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ON THE REGULAR LEAF SPACE OF THE CAULIFLOWER

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Abstract

We construct a pinching semiconjugacy from a quadratic polynomial $f_c(z) = z^2 + c$ with $c \in (0, 1/4)$ to $f_{1/4}(z) = z^2 + 1/4$ on the sphere. By lifting this semiconjugacy to their natural extensions, we investigate the structure of the regular leaf space of $f_{1/4}$ in detail.

1. Introduction

As an analogy to hyperbolic 3-orbifolds associated with Kleinian groups, Lyubich and Minsky [1] introduced hyperbolic orbifold 3-laminations associated with rational maps. For a given rational map of degree ≥ 2, considering its natural extension and regular leaf space is the first step to the construction of such a hyperbolic orbifold 3-lamination.

However, the global structures of the regular leaf spaces of rational maps are not precisely known except only a few examples. Here is one of such examples. For $f_c(z) = z^2 + c$ with c in the main cardioid of the Mandelbrot set, all regular leaf spaces of f_c are topologically the same as that of $f_0(z) = z^2$, which is 2-dimensional extension of 2-adic solenoid [2, Example 2] [1, §11].

Now we may expect the simplest deformation of these regular leaf spaces as c tends to $1/4$ along the real axis. Then the dynamics inside the Julia sets degenerate from "hyperbolic" to "parabolic", though the dynamics on and outside the Julia sets are still topologically the same as f_0. In this paper, we describe the structure of the regular leaf space of $f_{1/4}(z) = z^2 + 1/4$, whose Julia set is called *the cauliflower*, by using a pinching semiconjugacy from f_c with $c \in (0, 1/4)$ to $f_{1/4}$. We will see that the transversal structure of those regular leaf spaces are preserved, however, the dynamics on the invariant leaves corresponding to the fixed points are change from "hyperbolic" to "parabolic".

In this section, we first survey some basic notion of complex dynamics of quadratic maps and their natural extensions. In §2, we construct *tessellation* (or *tiling*) of the interior of the filled Julia sets of f_c with $c \in (0, 1/4]$ in a dynamically natural way. In §3, we construct a pinching semiconjugacy from f_c with $c \in (0, 1/4)$ to $f_{1/4}$ by gluing tile-to-tile homeomorphisms and the conjugacy

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on and outside the Julia sets. In §4, we lift such a semiconjugacy to their regular leaf spaces and describe their degeneration.

1.1. Preliminaries

The Julia set. Let us set $f_c(z) = z^2 + c$ $(c \in \boldsymbol{C})$ and consider it as a rational map on the Riemann sphere $\bar{\boldsymbol{C}} = \boldsymbol{C} \cup \{\infty\}$ with $f_c(\infty) = \infty$. The *filled Julia set* K_c of f_c is defined by

$$K_c := \{z \in \bar{\boldsymbol{C}} : \{f_c^n(z)\}_{n=0}^{\infty} \text{ is bounded}\}.$$

The *Julia set* J_c of f_c is the boundary of K_c. One can easily check that those sets are forward and backward invariant under the action of f_c.

Now suppose that K_c is connected. (Thus so is J_c.) We denote the unit disk by $\boldsymbol{D}$. For the outside of K_c, there exists a unique conformal map $\phi_c : \bar{\boldsymbol{C}} - K_c \to \bar{\boldsymbol{C}} - \bar{\boldsymbol{D}}$ such that

- $\phi_c(f_c(z)) = \phi_c(z)^2$; and
- $\phi_c(z)/z \to 1$ as $z \to \infty$.

Moreover, if J_c is a Jordan curve, ϕ_c continuously extends to $\bar{\phi}_c : \bar{\boldsymbol{C}} - K_c^{\circ} \to \bar{\boldsymbol{C}} - \boldsymbol{D}$.

Now let us restrict our interest to the case of $c \in (0, 1/4]$. In our particular situation, it is known that J_c are Jordan curves. Thus by looking through the map $\bar{\phi}_c$, the dynamics on and outside the Julia sets are topologically the same as $z \mapsto z^2$. For $\theta \in \boldsymbol{R}/\boldsymbol{Z}$, set $\gamma_c(\theta) := \bar{\phi}_c^{-1}(e^{2\pi i \theta})$. Then points on J_c are parameterized by angles in $\boldsymbol{R}/\boldsymbol{Z}$. See [3, §18] for more details.

We define some more notation:

- $\alpha_c := (1 - \sqrt{1-4c})/2$ which is the attracting (or parabolic iff $c = 1/4$) fixed point of f_c with multiplier $\lambda_c = 1 - \sqrt{1-4c}$.
- $\beta_c := (1 + \sqrt{1-4c})/2$ which is the repelling (or parabolic iff $c = 1/4$) fixed point of f_c with multiplier $\lambda_c' = 1 + \sqrt{1-4c}$.

The natural extension. Next we follow [1, §3]. For f_c as above, let us consider the set of all possible backward orbits

$$\mathcal{N}_c := \{\hat{z} = (z_0, z_{-1}, \ldots) : z_0 \in \bar{\boldsymbol{C}}, f_c(z_{-n-1}) = z_{-n}\}.$$

This set is called the *natural extension* of f_c, and is equipped with a topology from $\bar{\boldsymbol{C}} \times \bar{\boldsymbol{C}} \times \cdots$. On this natural extension, the lift of f_c and a natural projection are defined by

$$\hat{f}_c(\hat{z}) := (f_c(z_0), z_0, z_{-1}, \ldots) \quad \text{and}$$
$$\pi_c(\hat{z}) := z_0.$$

It is clear that $\hat{f}_c$ is a homeomorphism, and satisfies $\pi_c \circ \hat{f}_c = f_c \circ \pi_c$. For a fixed point $a \in \bar{\boldsymbol{C}}$ of f_c, set $\hat{a} := (a, a, \ldots) \in \mathcal{N}_c$.

The regular leaf space. An element $\hat{z} = (z_0, z_{-1}, \ldots) \in \mathcal{N}_c$ is *regular* if there exists a neighborhood U_0 of z_0 such that its pull-back U_{-n} along the backward orbit $\hat{z}$ are eventually univalent. For example, $\hat{\infty} = (\infty, \infty, \dots)$ is *not* regular for any $c \in \boldsymbol{C}$.

