J. Math. Soc. Japan
Vol. 72, No. 1 (2020) pp. 1–39
doi: 10.2969/jmsj/80808080

Chow rings of versal complete flag varieties

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(Received June 27, 2018)

Abstract. In this paper, we try to compute Chow rings of versal complete flag varieties corresponding to simple Lie groups, by using generalized Rost motives. As applications, we give new proofs of Totaro's results for the torsion indexes of simple Lie groups except for spin groups.

1. Introduction.

Let G and T be a connected compact Lie group and its maximal torus. Let G_k and T_k be a split reductive group and split maximal torus over a field k with $\mathrm{ch}(k) = 0$, corresponding to Lie groups G and T. Let B_k be a Borel subgroup containing T_k.

Moreover we take k such that there is a G_k-torsor $\mathbb{G}_k$ which is isomorphic to a versal G_k-torsor (for the definition of a versal G_k-torsor, see Section 4 below or see [**Ga-Me-Se**], [**Ka1**], [**Me-Ne-Za**], [**To1**]). Then $X = \mathbb{G}_k/B_k$ is thought as the *most twisted* complete flag variety. (We say that such X is a generically twisted or a versal flag variety [**Ka1**], [**Me-Ne-Za**].)

Let us fix a prime number p. In this paper, we study the p-localized Chow ring $CH^*(X)_{(p)} = CH^*(X) \otimes \mathbb{Z}_{(p)}$ and write it simply $CH^*(X)$, through this paper. We also use the notation $CH^*(X)/p$ for $CH^*(X) \otimes \mathbb{Z}/p$. By Petrov–Semenov–Zainoulline ([**Pe-Se-Za**], [**Se**], [**Se-Zh**]), it is known that the p-localized motive $M(X)_{(p)}$ of X is decomposed as

$$M(X)_{(p)} = M(\mathbb{G}_k/B_k)_{(p)} \cong \oplus_i R(\mathbb{G}_k) \otimes \mathbb{T}^{\otimes s_i}$$

where $\mathbb{T}$ is the Tate motive and $R(\mathbb{G}_k)$ is some motive called generalized Rost motive. (It is the original Rost motive ([**Ro1**], [**Ro2**], [**Vo2**], [**Vo3**]) when G is of type (I) as explained below [**Pe-Se-Za**]).

Let BB_k be the classifying space for B_k-bundles. (For an algebraic group H_k, we can approximate the classifying space BH_k by a colimit of algebraic varieties, and $CH^*(BH_k)$ is defined as a limit of Chow rings of these varieties, for details see [**Pe-Se**], [**To3**].) Since $\mathbb{G} \to X = \mathbb{G}/B_k$ is a B_k-bundle, we have the characteristic (classifying) map $X \to BB_k$. Hence we have maps

$$CH^*(BB_k) \stackrel{char.\ map}{\to} CH^*(X) \stackrel{split\ \ surj.}{\twoheadrightarrow} CH^*(R(\mathbb{G}_k)).$$

REMARK. In this paper, a map $A \to B$ (resp. $A \cong B$) for rings A, B means a ring map (resp. a ring isomorphism). However $CH^*(R(\mathbb{G}_k))$ does not have a canonical ring

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2010 *Mathematics Subject Classification.* Primary 57T15; Secondary 20G15, 14C15.
Key Words and Phrases. Chow ring, Rost motive, versal torsor, complete flag variety.

structure. Hence a map $A \to CH^*(R(\mathbb{G}_k))/p$ (resp. $A \cong CH^*(R(\mathbb{G}_k))/p$) means only a (graded) additive map (resp. additive isomorphism) even if $CH^*(R(\mathbb{G}_k))/p$ has some ring structure. For example, the above first map is a ring map but the second is not a ring map.

From Karpenko [**Ka1**], and Merkurjev–Neshitov–Zainoulline [**Me-Ne-Za**], we know that the first map is also surjective when $\mathbb{G}_k$ is a versal G_k-torsor. We study what elements in $CH^*(BB_k) \cong CH^*(BT_k)$ (Subsection 2.4, page 21 in [**To3**]) generate $CH^*(R(\mathbb{G}_k))$.

For example, Petrov (Theorem 1 in [**Pe**]) computed $CH^*(Y)$ for the versal maximal orthogonal Grassmannian Y corresponding to $G = SO(2\ell+1)$, $\ell > 0$. It is torsion free and is isomorphic to $CH^*(R(\mathbb{G}_k))$ (see Theorem 7.13 below). Hence the restriction map $CH^*(X) \to CH^*(G_k/B_k)$ is injective. Thus we know the ring structure of $CH^*(X)$ from that of $CH^*(G_k/B_k)$ ([**Tod-Wa**], [**Vi**]). These Petrov's results can be very simply written, when we consider the mod (2) Chow theories.

THEOREM 1.1. *Let $(G,p) = (SO(2\ell+1),2)$ and $X = \mathbb{G}_k/B_k$ be a versal flag variety. Then there are isomorphisms*

$$CH^*(R(\mathbb{G}_k))/2 \cong \mathbb{Z}/2[c_1,\dots,c_\ell]/(c_1^2,\dots,c_\ell^2) = \Lambda(c_1,\dots,c_\ell),$$
$$CH^*(X)/2 \cong S(t)/(2,c_1^2,\dots,c_\ell^2)$$

where $c_i = \sigma_i(t_1,\dots,t_\ell)$ is the i-th elementary symmetric function in

$$S(t) = CH^*(BB_k) \cong H^*(BT) \cong \mathbb{Z}[t_1,\dots,t_\ell].$$

REMARK. We have an isomorphism $CH^*(X)/2 \cong H^*(Sp(\ell)/T;\mathbb{Z}/2)$ for the symplectic group $Sp(\ell)$ (see Corollary 7.9).

We give a new proof of the above theorem, which can work for other groups such that Chow rings $CH^*(X)$ have p-torsion elements. The additive structures in the following theorem are known ([**Ka-Me**], [**Me-Su**], [**Ya4**]). However, the ring structure of $CH^*(X)/p$ was unknown except for $(G,p) = (G_2,2)$ ([**Ya3**]).

