, hence the existence of the family is assured. With this choice of the type of the cone one obtains the minimal dimension of $\mathrm{Aut}(\tilde{W})$, which can be computed as in the nonsingular case. One has $\dim \mathrm{Aut}(\tilde{W}) = 2a + 7 = \mu + 7$, thus the dimension of the family of the double covers up to isomorphisms is $40\mu + 175 - 1 - (\mu + 7) = 39\mu + 167$.

In the same way, for $\mu = 2a + 1$ and $W_1 = W(0, a, a+1)$ one has

$$\begin{aligned} &h^0(\tilde{W}_1, \mathcal{O}_{\tilde{W}_1}(8H - 2(\mu - 2)L)) \\ &\quad = h^0(\boldsymbol{P}^1, \mathrm{Sym}^8[\mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(a) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a+1))] \otimes \mathcal{O}_{\boldsymbol{P}^1}(-4a + 2)) \\ &\quad = 40\mu + 175. \end{aligned}$$

Again $\dim \mathrm{Aut}(\tilde{W}) = 2a + 8 = \mu + 7$, and the dimension of the family of the double covers up to isomorphisms is as above. □

Remark. Either for μ even or μ odd the dimension of the family of the double covers up to isomorphisms is greater than the dimension of the reduced component of the moduli space $\mathcal{M}^2_{2\mu, \mu+3}$ produced in theorem 4.11. Therefore this new family belongs to a new component of the moduli space.

References

[1] W. Barth, C. Peters, A. Van de Ven, Compact Complex Surfaces, Ergebnisse der Math. (3), 4, Springer Verlag, 1984.

[2] A. Beauville, L'application canonique pour les surfaces de type général, Invent. Math. **55** (1979), 121–140.

[3] G. Casnati, P. Supino, Construction of threefolds with finite canonical map, Glasgow Math. J. **44** (2002), 65–79.

[4] J. Harris, A bound on the geometric genus of projective varieties, Ann. Sc. Norm. Super. Pisa **8** (1981), 35–68.

[5] E. Horikawa, Algebraic surfaces of general type with small c_1^2, I, Annals of Math. **104** (1976), 357–387; II, Invent. Math. **37** (1976), 121–155.

[6] E. Horikawa, On deformations of Quintic surfaces, Invent. Math. **31** (1975), 43–85.

[7] M. Kobayashi, On Noether inequality for 3-folds, J. Math. Soc. Japan **44** (1992), 145–156.

[8] Y. Miyaoka, The Chern classes and Kodaira dimension of a minimal variety, in Algebraic Geometry, Sendai Adv. Stud. Pure Math., vol. 10, T. Oda (ed.) (1987), 449–476.

[9] R. Pardini, Abelian covers of algebraic varieties, J. Reine angew. Math. **417** (1991), 191–213.

[10] M. Reid, Canonical 3-folds, Geometrie Algebrique d'Angers, A. Beauville ed. (1979), 273–310.

[11] E. Viehweg, Weak positivity of certain Hilbert points. II, Invent. Math. **101** (1990), 191–223.

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