Let $\mathscr{R}_c$ denote the set of regular points in $\mathscr{N}_c$. $\mathscr{R}_c$ is called the *regular leaf space* of f_c. A *leaf* of $\mathscr{R}_c$ is a path connected component of $\mathscr{R}_c$. By [1, Lemma 3.1], leaves of $\mathscr{R}_c$ are Riemann surfaces:

LEMMA 1.1. *Leaves of $\mathscr{R}_c$ have the following properties*:
- *For each leaf L, we can introduce a complex structure such that $\pi_c : L \to \bar{\boldsymbol{C}}$ is an analytic map.*
- *π_c branches at $\hat{z} = (z_0, z_{-1}, \dots)$ if and only if $\hat{z}$ contains a critical point in $\{z_{-n}\}$.*
- *$\hat{f}_c$ maps a leaf to a leaf isomorphically.*

This lemma holds for any $c \in \boldsymbol{C}$. In our case, we have:

PROPOSITION 1.2. *If $c \in (0, 1/4]$, $\mathscr{R}_c$ has the following properties*:
- $\mathscr{R}_c = \mathscr{N}_c - \{\hat{\infty}, \hat{\alpha}_c\}$.
- *Each leaf of $\mathscr{R}_c$ is isomorphic to $\boldsymbol{C}$.*

This proposition is immediate from lemmas in [1, §3].

2. Making tiles

For the first step, we will decompose the interior K_c° of K_c with $c \in (0, 1/4]$ into countably many tiles and describe their relations.

Before making tiles, we introduce some notation. For $c \in (0, 1/4]$, let $I_c(0)$ denote the interval $[\alpha_c, \beta_c] \subset \boldsymbol{R}$. In particular, $I_{1/4}(0) = \{1/2\}$. Since $I_c(0)$ contains no critical orbit, preimages of $I_c(0)$ by f_c are univalently spread out with one of their endpoints on the Julia set. Set $\mathscr{I}_c := \bigcup_{i \ge 0} f_c^{-i}(I_c(0))$. For $\theta \in \boldsymbol{Q}/\boldsymbol{Z}$ of the form $k/2^n$, we denote the connected component of $\mathscr{I}_c$ containing $\gamma_c(\theta)$ by $I_c(\theta)$. As $c \to 1/4$, $\mathscr{I}_c$ degenerates into $\mathscr{I}_{1/4}$ which is the grand orbit of the parabolic fixed point $1/2$.

2.1. Tiles of K_c° with $c \in (0, 1/4)$

Suppose $c \in (0, 1/4)$. Then α_c is an attracting fixed point and K_c° is its attracting basin. On a neighborhood of α_c, there exists a linearizing coordinate Φ_c which analytically conjugates the action of f_c near α_c to $w \mapsto \lambda_c w$ near the origin. Moreover, we can extend this map to $\Phi_c : K_c^\circ \to \boldsymbol{C}$, and it is unique up to multiplication by a constant [3].

For the purpose of better understanding the difference between the cases of $c \in (0, 1/4)$ and $c = 1/4$, let us take an additional conjugation by $w \mapsto w + 1/(1 - \lambda_c)$ on $\boldsymbol{C}$. Then we can uniquely define $\Phi_c : K_c^\circ \to \boldsymbol{C}$ such that
- $\Phi_c(f_c(z)) = \lambda_c \Phi_c(z) + 1$;
- $\Phi_c(\alpha_c) = 1/(1 - \lambda_c)$, $\Phi_c(0) = 0$; and
- Φ_c is infinitely branched covering whose branch points are $\bigcup_{i \ge 0} f_c^{-i}(\{0\})$.

Now let us set $a_c := 1/(1 - \lambda_c)$, which tends to $+\infty$ as $c \to 1/4$.

For $m \in \boldsymbol{Z}$, set

$$A_c(m,+) := \{w \in \boldsymbol{C} : \lambda_c^{m+1} a_c \le |w - a_c| \le \lambda_c^m a_c, \operatorname{Im} w \ge 0\}$$
$$A_c(m,-) := \{w \in \boldsymbol{C} : \lambda_c^{m+1} a_c \le |w - a_c| \le \lambda_c^m a_c, \operatorname{Im} w \le 0\}$$

and we call them the *fundamental semi-annuli*. Let $\mathscr{A}_c$ denote the collection of all of the fundamental semi-annuli.

Note the following two facts:

- The vertices of fundamental semi-annuli on the interval $(-\infty, a_c)$ are the images of the grand orbit of 0. In particular, all of the ramified points (critical values) of Φ_c are on the interval $(-\infty, 0]$.
- All components of $\mathscr{I}_c \cap K_c^\circ$ are mapped univalently onto the interval $[a_c, \infty)$.

For the boundary of $A_c(m, \pm)$, we call the edge on the interval $(-\infty, a_c)$ (resp. $[a_c, \infty)$) the *critical edge* (*resp. degenerating edge*). We call the edges shared by $A_c(m-1, \pm)$ or $A_c(m+1, \pm)$ the *circular edges*.

By the facts above, $\Phi_c^{-1} : \boldsymbol{C} - (-\infty, 0] \to K_c^\circ$ is a multi-valued function with univalent branches. Such a branch $\Psi_c : \boldsymbol{C} - (-\infty, 0] \to K_c^\circ$ determines a unique angle $\theta \in \boldsymbol{Q}/\boldsymbol{Z}$ of the form $k/2^n$ such that $\Psi_c([a_c, \infty)) \cup \gamma_c(\theta) = I_c(\theta)$. Thus for $A_c(m,+) \in \mathscr{A}_c$, Ψ_c also determines a unique simply connected set $T = T_c(\theta, m, +) \subset K_c^\circ$ such that T is the closure of $\Psi_c(A_c(m,+) - (-\infty, 0])$. We call such a T a *tile*, and the triple $(\theta, m, +)$ the *address* of the tile T. We also define $T_c(\theta, m, -)$ in the same way. We denote the collection of all possible T by $\mathscr{T}_c$, and call it the *tessellation* of K_c°. Now it is clear that K_c is the closure of $\bigcup\{T \in \mathscr{T}_c\}$.