THEOREM 1.2. *Let G be of type (I) and $\mathrm{rank}(G) = \ell$. Then $2p-2 \le \ell$, and we can take $b_i \in S(t) = CH^*(BB_k)$ for $1 \le i \le \ell$ such that there are isomorphisms*

$$CH^*(R(\mathbb{G}_k))/p \cong \mathbb{Z}/p\{b_1,\dots,b_{2p-2}\},$$
$$CH^*(X)/p \cong S(t)/(p, b_ib_j, b_k | 0 \le i,j \le 2p-2 < k \le \ell)$$

where $\mathbb{Z}/p\{a,b,\dots\}$ is the $\mathbb{Z}/p$-free module generated by $a,b,\dots$. Moreover the ideal of torsion elements in $CH^*(X)$ is generated by $b_1, b_3, \dots, b_{2p-3}$.

Here $b_i \in H^*(BT)$ are transgression images in the spectral sequence induced from the fibering $G \to G/T \to BT$. These b_i are explicitly known ([**Na**], [**Tod2**], [**Tod-Wa**]), for example, when $(G,p) = (G_2,2)$, we can take $b_1 = t_1^2 + t_1t_2 + t_2^2$ and $b_2 = t_2^3$ in $H^*(BT) \cong \mathbb{Z}[t_1,t_2]$ with $|t_i| = 2$ (Theorem 5.3 in [**Ya3**]).

To explain the transgression and type (I) groups, we recall how to compute $H^*(G/T)$ in algebraic topology. By Borel ([**Bo**], [**Mi-Tod**]), its mod(p) cohomology is (for p odd)

$$H^*(G;\mathbb{Z}/p)\cong P(y)/p\otimes\Lambda(x_1,\dots,x_\ell),\quad |x_i|=odd$$

where $P(y)$ is a truncated polynomial ring generated by *even* dimensional elements y_i, and $\Lambda(x_1,\dots,x_\ell)$ is the $\mathbb{Z}/p$ exterior algebra generated by $x_1,\dots,x_\ell$. When $p=2$, we consider the graded ring $grH^*(G;\mathbb{Z}/2)$ which is isomorphic to the right hand side ring above.

When G is simply connected and $P(y)$ is generated by just one generator, we say that G is of type (I). Except for $(E_7,p=2)$ and $(E_8,p=2,3)$, all exceptional simple Lie groups are of type (I) (see [**Mi-Tod**], [**Pe-Se-Za**]). The groups $\mathrm{Spin}(n)$, $7\le n\le 10$ are also of type (I). Note that in these cases, it is known $\mathrm{rank}(G)=\ell\ge 2p-2$.

We consider the fibering ([**Mi-Ni**], [**Na**], [**Tod2**]) $G\stackrel{\pi}{\to}G/T\stackrel{i}{\to}BT$ and the induced spectral sequence

$$E_2^{*,*'}=H^*(BT;H^{*'}(G;\mathbb{Z}/p))\Longrightarrow H^*(G/T;\mathbb{Z}/p).$$

Here we can write $H^*(BT)\cong S(t)=\mathbb{Z}[t_1,\dots,t_\ell]$ with $|t_i|=2$.

It is well known that $y_i\in P(y)$ are permanent cycles (i.e., y_i exist as nonzero elements in $E_\infty^{0,*}$) and that there is a regular sequence $(\bar b_1,\dots,\bar b_\ell)$ in $H^*(BT)/(p)$ such that $d_{|x_i|+1}(x_i)=\bar b_i$ ([**Mi-Ni**], [**Tod2**]). The element $\bar b_i$ is called the transgression image of x_i. We know that G/T is a manifold such that $H^*(G/T)=H^{even}(G/T)$ and $H^*(G/T)$ is torsion free. We also see that there is a filtration in $H^*(G/T)_{(p)}$ such that

$$grH^*(G/T)_{(p)}\cong P(y)\otimes S(t)/(b_1,\dots,b_\ell)$$

where $b_i\in S(t)$ with $b_i=\bar b_i \mod (p)$. Here we note that we can take $b_i=0\in H^*(G/T)/p$, since $b_i=0\in E_\infty^{*,0}$ in the spectral sequence.

The transgression images b_i in Theorem 1.2 are just b_i above. When $(G,p)=(SO(2\ell+1),2)$ we can take $b_i=c_i$. Hence $b_1,\dots,b_\ell$ generate the kernel $I(p)$ of the map

$$H^*(BT)/p\cong S(t)/p\to S(t)/(p,b_1,\dots,b_\ell)\subset H^*(G/T)/p$$

(it is also isomorphic to the kernel of $CH^*(BB_k)/p\to CH^*(G_k/B_k)/p$).

By giving the filtration on $S(t)$ by b_i, we can write

$$grS(t)/p\cong A\otimes S(t)/(b_1,\dots,b_\ell)\quad\text{for}\quad A=\mathbb{Z}/p[b_1,\dots,b_\ell].$$

In particular, we have maps $A\stackrel{i_A}{\to}CH^*(X)/p\to CH^*(R(\mathbb{G}_k))/p$. We easily see that $i_A(A)\supset CH^*(R(\mathbb{G}_k))/p$. In particular the above composition map is surjective. Suppose that there are $f_1(b),\dots,f_s(b)\in A$ such that $CH^*(R(\mathbb{G}_k))/p\cong A/(f_1(b),\dots,f_s(b))$. Moreover if $f_i(b)=0$ also in $CH^*(X)/p$, then we have the isomorphism

$$CH^*(X)/p\cong S(t)/(f_1(b),\dots,f_s(b)).$$

The first isomorphism of Theorem 1.1 (resp. Theorem 1.2 when $\ell=2p-2$) can be rewritten

$$CH^*(X)/2\cong S(t)/(I(2)^{[2]}),\quad (\text{resp. } CH^*(X)/p\cong S(t)/(I(p)^2))$$

where $I(2)^{[2]} = \mathrm{Ideal}(x^2|x \in I(2))$.

For other simple groups G, it seems that only few facts were known for $CH^*(R(\mathbb{G}_k))/p$ when $* > 3$. Hence we write down the fundamental facts here.

THEOREM 1.3. *Let $(G,p) = (SO(2\ell+1),2)$, $(G',p) = (\mathrm{Spin}(2\ell+1),2)$, and $\pi : G' \to G$ be the natural projection. Let $c_i' = \pi^*(c_i)$. Then π^* induces maps such that their composition map is surjective*

$$CH^*(R(\mathbb{G}_k)/(2,c_1) \cong \Lambda(c_2,\dots,c_\ell) \stackrel{\pi^*}{\to} CH^*(R(\mathbb{G}_k'))/2 \twoheadrightarrow \mathbb{Z}/2\{1,c_2',\dots,c_{\bar{\ell}}'\}$$

where $\bar{\ell} = \ell - 1$ if $\ell = 2^j$ for some $j > 0$, otherwise $\bar{\ell} = \ell$. Moreover $c_{2^k}' - 2c_1^{2^k}$, $k > 0$ are torsion elements in $CH^*(X)$.

The right hand side module in the above map seems some fundamental parts in $CH^*(R(\mathbb{G}_k))/2$. For example, the groups Spin(7), Spin(9) are of type (I) and $CH^*(R(\mathbb{G}_k))/2 \cong \mathbb{Z}/2\{1,c_2',c_3'\}$. However, the group Spin(11) is not of type (I).

LEMMA 1.4. *For $(G',p) = (\mathrm{Spin}(11),2)$, we have the surjection*

$$CH^*(R(\mathbb{G}_k'))/2 \twoheadrightarrow \mathbb{Z}/2\{1,c_2',c_3',c_4',c_5',c_2'c_4',c_1^8\}.$$

REMARK. Quite recently, Karpenko [**Ka2**] proved that the above surjection is an isomorphism.

THEOREM 1.5. *Let $(G,p) = (E_7,2)$, $(E_8,2)$ or $(E_8,3)$ so that $\ell = 7$ for E_7 and $\ell = 8$ for E_8. Then we have the surjective map*

$$CH^*(R(\mathbb{G}_k))/p \twoheadrightarrow \mathbb{Z}/p\{1,b_1,\dots,b_\ell\}.$$

Moreover for $(G,p) = (E_7,2)$, $(E_8,3)$, we have

$$(CH^*(R(\mathbb{G}_k))/(\mathrm{Tor})) \otimes \mathbb{Z}/p \cong \mathbb{Z}/p\{1,b_2,\dots,b_\ell,b_2b_\ell\}$$

where Tor *is the submodule of $CH^*(R(\mathbb{G}_k))$ generated by torsion elements.*

Note that the above $b_i \neq 0$ is not a trivial fact. Indeed for groups of type (I), we see $b_i = 0$ when $2p-2 < i \leq \ell$.

To see the above elements are nonzero, we mainly use the torsion index $t(G)_{(p)}$. For $\dim_{\mathbb{R}}(G/T) = 2d$, the torsion index is defined as

$$t(G) = |H^{2d}(G/T;\mathbb{Z})/i^*H^{2d}(BT;\mathbb{Z})| \qquad \text{for } \ i : G/T \to BT.$$

Let $n(\mathbb{G}_k)$ be the greatest common divisor of the degrees of all finite field extension k' of k such that $\mathbb{G}_k$ becomes trivial over k'. Then by Grothendieck [**Gr**], it is known that $n(\mathbb{G}_k)$ divides $t(G)$. Moreover, when $\mathbb{G}_k$ is a versal G_k-torsor, we have $n(\mathbb{G}_k) = t(G)$ ([**Ga-Me-Se**], [**To2**]). Totaro determined ([**To1**], [**To2**]) torsion indexes for all simply connected compact Lie groups G. For example, $t(E_8) = 2^6 3^2 5$.

For all exceptional simple groups G, we give another proofs of Totaro's results by using arguments of the above transgression images b_i (e.g., Lemma 11.11). However we can not compute $t(G)$ for $G = \mathrm{Spin}(2\ell+1)$ by our arguments.

We also consider a field K of an extension of k such that $R(\mathbb{G}_k)|_K$ is a direct sum of the original Rost motives, and study the restriction map $CH^*(R(\mathbb{G}_k))/p \to CH^*(R(\mathbb{G}_k)|_K)/p$ (Theorems 7.12, 11.13, Propositions 10.8, 12.8). The first two theorems relate to recent results by Smirnov–Vishik [**Sm-Vi**] and Semenov [**Se**] respectively.

The plan of this paper is the following. In Section 2, Section 3, we recall and prepare the topological arguments for $H^*(G/T)$ and $BP^*(G/T)$. In Section 4, we recall the decomposition of the motive of a versal flag variety. In Section 5, we recall the torsion index briefly. In Section 6, we study $U(m), Sp(m)$ and $PU(p)$ for each p. In Section 7, Section 8 we study $SO(m)$ and $\mathrm{Spin}(m)$ for $p=2$. In Section 9, we study the cases that G is of type (I). In Section 10, Section 11, Section 12, we study the cases $(G,p) = (E_8,3), (E_8,2)$ and $(E_7,2)$ respectively.

The author thanks the referees for their kind comments and suggestions.

2. Lie groups G and the flag manifolds G/T.

Let G be a connected compact Lie group. By Borel ([**Bo**], [**Mi-Tod**]), its mod(p) cohomology is (for p odd)

$$H^*(G;\mathbb{Z}/p) \cong P(y)/p \otimes \Lambda(x_1,\dots,x_\ell), \quad \ell = \mathrm{rank}(G) \tag{2.1}$$
$$\text{with} \quad P(y) = \mathbb{Z}_{(p)}[y_1,\dots,y_k]/(y_1^{p^{r_1}},\dots,y_k^{p^{r_k}})$$

where the degree $|y_i|$ of y_i is even and $|x_j|$ is odd. When $p=2$, a graded ring $grH^*(G;\mathbb{Z}/2)$ is isomorphic to the right hand side ring, e.g., $x_j^2 = y_{i_j}$ for some y_{i_j}. In this paper, $H^*(G;\mathbb{Z}/2)$ means this $grH^*(G;\mathbb{Z}/2)$ so that (2.1) is satisfied also for $p=2$.