For each $T \in \mathscr{T}_c$, Φ_c maps T to an $A \in \mathscr{A}_c$ homeomorphically. For the boundary of T, *critical* (*resp. degenerating*, *circular*) edges are defined by the boundary arcs corresponding to the critical (resp. degenerating, circular) edges of A.

Now we give relations among tiles:

Proposition 2.1. *For $T = T_c(\theta, m, +) \in \mathscr{T}_c$, we have the following properties*:

(1) $f_c(T) = T_c(2\theta, m+1, +)$. *Moreover*, $f_c^{-1}(T) = T_c(\theta/2, m-1, +) \cup T_c(\theta/2 + 1/2, m-1, +)$.

(2) *T shares the circular edges with $T_c(\theta, m-1, +)$ and $T_c(\theta, m+1, +)$.*

(3) *T shares the degenerating edge with $T_c(\theta, m, -)$.*

(4) *T shares the critical edge with $T_c(\theta + 2^m, m, -)$.*

The similar holds if we replace T by $T_c(\theta, m, -)$. In particular, $T_c(\theta, m, -)$ shares the critical edge with $T_c(\theta - 2^m, m, +)$.

Proof. The first three properties come from the definition of tiles. One can easily check property (4) in the case of $\theta = 0$. For general θ of the form $k/2^n \in \boldsymbol{Q}/\boldsymbol{Z}$, suppose that $T = T_c(\theta, m, +)$ shares the critical edge with $T' \in \mathscr{T}_c$. Then T' is the form $T_c(\theta', m, -)$ since Φ_c maps T and T' to $A_c(m,+)$ and $A_c(m,-)$ respectively. Now f^n sends T univalently to $T_c(0, m+n, +)$ which shares the critical edge with $T_c(2^{m+n}, m+n, -)$. Thus $f^n(T') = T_c(2^{m+n}, m+n, -)$. Note that $f^n(T \cup T')$ joins $I_c(0)$ and $I_c(2^{m+n})$. By pulling it back by a suitable univalent branch of f^n, $T \cup T'$ must join $I_c(\theta)$ and $I_c(\theta + 2^m)$. This implies $\theta' = \theta + 2^m$. ■

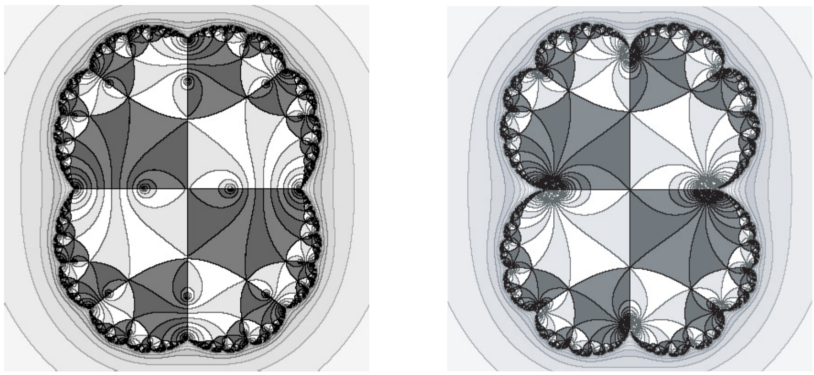

FIGURE 1. The tessellations of $z^2 + 0.2$ and $z^2 + 1/4$

2.2. Tiles of $K_{1/4}^{\circ}$

Next we make the tessellation $\mathscr{T}_{1/4}$ for $K_{1/4}^{\circ}$ in the same way as above. Now $\alpha_{1/4} = 1/2$ is the parabolic fixed point and $K_{1/4}^{\circ}$ is its parabolic basin. On an attracting petal of $\alpha_{1/4}$, there exists a Fatou coordinate $\Phi_{1/4}$ which analytically conjugates the action of $f_{1/4}$ to $w \mapsto w + 1$. Moreover, we can uniquely extend this Fatou coordinate to $\Phi_{1/4} : K_{1/4}^{\circ} \to \boldsymbol{C}$ such that

- $\Phi_{1/4}(f_{1/4}(z)) = \Phi_{1/4}(z) + 1$;
- $\Phi_{1/4}(0) = 0$; and
- $\Phi_{1/4}$ is infinitely branched covering whose branch points are $\bigcup_{n \geq 0} f_{1/4}^{-n}(\{0\})$.

(See [3] again.)

For $m \in \boldsymbol{Z}$, set

$$A_{1/4}(m, +) := \{w \in \boldsymbol{C} : m \leq \operatorname{Re} w \leq m + 1, \operatorname{Im} w \geq 0\}$$

$$A_{1/4}(m, -) := \{w \in \boldsymbol{C} : m \leq \operatorname{Re} w \leq m + 1, \operatorname{Im} w \leq 0\}$$

and we call them the *fundamental semi-cylinders*. Let $\mathscr{A}_{1/4}$ denote the collection of all of the fundamental semi-cylinders.

Note the following two facts, and compare with the case of $c \in (0, 1/4)$:

- The vertices of fundamental semi-cylinders on the real axis $(-\infty, \infty)$ are the images of the grand orbit of 0. In particular, all of the ramified points (critical values) of $\Phi_{1/4}$ are on the interval $(-\infty, 0]$.
- All components of $\mathscr{I}_{1/4}$ are outside of the domain of $\Phi_{1/4}$.

For the boundary of $A_{1/4}(m, \pm)$, we call the edge on the real axis the *critical edge*. We also call the edges shared by $A_{1/4}(m - 1, \pm)$ or $A_{1/4}(m + 1, \pm)$ the *circular edges*. Note that $A_{1/4}(m, \pm)$ has no edges corresponding to degenerating edges of fundamental semi-annuli.

Now $\Phi_{1/4}^{-1} : \boldsymbol{C} - (-\infty, 0] \to K_{1/4}^{\circ}$ is a multi-valued function with univalent branches. Such a branch $\Psi_{1/4} : \boldsymbol{C} - (-\infty, 0] \to K_{1/4}^{\circ}$ determines a unique angle $\theta \in \boldsymbol{Q}/\boldsymbol{Z}$ of the form $k/2^n$ such that $I_{1/4}(\theta) \in \mathscr{I}_{1/4}$ is the limit of $\Psi_{1/4}(t)$ as t tends to $+\infty$ along the real axis. Thus for $A_{1/4}(m,+) \in \mathscr{A}_{1/4}$, $\Psi_{1/4}$ maps the interior of $A_{1/4}(m,+)$ to a simply connected subset of $K_{1/4}^{\circ}$. Since $\Psi_{1/4}$ extends continuously to the boundary of $A_{1/4}(m,+)$, we denote its image by $T = T_{1/4}(\theta, m, +) \subset K_{1/4}^{\circ}$. We also define $T_{1/4}(\theta, m, -)$ in the same way. We denote the collection of all possible T by $\mathscr{T}_{1/4}$, and call it the *tessellation* of $K_{1/4}^{\circ}$. Now it is also clear that $K_{1/4}$ is the closure of $\bigcup\{T \in \mathscr{T}_{1/4}\}$.

For each $T \in \mathscr{T}_{1/4}$, Φ_c maps T to an $A \in \mathscr{A}_{1/4}$ homeomorphically. For the boundary of T, *critical* (*resp. circular*) *edges* are defined by the boundary edges corresponding to the critical (resp. circular) edges of A. Note that $T_{1/4}(\theta, m, \pm)$ does not contain the point $I_{1/4}(\theta) \in J_{1/4}$, and has no edges corresponding to degenerating edges.

The relations among tiles are given in the same way as Proposition 2.1:

Proposition 2.2. *For $T = T_{1/4}(\theta, m, +) \in \mathscr{T}_{1/4}$, we have the following properties*:

(1) $f_{1/4}(T) = T_{1/4}(2\theta, m+1, +)$. *Moreover*, $f_{1/4}^{-1}(T) = T_{1/4}(\theta/2, m-1, +) \cup T_{1/4}(\theta/2 + 1/2, m-1, +)$.

(2) *T shares the circular edges with $T_{1/4}(\theta, m \pm 1, +)$.*

(3) *For any $n \in \boldsymbol{Z}$, $\overline{T}$ shares a point $I_{1/4}(\theta) \in \mathscr{I}_{1/4}$ with $\overline{T_{1/4}(\theta, n, +)}$ and $\overline{T_{1/4}(\theta, n, -)}$.*

(4) *T shares the critical edge with $T_{1/4}(\theta + 2^m, m, -)$.*

The similar holds if we replace T by $T_{1/4}(\theta, m, -)$.

3. Construction of the semiconjugacy

Here we construct a semiconjugacy which pinches $\mathscr{I}_c$ to $\mathscr{I}_{1/4}$:

Theorem 3.1. *For $c \in (0, 1/4)$, there exists a semiconjugacy $H_c : \bar{\boldsymbol{C}} \to \bar{\boldsymbol{C}}$ from f_c to $f_{1/4}$ such that*

- *H_c maps $\bar{\boldsymbol{C}} - \mathscr{I}_c$ to $\bar{\boldsymbol{C}} - \mathscr{I}_{1/4}$ homeomorphically and is a topological conjugacy between $f_c \,|\, (\bar{\boldsymbol{C}} - \mathscr{I}_c)$ and $f_{1/4} \,|\, (\bar{\boldsymbol{C}} - \mathscr{I}_{1/4})$;*
- *For each $\theta = k/2^n \in \boldsymbol{Q}/\boldsymbol{Z}$, H_c maps an arc $I_c(\theta)$ onto a point $I_{1/4}(\theta)$.*

Proof. The rest of this section is devoted to the proof of this theorem. Let us fix $c \in (0, 1/4)$.

Conjugation on tiles. First we make a homeomorphism between $A_c(0,+) - [a_c, \infty)$ (a tile minus its degenerating edge) and $A_{1/4}(0,+)$. Take $w \in A_c(0,+) - [a_c, \infty)$, and set $re^{it} := w - a_c$ with $(a_c - 1 =)\lambda_c a_c \le r \le a_c$ and $0 < t \le \pi$. Now set

$$h_c(w) := (a_c - r) + i \tan \frac{\pi - t}{2} \in A_{1/4}(0,+).$$

Then h_c is a homeomorphism between $A_c(0,+) - [a_c, \infty)$ and $A_{1/4}(0,+)$ which preserves the action of $S_c(w) = \lambda_c w + 1$ on the (outer) circular edge of $A_c(0,+) - [a_c, \infty)$ to that of $S_{1/4}(w) = w + 1$ on the (left) circular edge of $A_{1/4}(0,+)$.

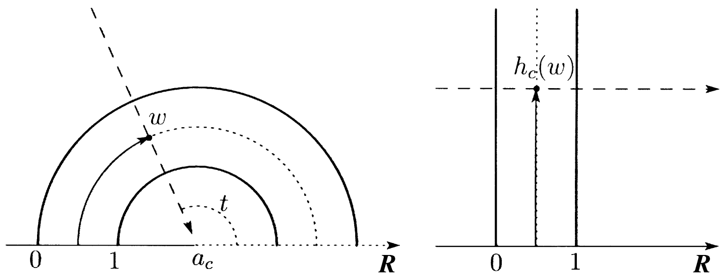

FIGURE 2. $h_c : A_c(0,+) \to A_{1/4}(0,+)$

By using this property, for any $m \in \boldsymbol{Z}$, we can extend h_c to $A_c(m,+) - [a_c, \infty)$ by

$$h_c : A_c(m,+) - [a_c, \infty) \to A_{1/4}(m,+)$$
$$w \mapsto h_c(S_c^{-m}(w)) + m.$$

Then h_c gives a homeomorphism between $A_c(m,+) - [a_c, \infty)$ and $A_{1/4}(m,+)$. Similarly, we define h_c on $A_c(m,-) - [a_c, \infty)$. Then we obtain a homeomorphism $h_c : \boldsymbol{C} - [a_c, \infty) \to \boldsymbol{C}$, moreover, h_c is a topological conjugacy between $S_c \,|\, (\boldsymbol{C} - [a_c, \infty))$ and $S_{1/4}$.