Let T be the maximal torus of G and BT be the classifying space of T. We consider the fibering ([**Mi-Ni**], [**Tod2**]) $G \xrightarrow{\pi} G/T \xrightarrow{i} BT$ and the induced spectral sequence

$$E_2^{*,*'} = H^*(BT; H^{*'}(G;\mathbb{Z}/p)) \Longrightarrow H^*(G/T;\mathbb{Z}/p).$$

The cohomology of the classifying space of the torus is given by $H^*(BT) \cong S(t) = \mathbb{Z}[t_1,\dots,t_\ell]$ with $|t_i| = 2$, where $t_i = \mathrm{pr}_i^*(c_1)$ is the 1-st Chern class induced from

$$T = S^1 \times \dots \times S^1 \xrightarrow{\mathrm{pr_i}} S^1 \subset U(1)$$

for the i-th projection $\mathrm{pr_i}$. Note that $\ell = \mathrm{rank(G)}$ is also the number of the odd degree generators x_i in $H^*(G;\mathbb{Z}/p)$.

It is well known that y_i are permanent cycles (i.e., $d_r(y_i) = 0$ for $r \geq 2$) and that there is a regular sequence ([**Mi-Ni**], [**Tod2**]) $(\bar{b}_1,\dots,\bar{b}_\ell)$ in $H^*(BT)/(p)$ such that $d_{|x_i|+1}(x_i) = \bar{b}_i$. Thus we get

$$E_\infty^{*,*'} \cong grH^*(G/T;\mathbb{Z}/p) \cong P(y)/p \otimes S(t)/(\bar{b}_1,\dots,\bar{b}_\ell).$$

Moreover we know that G/T is a manifold such that $H^*(G/T)$ is torsion free, and hence

$$H^*(G/T)_{(p)} \cong \mathbb{Z}_{(p)}[y_1,\dots,y_k]\otimes S(t)/(f_1,\dots,f_k,b_1,\dots,b_\ell) \qquad (2.2)$$

where $b_i=\bar{b}_i \mod (p)$ and $f_i=y_i^{p^{r_i}} \mod (t_1,\dots,t_\ell)$.

Let $BP^*(-)$ be the Brown–Peterson theory with the coefficients ring $BP^*\cong \mathbb{Z}_{(p)}[v_1,v_2,\dots]$, $|v_i|=-2(p^i-1)$ ([**Ha**], [**Ra**]). Since $H^*(G/T)$ is torsion free, the Atiyah–Hirzebruch spectral sequence collapses. Hence we also know

$$BP^*(G/T)\cong BP^*[y_1,\dots,y_k]\otimes S(t)/(\tilde{f}_1,\dots,\tilde{f}_k,\tilde{b}_1,\dots,\tilde{b}_\ell) \qquad (2.3)$$

where $\tilde{b}_i=b_i \mod (BP^{<0})$ and $\tilde{f}_i=f_i \mod (BP^{<0})$.

Let G_k be the split reductive algebraic group corresponding to G, and T_k be the split maximal torus corresponding to T. Let B_k be the Borel subgroup with $T_k\subset B_k$. Note that G_k/B_k is cellular, and $CH^*(G_k/T_k)\cong CH^*(G_k/B_k)$, since the fiber of the map $G_k/T_k\to G_k/B_k$ is a unipotent group. Hence we have

$$CH^*(G_k/B_k)\cong H^*(G/T)_{(p)},\quad CH^*(BB_k)\cong H^*(BT)_{(p)}.$$

Let $\Omega^*(-)$ be the BP-version of the algebraic cobordism ([**Le-Mo1**], [**Le-Mo2**], [**Ya2**], [**Ya4**])

$$\Omega^*(X)=MGL^{2*,*}(X)_{(p)}\otimes_{MU^*_{(p)}}BP^*,\quad \Omega^*(X)\otimes_{BP^*}\mathbb{Z}_{(p)}\cong CH^*(X)$$

where $MGL^{*,*'}(X)$ is the algebraic cobordism theory defined by Voevodsky with $MGL^{2*,*}(pt.)\cong MU^*$ the complex cobordism ring. There is a natural (realization) map $\Omega^*(X)\to BP^*(X(\mathbb{C}))$. In particular, we have $\Omega^*(G_k/B_k)\cong BP^*(G/T)$. Let $I_n=(p,v_1,\dots,v_{n-1})$ and $I_\infty=(p,v_1,\dots)$ be the (prime invariant) ideals in BP^*. We also note

$$\Omega^*(G_k/B_k)/I_\infty\cong BP^*(G/T)/I_\infty\cong H^*(G/T)/p.$$

3. The Brown-Peterson theory $BP^*(G/T)$.

Recall that $k(n)^*(X)$ is the connected Morava K-theory with the coefficients ring $k(n)^*\cong \mathbb{Z}/p[v_n]$ and $\rho: k(n)^*(X)\to H^*(X;\mathbb{Z}/p)$ is the natural (Thom) map. Recall that there is an exact sequence (Sullivan exact sequence [**Ra**], [**Ya2**])

$$\cdots\to k(n)^{*+2(p^n-1)}(X)\xrightarrow{v_n}k(n)^*(X)\xrightarrow{\rho}H^*(X;\mathbb{Z}/p)\xrightarrow{\delta}\cdots$$

such that $\rho\cdot\delta(x)=Q_n(x)$. Here the Milnor Q_i operation

$$Q_i: H^*(X;\mathbb{Z}/p)\to H^{*+2p^i-1}(X;\mathbb{Z}/p)$$

is defined by $Q_0=\beta$ and $Q_{i+1}=P^{p^i}Q_i-Q_iP^{p^i}$ for the Bockstein operation β and the reduced power operation P^j.

We consider the Serre spectral sequence

$$E_2^{*,*'} \cong H^*(B; H^{*'}(F;\mathbb{Z}/p)) \Longrightarrow H^*(E;\mathbb{Z}/p).$$

induced from the fibering $F \xrightarrow{i} E \xrightarrow{\pi} B$ with $H^*(B) \cong H^{even}(B)$.

LEMMA 3.1 (Lemma 4.3 in [**Ya1**]). *In the spectral sequence $E_r^{*,*'}$ above, suppose that there is $x \in H^*(F;\mathbb{Z}/p)$ such that*

$$(*) \quad y = Q_n(x) \neq 0 \quad \text{and} \quad b = d_{|x|+1}(x) \neq 0 \in E_{|x|+1}^{*,0}.$$

Moreover suppose that $E_{|x|+1}^{0,|x|} \cong \mathbb{Z}/p\{x\} \cong \mathbb{Z}/p$. Then there are $y' \in k(n)^*(E)$ and $b' \in k(n)^*(B)$ such that $i^*(y') = y$, $\rho(b') = b$ and that in $k(n)^*(E)$,

$$(**) \quad v_n y' = \lambda \pi^*(b') \quad \text{for} \quad \lambda \neq 0 \in \mathbb{Z}/p.$$

Conversely if $(**)$ holds in $k(n)^*(E)$ for $y = i^*(y') \neq 0$ and $b = \rho(b') \neq 0$, then there is $x \in H^*(F;\mathbb{Z}/p)$ such that $(*)$ holds.