Next, let us take θ of the form $k/2^n \in \boldsymbol{Q}/\boldsymbol{Z}$, and take $m \in \boldsymbol{Z}$. We define a map between $T_c(\theta, m, +) - I_c(\theta)$ (a tile minus its degenerating edge) and $T_{1/4}(\theta, m, +)$ as follows: There is a unique branch $\Psi_{1/4}^{\theta}$ of $\Phi_{1/4}^{-1}$ which maps the interior of $A_{1/4}(m,+)$ to the interior of $T_{1/4}(\theta, m, +)$. It is clear that $\Psi_{1/4}^{\theta}$ extends continuously to the boundary of $A_{1/4}(m,+)$. For $z \in T_c(\theta, m, +) - I_c(\theta)$, set

$$H_c : T_c(m,+) - I_c(\theta) \to T_{1/4}(\theta, m, +)$$
$$z \mapsto \Psi_{1/4}^{\theta} \circ h_c \circ \Phi_c(z).$$

This definition gives a homeomorphism $H_c : K_c^{\circ} - \mathscr{I}_c \to K_{1/4}^{\circ}$, moreover, by the definition and the combinatorics of tiles, H_c is a topological conjugacy between $f_c \,|\, (K_c^{\circ} - \mathscr{I}_c)$ and $f_{1/4}|K_{1/4}^{\circ}$.

Continuous extension. For $\theta \in \boldsymbol{Q}/\boldsymbol{Z}$ of the form $k/2^n$, set $H_c(I_c(\theta)) := I_{1/4}(\theta) = \gamma_{1/4}(\theta)$. Then $H_c : K_c^{\circ} \cup \mathscr{I}_c \to K_{1/4}^{\circ} \cup \mathscr{I}_{1/4}$ is a semiconjugacy which pinches the arcs in $\mathscr{I}_c$ to the points in $\mathscr{I}_{1/4}$. *Now we claim that we can continuously extend this H_c to $H_c : K_c \to K_{1/4}$.* Fix a point $\zeta \in J_c$. Since J_c is a Jordan curve, there exists an angle $\theta \in \boldsymbol{R}/\boldsymbol{Z}$ such that $\zeta = \gamma_c(\theta)$. Let $z_n \in K_c^{\circ} \cup \mathscr{I}_c$ be a sequence converging to ζ. We show that $w_n := H_c(z_n) \in K_{1/4}^{\circ} \cup \mathscr{I}_{1/4}$ converges to $\gamma_{1/4}(\theta)$.

Take a small interval of angle $[t, t']$ containing θ, where t and t' are of the forms $(2k-1)/2^m$ and $(2k+1)/2^m$ respectively with the same k and $m \gg 0$. Then $\gamma_c(t)$ and $\gamma_c(t')$ bound a small piece of J_c, and the piece, say J_c', is a Jordan arc containing ζ. Take an open arc $C \subset K_c^\circ$ such that C only passes though tiles of angles in $[t, t']$ and $\bar{C} \cap J_c = \{\gamma_c(t), \gamma_c(t')\}$. (See Remark 2 below.) Let V denote the small open set with $\partial V = C \cup J_c'$. By the definition of H_c, $\overline{H_c(V)} \cap J_{1/4} =: J_{1/4}'$ is a piece of $J_{1/4}$ which is a small Jordan arc with endpoints $\gamma_{1/4}(t)$ and $\gamma_{1/4}(t')$.

Since $z_n \in V \cup J_c'$ for all $n \gg 0$, $w_n \in H_c(V) \cup J_{1/4}'$ for all $n \gg 0$. If there exists a subsequence $\{n_i\} \subset \{n\}$ such that w_{n_i} converges to a point in $K_{1/4}^\circ$, then $z_{n_i} \to \zeta \in K_c^\circ$ by the definition of H_c. This contradicts $\zeta \in J_c$. Thus w_n accumulates on $J_{1/4}'$. Since t and t' are arbitrarily close to θ, w_n must converges to $\gamma_{1/4}(\theta)$.

Extension to a global semiconjugacy. Finally we define H_c outside the Julia set by

$$\begin{aligned} H_c : \bar{\boldsymbol{C}} - K_c &\to \bar{\boldsymbol{C}} - K_{1/4} \\ z &\mapsto \phi_{1/4}^{-1} \circ \phi_c(z), \end{aligned}$$

which gives a topological conjugacy on the domain, and continuously extends to the conjugacy $H_c : \bar{\boldsymbol{C}} - K_c^\circ \to \bar{\boldsymbol{C}} - K_{1/4}^\circ$. Then H_c inside and outside J_c are continuously glued along J_c. Now $H_c : \bar{\boldsymbol{C}} \to \bar{\boldsymbol{C}}$ is a desired semiconjugacy. ■

Remark. There are other possible choices of the conjugacy $h_c : \boldsymbol{C} - [a_c, \infty) \to \boldsymbol{C}$ with better regularity. For example, for $w = a_c + re^{it}$ with $r > 0$ and $0 < t < 2\pi$, one may also use

$$h_c(w) = \frac{\log r - \log a_c}{\log \lambda_c} + i \tan \frac{\pi - t}{2}.$$

Remark 2. The arc C in the proof above exists. First, take tiles $T = T_c(t, -m, +)$ and $T' = T_c(t', -m, -)$. By Proposition 2.1, T and T' share the critical edges with $T_1 = T_c(k/2^{m-1}, -m, -)$ and $T_1' = T_c(k/2^{m-1}, -m, +)$ respectively, and then T_1 and T_1' share the degenerating edges. Now we can join $\gamma_c(t)$ and $\gamma_c(t')$ by an arc via $I_c(t)$, T, T_1, T_1', T', and $I_c(t')$.

4. Degeneration of the regular leaf spaces

Finally we investigate the structure of $\mathscr{R}_{1/4}$, the regular leaf space of $f_{1/4}$, and the degeneration of the invariant leaf corresponding to $\hat{\beta}_c$.