PROOF. Let $B' = BT^{|b|-1}$ be the $|b|-1$ dimensional skeleton of BT, and $E' = \pi^{-1}(B')$. Consider the Serre spectral sequence

$$E_2^{*,*'} \cong H^*(B'; H^{*'}(F;\mathbb{Z}/p)) \Longrightarrow H^*(E';\mathbb{Z}/p).$$

Since $d_r(x) = b = 0 \in H^*(B';\mathbb{Z}/p)$, there is $x' \in H^*(E';\mathbb{Z}/p)$ such that $i^*(x') = x$. Let $Q_n(x') = y'$ so that $i^*y' = y$. Then y' can be identified as $\delta x' \in k(n)^*(E')$ from $Q_n = \rho\delta$. By the Sullivan exact sequence, we see $v_n y' = 0$ in $k(n)^*(E')$.

On the other hand, let $B'' = B^{|b|-1} \cup e_b$ and $E'' = \pi^{-1}B''$ where e_b is the normal cell representing b. Then $d_r x = b \neq 0 \in H^*(B'';\mathbb{Z}/p)$. By the supposition in this lemma, there does not exist $x'' \in H^*(E'';\mathbb{Z}/p)$ such that $i^*(Q_n x'') = y$, that is, for each $y'' \in H^*(E'';\mathbb{Z}/p)$ with $\pi^* y'' = y$, we see $v_n y'' \neq 0 \in k(n)^*(E'')$.

For $j : E' \subset E''$, we can take an element y'' with $j^*(y'') = y'$ by the following reason. Consider the long exact sequence

$$\cdots \to H^*(E'';\mathbb{Z}/p) \xrightarrow{j^*} H^*(E';\mathbb{Z}/p) \xrightarrow{\partial} H^*(E''/E';\mathbb{Z}/p) \to \cdots$$

Since x' does not exist in $H^*(E'';\mathbb{Z}/p)$, we see $\partial(x') \neq 0$. Hence $\partial(x') = b$ from $H^{|b|}(E''/E';\mathbb{Z}/p) \cong \mathbb{Z}/p\{b\}$. So we see

$$\partial(y') = \partial(Q_n(x')) = Q_n(b) = 0,$$

since $b \in H^*(B)$. Hence $y' \in \mathrm{Im}(j^*)$.

Hence $v_n y' = v_n j^*(y'') = 0 \in k(n)^*(E')$ but $v_n y'' \neq 0 \in k(n)^*(E'')$. By dimensional reason, $v_n y'' = \lambda b$ for $\lambda \neq 0 \in \mathbb{Z}/p$.

Conversely, suppose that $v_n y' = \pi^*(b') \neq 0$ in $k(n)^*(E)$. Then $v_n y' = 0$ in $k(n)^*(E')$ and there is $\tilde{x} \in H^*(E';\mathbb{Z}/p)$ with $Q_n \tilde{x} = y'$. Then for $i^*(y') = y$ and $i^*(\tilde{x}) = x$, we see $Q_n(x) = y$. But $\tilde{x}$ does not exist in $H^*(E'';\mathbb{Z}/p)$. Hence $d_{|x|+1}(x) = \lambda b$ for $\lambda \neq 0 \in \mathbb{Z}/p$, by dimensional reason. □

REMARK (Remark 4.8 in [**Ya1**]). The above lemma also holds when $k(0)^*(X) = H^*(X;\mathbb{Z}_{(p)})$ and $v_0 = p$. This fact is well known (Lemma 2.1 in [**Tod2**]).

COROLLARY 3.2. *In the spectral sequence converging to* $H^*(G/T;\mathbb{Z}/p)$, *let* $b \neq 0$ *be the transgression image of* x, *i.e.,* $d_{|x|+1}(x) = b$. *Then we have the relation in* $BP^*(G/T)/I_\infty^2$ *such that*

$$b = \sum_{i=0} v_i y(i)$$

where $y(i) \in H^*(G/T;\mathbb{Z}/p)$ *with* $\pi^* y(i) = Q_i x$.

PROOF. Since $b = 0 \in H^*(G/T;\mathbb{Z}/p)$, in $BP^*(G/T)/I_\infty^2$, we can write

$$b = py(0) + v_1 y(1) + v_2 y(2) + \cdots .$$

If $Q_i(x) = y(i)' \neq 0$, then $b = v_i y(i)'$ and take $y(i) = y(i)'$. If $Q_i(x) = 0$, then $b = 0$ mod (v_i^2) in $k(i)^*(G/T)$. Otherwise $b = v_i y(i)'$ with $y(i)' \neq 0$ in $H^*(G/T;\mathbb{Z}/p)$ by Sullivan exact sequence. Then $Q_i(x) = y(i)'$ from the converse of the preceding lemma. This is a contradiction. So let $y(i) = 0$ when $Q_i(x) = 0$. □

Let G be a *simply* connected Lie group such that $H^*(G)$ has p-torsion. Then it is known ([**Mi-Tod**]) that $H^*(G)$ has just (not higher) p-torsion in $H^*(G)_{(p)}$. It is also known that there is $m \geq 1$ with

$$(*) \quad P^{p^i}(y_i) = y_{i+1} \quad \text{for} \quad 1 \leq i \leq m-1, \ \text{and} \quad P^{p^m}(y_m) = 0.$$

(Here suffix i is changed adequately from that defined in the preceding section (2.1). Note $m = 1$ for type (I) groups.) Moreover $|x_1| = 3$ and $P^1(x_1) = -x_2$, and $\beta(x_2) = y_1$. We can also take x_{i+1} such that

$$(**) \quad Q_i(x_1) = y_i, \quad Q_0(x_{i+1}) = y_i.$$

Therefore from the preceding corollary, in $BP^*(G/T)/I_\infty^2$, we have

$$b_1 = v_1 y(1) + \cdots + v_m y(m)$$

with $\pi^*(y(i)) = y_i$. We will study the above equation in more details.

Here we recall the Quillen (Landweber–Novikov) operation ([**Ha**], [**Ra**]). For a sequence $\alpha = (a_1, a_2, \ldots)$, $a_i \geq 0$ with $|\alpha| = \sum_i 2(p^i - 1)a_i$, we have the Quillen operation $r_\alpha : BP^*(X) \to BP^{*+|\alpha|}(X)$ such that

$$(1) \quad \rho(r_\alpha(x)) = \chi P^\alpha(\rho(x)) \quad \text{for} \quad \rho : BP^*(X) \to H^*(X;\mathbb{Z}/p),$$

where χ is the anti-automorphism in the Steenrod algebra,

$$(2) \qquad r_\alpha(xy) = \sum_{\alpha=\alpha'+\alpha''} r_{\alpha'}(x) r_{\alpha''}(y) \qquad \text{Cartan formula},$$

$$(3) \qquad r_\alpha(v_n) = \begin{cases} v_i \quad \mod (I_\infty^2) & \text{if } \alpha = p^i\Delta_{n-i} = (0,\dots,0,\overset{n-i}{p^i},0,\dots,0). \\ 0 \quad \mod (I_\infty^2) & \text{otherwise.} \end{cases}$$

We also note that $\Omega^*(X)$ has the same operation r_α satisfying (2), (3) and (1) for $\rho : \Omega^*(X) \to CH^*(X)/p$ and the reduced power operation P^α on $CH^*(X)/p = H^{2*,*}(X;\mathbb{Z}/p)$ defined by Voevodsky.

LEMMA 3.3. *If $|\alpha| < 2(p^i - p^{i-1})$, then r_α acts on $BP^*(X)/(I_\infty^2, v_i, \dots)$.*

PROOF. Here note $|v_{i-1}| - |v_i| = 2(p^i - p^{i-1})$. In this case, we have $r_\alpha(v_s) \in I_\infty^2$ for all $s \geq i$. □

Let $h^*(-)$ be a mod(p) cohomology theory (e.g., $H^*(-;\mathbb{Z}/p)$, $k(n)^*(-)$). The product $G \times G \to G$ induces the map

$$\mu : G \times G/T \to G/T.$$

Here note $h^*(G\times G/T) \cong h^*(G)\otimes_{h^*} h^*(G/T)$, since $h^*(G/T)$ is h^*-free. For $x \in h^*(G/T)$, we say that x is *primitive* ([**Mi-Ni**], [**Mi-Tod**]) if

$$\mu^*(x) = \pi^*(x) \otimes 1 \ + \ 1 \otimes x \quad \text{where} \quad \pi : G \to G/T.$$

It is immediate that if x is primitive, then so is $r_\alpha(x)$. Of course $b \in BP^*(BT)$ are primitive but by_i are not, in general. We can take y_1 as a primitive element (adding elements if necessary) in $BP^*(G/T)$.

LEMMA 3.4. *Let G be a simply connected Lie group satisfying $(*)$. Let y_1 be a primitive element in $BP^*(G/T)$, and define $y_{i+1} = r_{p^i\Delta_1}(y_i)$. Then we have*

$$v_1y_1 + v_2y_2 + \cdots + v_my_m = b_1 \qquad \mod (I_\infty^2).$$

PROOF. Note that $v_ny(n) = b_1 \in k(n)^*(G/T)$ is primitive. We prove $y_n = y(n)$ mod (I_∞^2). Let us write

$$y(n) = y_n + \sum yt$$

with $y \in P(y)$, $t \in S(t)$, $|t| \geq 2$.

We will prove $t = 0$. Consider the Atiyah–Hirzebruch spectral sequence

$$E_2^{*,*'} \cong H^*(G; k(n)^{*'}) \Longrightarrow k(n)^*(G).$$

The first non-zero differential is $d_{2p^n-1}(x) = v_nQ_n(x)$. Since $|y| \leq |y_n| - 2 = 2p^n$, we see that y is v_n-torsion free in $k(n)^*(G)$. This means if $t \neq 0$, then

$$v_ny \otimes t \neq 0$$

in $k(n)^*(G) \otimes_{k^*(n)} k(n)^*(G/T)$. Therefore $t = 0$ since y_n and $v_ny(n)$ are primitive. □

Applying r_{Δ_1} to the equation in Lemma 3.4, we have

LEMMA 3.5. *In $BP^*(G/T)/(I_\infty^2)$, we have*

$$py_1 + v_1P^1(y_1) + v_2P^1(y_2) + \cdots + v_mP^1(y_m) = P^1(b_1) = b_2.$$

4. Versal flag varieties.

Recall that $\mathbb{G}_k$ is a nontrivial G_k-torsor. We can construct a twisted form of G_k/B_k by

$$(\mathbb{G}_k \times G_k/B_k)/G_k \cong \mathbb{G}_k/B_k.$$

We will study the twisted flag variety $X = \mathbb{G}_k/B_k$.

Let $P \supset T$ be a parabolic subgroup of G. Petrov, Semenov and Zainoulline developed the theory of decompositions of motives $M(\mathbb{G}_k/P_k)$. They develop the theory of generically split varieties. We say that L is a splitting field of a variety of X if $M(X|_L)$ is isomorphic to a direct sum of twisted Tate motives $\mathbb{T}^{\otimes i}$ and the restriction map $i_L : M(X) \to M(X|_L)$ is isomorphic after tensoring $\mathbb{Q}$. A smooth scheme X is said to be generically split over k if its function field $L = k(X)$ is a splitting field. Note that (the complete flag) $X = \mathbb{G}_k/B_k$ is always generically split, i.e., $X|_L$ is cellular.