We begin with some notation and preliminary remarks. For $c \in (0, 1/4]$, let us set $\hat{\mathscr{I}}_c := \pi_c^{-1}(\mathscr{I}_c)$. Now we take a sequence of angles $\hat{\theta} := (\theta_0, \theta_{-1}, \ldots)$ such that θ_0 is of the form $k/2^m$ and $\theta_{-n} = 2\theta_{-n-1}$. Since $\mathscr{I}_c$ contains no critical orbit, $\hat{\theta}$ corresponds bijectively to a connected component of $\hat{\mathscr{I}}_c$ which consists of the backward orbits $\{z_{-n}\}_{n=0}^{\infty}$ satisfying $z_{-n} \in I_c(\theta_{-n})$. We denote such a component $\hat{I}_c(\hat{\theta})$. Note that $\hat{I}_c(\hat{\theta})$ is an arc if $c \in (0, 1/4)$, or a point if $c = 1/4$.

Note also that for $\hat{0} = (0, 0, \ldots)$, $\hat{I}_c(\hat{0})$ is the component corresponding to the backward orbits which are always in the interval $I_c(0) = [\alpha_c, \beta_c]$. Thus $\hat{I}_c(\hat{0})$ contains $\hat{\alpha}_c$, one of the two irregular points of $\mathcal{N}_c$.

For $c \in (0, 1/4]$, the set

$$L_c := \{\hat{z} = (z_0, z_{-1}, \ldots) \in \mathcal{R}_c : z_{-n} \to \beta_c\}$$

is invariant under the action of $\hat{f}_c$ and is a leaf isomorphic to $\boldsymbol{C}$. (We will construct the isomorphism later.) Note that $\hat{I}_{1/4}(\hat{0}) = \hat{\beta}_{1/4} = \hat{\alpha}_{1/4}$ does not belong to $L_{1/4}$. On the other hand, for $c \in (0, 1/4)$, $\hat{I}_c(\hat{0}) - \{\hat{\alpha}_c\}$ is a subset of L_c.

Now the main result is:

Theorem 4.1. *For $c \in (0, 1/4)$, there exists a semiconjugacy $\hat{H}_c : \mathcal{N}_c \to \mathcal{N}_{1/4}$ from $\hat{f}_c$ to $\hat{f}_{1/4}$ with the following properties*:

(1) *$\hat{H}_c : \mathcal{N}_c - \hat{\mathcal{I}}_c \to \mathcal{N}_{1/4} - \hat{\mathcal{I}}_{1/4}$ is a topological conjugacy between $\hat{f}_c \mid (\mathcal{N}_c - \hat{\mathcal{I}}_c)$ and $\hat{f}_{1/4} \mid (\mathcal{N}_{1/4} - \hat{\mathcal{I}}_{1/4})$.*
(2) *For $\hat{\theta}$ as above, $\hat{H}_c$ maps the arc $\hat{I}_c(\hat{\theta})$ to the point $\hat{I}_{1/4}(\hat{\theta})$.*
(3) *$\hat{H}_c^{-1}(L_{1/4} \cup \{\hat{\alpha}_{1/4}\}) = L_c \cup \{\hat{\alpha}_c\}$.*
(4) *For a leaf L in $\mathcal{R}_c - L_c$, $\hat{H}_c(L)$ is a leaf in $\mathcal{R}_{1/4} - L_{1/4}$.*
(5) *For a leaf L in $\mathcal{R}_{1/4} - L_{1/4}$, $\hat{H}_c^{-1}(L)$ is a leaf in $\mathcal{R}_c - L_c$.*

Proof. For $\hat{z} = (z_0, z_{-1}, \ldots) \in \mathcal{N}_c$, set

$$\hat{H}_c(\hat{z}) := (H_c(z_0), H_c(z_{-1}), \ldots) \in \mathcal{N}_{1/4}.$$

Since H_c is a semiconjugacy from f_c to $f_{1/4}$, one can easily check that $\hat{H}_c$ is surjective, continuous, and satisfies $\hat{H}_c \circ \hat{f}_c = \hat{f}_{1/4} \circ \hat{H}_c$. Thus $\hat{H}_c$ is a semiconjugacy from $\hat{f}_c$ to $\hat{f}_{1/4}$ on their respective natural extensions. In particular, since $H_c : \bar{\boldsymbol{C}} - \mathcal{I}_c \to \bar{\boldsymbol{C}} - \mathcal{I}_{1/4}$ is a topological conjugacy, corresponding lift to the natural extensions $\hat{H}_c : \mathcal{N}_c - \hat{\mathcal{I}}_c \to \mathcal{N}_{1/4} - \hat{\mathcal{I}}_c$ is also a topological conjugacy. Thus we obtain property (1).

Property (2) comes from the definition of $\hat{H}_c$ above and the one-to-one correspondence

$$\hat{I}_c(\hat{\theta}) \leftrightarrow \hat{\theta} \leftrightarrow \hat{I}_{1/4}(\hat{\theta}).$$

Let us show property (3). Take $\hat{z} = \{z_{-n}\} \in L_c$. Then $z_{-n} \to \beta_c$ implies $H_c(z_{-n}) \to H_c(\beta_c) = \beta_{1/4} = \alpha_{1/4}$ and thus $\hat{H}_c(L_c) \subset L_{1/4} \cup \{\hat{\alpha}_{1/4}\}$. Since $\hat{H}_c(\hat{\alpha}_c) = \hat{\alpha}_{1/4}$, we have $\hat{H}_c(L_c \cup \{\hat{\alpha}_c\}) \subset L_{1/4} \cup \{\hat{\alpha}_{1/4}\}$. *On the other hand, we claim that for $\hat{w} = \{w_{-n}\} \in L_{1/4}$ and $\hat{z} = \{z_{-n}\} \in \hat{H}_c^{-1}(\hat{w})$, $w_{-n} \to \beta_{1/4}$ implies $z_{-n} \to \beta_c$, and thus $\hat{H}_c^{-1}(L_{1/4}) \subset L_c$.* Take a subsequence z_{-n_i} converging to a point ζ. Note that ζ must be in the Julia set, since otherwise one can show that $\hat{z} = \hat{\alpha}$ or $\hat{\infty}$ and it contradicts to $\hat{H}_c(\hat{z}) = \hat{w} \in \mathcal{R}_{1/4}$. By continuity of H_c, $H_c(z_{-n_i}) = w_{-n_i} \to H_c(\zeta)$ and this implies $\zeta \in H_c^{-1}(\beta_{1/4}) = I_c(0) = [\alpha_c, \beta_c]$. Since $\zeta \in J_c$, $\zeta = \beta_c$ and we conclude the claim. Since $\hat{H}_c^{-1}(\hat{\alpha}_{1/4}) = \hat{H}_c^{-1}(\hat{I}_{1/4}(\hat{0})) = \hat{I}_c(\hat{0}) \subset L_c \cup \{\hat{\alpha}_c\}$, we have

$$\hat{H}_c^{-1}(L_{1/4} \cup \{\hat{\alpha}_{1/4}\}) = \hat{H}_c^{-1}(L_{1/4}) \cup \hat{H}_c^{-1}(\hat{\alpha}_{1/4}) \subset L_c \cup \{\hat{\alpha}_c\}.$$

Now one can easily check property (3).

To show properties (4) and (5), for $* \in \{c, 1/4\}$ and $\hat{\theta}$ as above, we define open arcs $\eta_*[\hat{\theta}] : (1,2) \to \mathscr{R}_* - \hat{\mathscr{I}}_*$ by

$$\eta_*[\hat{\theta}](r) := (\phi_*^{-1}(re^{2\pi i\theta_0}), \phi_*^{-1}(r^{1/2}e^{2\pi i\theta_{-1}}), \phi_*^{-1}(r^{1/4}e^{2\pi i\theta_{-2}}), \dots).$$

Note that the points

$$\hat{z}_*[\hat{\theta}] = (\gamma_*(\theta_0), \gamma_*(\theta_{-1}), \dots) \in \hat{\mathscr{I}}_*$$

are accessible by $\eta_*[\hat{\theta}](r)$ by letting r tend to 1. Thus for each $* \in \{c, 1/4\}$, $\hat{z}_*[\hat{\theta}]$ and $\eta_*[\hat{\theta}]$ are in the same leaf of $\mathscr{R}_*$, except when $* = 1/4$ and $\hat{\theta} = \hat{0}$, that is, $\hat{z}_*[\hat{\theta}] = \hat{\alpha}_{1/4}$.

We show property (5) first. Take a leaf $L \neq L_{1/4}$ from $\mathscr{R}_{1/4}$. By property (1), $\hat{H}_c^{-1}(L - \hat{\mathscr{I}}_{1/4})$ is homeomorphic to $L - \hat{\mathscr{I}}_{1/4}$, which is path connected. Now any connected component of $\hat{H}_c^{-1}(L \cap \hat{\mathscr{I}}_{1/4}) \subset \hat{\mathscr{I}}_c$ is an arc and has an endpoint of the form $\hat{z}_c[\hat{\theta}]$, which is accessible by $\eta_c[\hat{\theta}]$. Since $\hat{H}_c(\hat{z}_c[\hat{\theta}]) = \hat{z}_{1/4}[\hat{\theta}] \in L$ is accessible by $\eta_{1/4}[\hat{\theta}]$, we have $\eta_{1/4}[\hat{\theta}] \subset L - \hat{\mathscr{I}}_{1/4}$ and thus $\eta_c[\hat{\theta}] \subset \hat{H}_c^{-1}(L - \hat{\mathscr{I}}_{1/4})$. Then we conclude that any component of $\hat{H}_c^{-1}(L \cap \hat{\mathscr{I}}_{1/4})$ is attached to an arc in $\hat{H}_c^{-1}(L - \hat{\mathscr{I}}_{1/4})$, and thus $\hat{H}_c^{-1}(L)$ is path connected.

Since $\hat{H}_c^{-1}(\hat{\infty}) = \hat{\infty}$, $\hat{H}_c^{-1}(\hat{\alpha}_{1/4}) = \hat{I}_c(\hat{0})$, and $L \subset \mathscr{R}_{1/4} = \mathscr{N}_{1/4} - \{\hat{\infty}, \hat{\alpha}_{1/4}\}$ (Proposition 1.2), we have

$$\hat{H}_c^{-1}(L) \subset \mathscr{N}_c - \{\hat{\infty}\} \cup \hat{I}_c(\hat{0}) \subset \mathscr{R}_c$$

and thus there is a leaf L' of $\mathscr{R}_c$ which contains $\hat{H}_c^{-1}(L)$. By property (3) and $L \neq L_{1/4}$, L' is not L_c. Now $L \subset \hat{H}_c(L') \subset \mathscr{R}_{1/4} - L_{1/4}$ and $\hat{H}_c(L')$ is path connected by the continuity of $\hat{H}_c$. Thus $\hat{H}_c(L')$ is contained by a leaf in $\mathscr{R}_{1/4} - L_{1/4}$, which must be L. This implies $\hat{H}_c(L') = L$ and we have

$$L' \subset \hat{H}_c^{-1}(\hat{H}_c(L')) = \hat{H}_c^{-1}(L) \subset L'.$$

Thus $\hat{H}_c^{-1}(L)$ is L', a leaf in $\mathscr{R}_c - L_c$.

Property (4) comes from property (5). Take a leaf L' from $\mathscr{R}_c - L_c$. Since $\hat{H}_c(L')$ contains no irregular point and is path connected, there is a leaf $L \neq L_{1/4}$ of $\mathscr{R}_{1/4}$ containing $\hat{H}_c(L')$. Then $L' \subset \hat{H}_c^{-1}(L)$. By property (5), $\hat{H}_c^{-1}(L)$ is a leaf $\neq L_c$, which must be L'. Thus $L' = \hat{H}_c^{-1}(L)$ and this implies $\hat{H}_c(L')$ is L, a leaf in $\mathscr{R}_{1/4} - L_{1/4}$. ■

Dynamics on the invariant leaves. Let us describe property (3) in further detail. Now the semiconjugacy $\hat{H}_c$ maps $L_c \cup \{\hat{\alpha}_c\}$ onto $L_{1/4} \cup \{\hat{\alpha}_{1/4}\}$. Within $\mathscr{R}_c$ and $\mathscr{R}_{1/4}$, we observe this as following.