THEOREM 4.1 (Theorem 3.7 in [**Pe-Se-Za**]). *Let $Q_k \subset P_k$ be parabolic subgroups of G_k which are generically split over k. Then there is a decomposition of motive $M(\mathbb{G}/Q_k) \cong M(\mathbb{G}_k/P_k) \otimes H^*(P/Q)$.*

By extending the arguments by Vishik [**Vi**] for quadrics to that for flag varieties, Petrov, Semenov and Zainoulline define the J-invariant of $\mathbb{G}_k$. Recall the expression in Section 2

$$(*) \quad H^*(G;\mathbb{Z}/p) \cong \mathbb{Z}/p[y_1,\dots,y_s]/(y_1^{p^{r_1}},\dots,y_s^{p^{r_s}}) \otimes \Lambda(x_1,\dots,x_\ell).$$

Roughly speaking (for the accurate definition, see [**Pe-Se-Za**]), the J-invariant is defined as $J_p(\mathbb{G}_k) = (j_1,\dots,j_s)$ if j_i is the minimal integer such that

$$y_i^{p^{j_i}} \in \mathrm{Im}(\mathrm{res}_{\mathrm{CH}}) \quad \mathrm{mod}\ (y_1,\dots,y_{i-1},t_1,\dots,t_\ell)$$

for $\mathrm{res}_{\mathrm{CH}} : CH^*(\mathbb{G}_k/B_k) \to CH^*(G_k/B_k)$. Here we take $|y_1| \le |y_2| \le \cdots$ in $(*)$. Hence $0 \le j_i \le r_i$ and $J_p(\mathbb{G}_k) = (0,\dots,0)$ if and only if $\mathbb{G}_k$ splits by an extension of the index coprime to p. One of the main results in [**Pe-Se-Za**] is

THEOREM 4.2 (Theorem 5.13 in [**Pe-Se-Za**] and Theorem 4.3 in [**Se-Zh**]). *Let $\mathbb{G}_k$ be a G_k-torsor over k, $X = \mathbb{G}_k/B_k$ and $J_p(\mathbb{G}_k) = (j_1,\dots,j_s)$. Then there is a p-localized motive $R(\mathbb{G}_k)$ such that*

$$M(X)_{(p)} \cong \oplus_u R(\mathbb{G}_k) \otimes \mathbb{T}^{\otimes u}.$$

Here $\mathbb{T}^{\otimes u}$ are Tate motives with $CH^*(\oplus_u \mathbb{T}^{\otimes u})/p \cong P'(y) \otimes S(t)/(b)$ where

$$P'(y) = \mathbb{Z}/p[y_1^{p^{j_1}}, \dots, y_s^{p^{j_s}}]/(y_1^{p^{r_1}}, \dots, y_s^{p^{r_s}}) \subset P(y)/p,$$
$$S(t)/(b) = S(t)/(b_1, \dots, b_\ell).$$

The mod (p) *Chow group of* $\bar{R}(\mathbb{G}_k) = R(\mathbb{G}_k) \otimes \bar{k}$ *is given by*

$$CH^*(\bar{R}(\mathbb{G}_k))/p \cong \mathbb{Z}/p[y_1, \dots, y_s]/(y_1^{p^{j_1}}, \dots, y_s^{p^{j_s}}).$$

Hence we have $CH^*(\bar{X})/p \cong CH^*(\bar{R}(\mathbb{G}_k)) \otimes P'(y) \otimes S(t)/(b)$ *and*

$$CH^*(X)/p \cong CH^*(R(\mathbb{G}_k)) \otimes P'(y) \otimes S(t)/(b).$$

Let P_k be a special parabolic subgroup of G_k (i.e., any extension is split, e.g., B_k). Let us consider an embedding of G_k into the general linear group GL_N for some N. This makes GL_N a G_k-torsor over the quotient variety $S = GL_N/G_k$. We define F to be the function field $k(S)$ and define the *versal* G_k*-torsor* E to be the G_k-torsor over F given by the generic fiber of $GL_N \to S$. (For details, see [**Ga-Me-Se**], [**Ka1**], [**Me-Ne-Za**], [**To2**].)

$$\begin{array}{ccc} E & \longrightarrow & GL_N \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k(S)) & \longrightarrow & S = GL_N/G_k \end{array}$$

The corresponding flag variety E/P_k is called *generically twisted* or *versal* flag variety, which is considered as the most complicated twisted flag variety (for given G_k, P_k). It is known that the Chow ring $CH^*(E/P_k)$ is not dependent to the choice of generic G_k-torsors E (Remark 2.3 in [**Ka1**]).

Karpenko [**Ka1**] proved the following theorem for $CH^*(X)$. Merkurjev–Neshitov–Zainoulline [**Me-Ne-Za**] also stated this theorem.

Theorem 4.3 (Karpenko Lemma 2.1 in [**Ka1**], [**Me-Ne-Za**]). *Let* $h^*(X)$ *be an oriented cohomology theory (e.g.,* $CH^*(X)$*,* $\Omega^*(X)$*). Let* P_k *be a parabolic subgroup of* G_k *and* $\mathbb{G}_k/P_k$ *be a versal flag variety. Then the natural map* $h^*(BP_k) \to h^*(\mathbb{G}_k/P_k)$ *is surjective.*

Corollary 4.4. *The Chow ring* $CH^*(\mathbb{G}_k/B_k)$ *is generated by elements* t_i *in* $S(t)$*. In particular, for each* $x \in CH^*(G_k/B_k)$*, the element* $p^s x$ *is represented by elements in* $S(t)$ *for a sufficient large* s*.*

Proof. For some extension F/k of order ap^s with a coprime to p (i.e., $(a,p) = 1$), the G_k-torsor $\mathbb{G}_k$ splits. Hence $p^s y^i \in \mathrm{Im}(\mathrm{res}_{\mathrm{CH}} : CH^*(\mathbb{G}_k/B_k) \to CH^*(G_k/B_k))$, which is written by elements in $S(t)$ by the above Karpenko theorem. □

Corollary 4.5. *If* $\mathbb{G}_k$ *is versal, then* $J(\mathbb{G}_k) = (r_1, \dots, r_s)$*, i.e.,* $r_i = j_i$*.*

Proof. If $j_i < r_i$, then $0 \neq y_i^{p^{j_i}} \in \mathrm{res}(CH^*(X) \to CH^*(G_k/B_k))$, which is in the image from $S(t)$ by the preceding theorem. This induces a contradiction since

$CH^*(G_k/T_k;\mathbb{Z}/p)\cong P(y)/p\otimes S(t)/(b)$ and $0\neq y_i^{p^{j_i}}\in P(y)/p$. □