For $c \in (0, 1/4)$, L_c compactly contains all but one component of $\pi_c^{-1}(\mathscr{I}_c) \cap L_c$. The exception is $\hat{I}_c(\hat{0}) - \{\hat{\alpha}_c\}$. Since $\hat{I}_c(\hat{0})$ (resp. $\hat{\alpha}_{1/4}$) is invariant under the action of $\hat{f}_c$ (resp. $\hat{f}_{1/4}$) and $\hat{H}_c^{-1}(\hat{\alpha}_{1/4}) = \hat{I}_c(\hat{0})$, the map

$$\hat{H}_c : L_c - \hat{I}_c(\hat{0}) \to L_{1/4}$$

is a semiconjugacy from $\hat{f}_c \,|\, (L_c - \hat{I}_c(\hat{0}))$ to $\hat{f}_{1/4}|L_{1/4}$.

Let us describe this semiconjugacy more precisely. For $c \in (0, 1/4)$ (resp. $c = 1/4$), take a linearizing (resp. Fatou) coordinate Φ_c^+ on a neighborhood (resp. repelling petal) Π_c of β_c such that the action of f_c is conjugate to $S_c^+(w) = \lambda_c' w + 1$. (Recall that $\lambda_{1/4}' = 1$.) In particular, we may assume that Π_c contains $z = 1$ and $\Phi_c^+(1) = 0$. Then for any $\hat{z} = (z_0, z_{-1}, \ldots) \in L_c$, there exists an N such that $z_{-n} \in \Pi_c$ for $n \geq N$. By [1, §4], the isomorphism between L_c and $\boldsymbol{C}$ is given by:

$$\hat{\Phi}_c^+(\hat{z}) := (S_c^+)^N(\Phi_c^+(z_{-N})).$$

One can easily check that $\hat{\Phi}_c^+(\hat{z})$ does not depend on the choice of N. Then the isomorphism $\hat{\Phi}_c^+ : L_c \to \boldsymbol{C}$ has the following properties:

- $\hat{\Phi}_c^+(\hat{f}_c(\hat{z})) = \lambda_c' \hat{\Phi}_c^+(\hat{z}) + 1$;
- $(\hat{\Phi}_c^+)^{-1}(0)$ is the backward orbit of $z = 1$ along the interval $(\beta_c, 1]$;
- if $c \in (0, 1/4)$, $b_c := \hat{\Phi}_c^+(\hat{\beta}_c) = 1/(1 - \lambda_c')$ tends to $-\infty$ as $c \to 1/4$; and
- if $c \in (0, 1/4)$, $\hat{\Phi}_c^+(\hat{I}_c(\hat{0}) - \{\hat{\alpha}_c\}) = (-\infty, b_c]$.

Now let us consider the map

$$\hat{\Phi}_{1/4}^+ \circ \hat{H}_c \circ (\hat{\Phi}_c^+)^{-1} : \boldsymbol{C} - (-\infty, b_c] \to \boldsymbol{C}$$

for $c \in (0, 1/4)$, which is a semiconjugacy from $S_c^+ \,|\, (\boldsymbol{C} - (-\infty, b_c])$ to $S_{1/4}^+$. The slit $(-\infty, b_c]$ is just like pinched and pushed away to "infinity". Topologically the same thing happens on the invariant leaves. By $\hat{H}_c$, a slit $\hat{I}_c(\hat{0}) \cap L_c$ is pinched, and pushed away to $\hat{\beta}_{1/4}$. As a result, $\pi_{1/4}^{-1}(J_{1/4}) \cap L_{1/4}$ is split into two components. (See Figure 3)

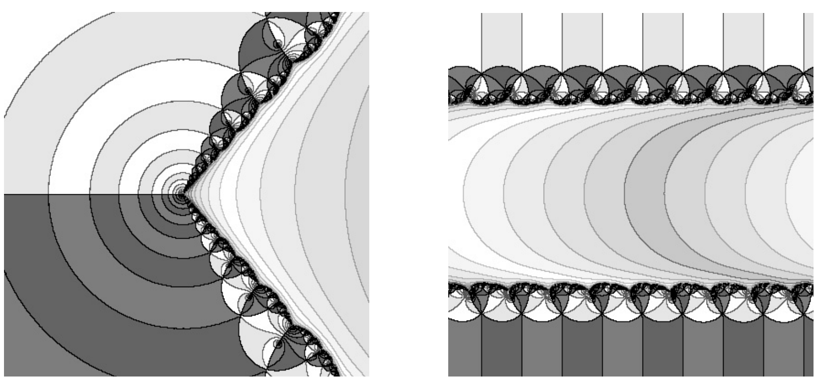

FIGURE 3. L_c for $c = 0.2$ and $L_{1/4}$

Notes.

1. For $c \in (0, 1/4]$, $\mathscr{R}_c$ has the structure of Riemann surface lamination. More precisely, each point of $\mathscr{R}_c$ has a neighborhood homeomorphic to $D \times T$, where D is a topological disk and T is a Cantor set, and each $t \in T$, $D \times \{t\}$ corresponds to a topological disk on a leaf of $\mathscr{R}_c$. (See [1, §2].) For

$c \in (0, 1/4)$, $\hat{H}_c$ preserves the Cantor set direction of such neighborhoods, and the holonomies of fibers of π_c and $\pi_{1/4}$.

2. The hyperbolic 3-lamination of f_c is constructed by adding "height" to the leaves of $\mathscr{R}_c$ to obtain leaves isomorphic to $\boldsymbol{H}^3$. Though the actual construction in [1] is very complicated, we may hope that the pinching $\hat{H}_c$ will naturally extend to this hyperbolic 3-lamination and describe the degeneration as c tends to $1/4$.
3. For a quadratic polynomial with an attracting cycle, we can consider its degeneration to a parabolic cycle with multiple petals. To investigate the associated degeneration of the regular leaf spaces, the method developed in this paper would be useful. Make a semiconjugation between the maps, and lift it to their natural extensions. Then the lifted semiconjugation would give us essential information about the degeneration (or bifurcation) of the regular leaf spaces.

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