Here we recall the (original) Rost motive R_a (we write it by R_n) defined from a nonzero pure symbol a in the mod(p) Milnor K-theory $K_{n+1}^M(k)/p$. When $J(\mathbb{G}_k)=(1)$ (and G is simply connected), we know $R(\mathbb{G}_k)\cong R_2$ from [**Pe-Se-Za**]. We write $\bar{R}_n=R_n\otimes\bar{k}$. The Rost motive R_n is defined as a non-split motive but split over a field of degree ap with $(a,p)=1$, and for $|y|=2b_n=2(p^n-1)/(p-1)$

$$CH^*(\bar{R}_n)\cong\mathbb{Z}[y]/(y^p),\quad \Omega^*(\bar{R}_n)\cong BP^*[y]/(y^p).$$

THEOREM 4.6 ([**Me-Su**], [**Vi-Ya**], [**Ya4**]). *Let R_n be the (original) Rost motive defined by Rost and Voevodsky* ([**Ro1**], [**Ro2**], [**Vo2**], [**Vo3**]). *Then the restriction* $res_\Omega:\Omega^*(R_n)\to\Omega^*(\bar{R}_n)$ *is injective. Recall $I_n=(p,\dots,v_{n-1})\subset BP^*$. The restriction image* $\mathrm{Im}(res_\Omega)$ *is isomorphic to*

$$\begin{aligned}&BP^*\{1\}\oplus I_n\otimes\mathbb{Z}_{(p)}[y]^+/(y^p)\\ &\quad\cong BP^*\{1,v_jy^i\mid 0\le j\le n-1,\ 1\le i\le p-1\}\subset BP^*[y]/(y^p).\end{aligned}$$

Hence writing $v_jy^i=c_j(y^i)$, $|c_j(y^i)|=2ib_n-2(p^j-1)$, we have

$$CH^*(R_n)/p\cong\mathbb{Z}/p\{1,c_j(y^i)\mid 0\le j\le n-1,\ 1\le i\le p-1\}.$$

EXAMPLE. In particular, we have isomorphisms

$$CH^*(R_1)/p\cong\mathbb{Z}/p\{1,c_0(y),\dots,c_0(y^{p-1})\},$$
$$CH^*(R_2)/p\cong\mathbb{Z}/p\{1,c_0(y),c_1(y),\dots,c_0(y^{p-1}),c_1(y^{p-1})\}.$$

5. Torsion index.

Let $\dim_{\mathbb{R}}(G/T)=2d$. Then the torsion index is defined as

$$t(G)=|H^{2d}(G/T;\mathbb{Z})/i^*H^{2d}(BT;\mathbb{Z})|$$

for $i:G/T\to BT$. Let $n(\mathbb{G}_k)$ be the greatest common divisor of the degrees of all finite field extension k' of k such that $\mathbb{G}_k$ becomes trivial over k'. Then by Grothendieck [**Gr**], it is known that $n(\mathbb{G}_k)$ divides $t(G)$. Moreover, there is a G_k-torsor $\mathbb{G}_F$ over some extension field F of k such that $n(\mathbb{G}_F)=t(G)$ (in fact, this holds for each versal G_k-torsor [**Ka1**], [**Me-Ne-Za**], [**To2**]). Note that $t(G_1\times G_2)=t(G_1)\cdot t(G_2)$. It is well known that if $H^*(G)$ has a p-torsion, then p divides the torsion index $t(G)$. Torsion indexes for simply connected compact Lie groups are completely determined by Totaro [**To1**], [**To2**]. For example, $t(E_8)=2^63^25$.

Hereafter in this paper, we assume that $\mathbb{G}_k$ is a *versal* G_k-torsor and $X=\mathbb{G}_k/B_k$ is the *versal* flag variety. Recall that

$$grH^*(G/T;\mathbb{Z}/p)\cong P(y)/p\otimes S(t)/(b)$$

where $S(t)/(b) = S(t)/(b_1,\dots,b_\ell)$, $P(y)/p \cong \mathbb{Z}/p[y_1,\dots,y_s]/(y_1^{p^{r_1}},\dots,y_s^{p^{r_s}})$. Recall Corollary 4.5, and then we see $J(\mathbb{G}_k)=(r_1,\dots,r_s)$, e.g., $y_i^{p^{r_i-1}} \not\in S(t)$.

Giving the filtration on $S(t)$ by b_i, we have the isomorphism

$$grS(t)/p \cong \mathbb{Z}/p[b_1,\dots,b_\ell]\otimes S(t)/(b_1,\dots,b_\ell).$$

Let us write for $N>0$

$$A_N = \mathbb{Z}/p\{b_{i_1}\cdots b_{i_k} |\ |b_{i_1}|+\cdots+|b_{i_k}|\le N\} \subset grS(t).$$

Of course $H^*(G/T)=0$ for $*>2d=\dim_{\mathbb{R}}(G/T)$, so we have a map

$$grS(t)/p \to A_{2d}\otimes S(t)/(b) \to grCH^*(X)/p.$$

Lemma 5.1. *The composition map is a surjection*

$$A_{2d}\to CH^*(X)/p \stackrel{\mathrm{pr}}{\twoheadrightarrow} CH^*(R(\mathbb{G}_k))/p.$$

Proof. Recall the decomposition $M(X)_{(p)}\cong \oplus_i R(\mathbb{G}_k)\otimes \mathbb{T}^{s_i}$. Since the restriction map $\mathrm{res}_{\mathrm{CH}}: CH^*(\mathbb{T}^{s_i})/p \to CH^*(\bar{\mathbb{T}}^{s_i})/p$ is an isomorphism, we have

$$\begin{aligned} CH^*(\oplus_i\mathbb{T}^{s_i})/p &\cong CH^*(\oplus_i\bar{\mathbb{T}}^{s_i})/p \\ &\cong CH^*(G_k/T_k)/(p,P(y)^+)\cong S(t)/(p,b). \end{aligned}$$

Thus we can write $CH^*(\mathbb{T}^{s_i})\cong \mathbb{Z}_{(p)}\{u_i\}$ for some $u_i\ne 0\in S(t)/(p,b)$. Hence $CH^*(X)/p$ is generated by elements which are product $b\cdot u$ in $CH^*(X)/p$ for $b\in CH^*(R(\mathbb{G}_k))\subset CH^*(X)/p$ and $u\in S(t)/(p,b)$. Note $bu\ne 0$ if $b\ne 0$ in $CH^*(X)/p$.

On the other hand, since $CH^*(X)$ is versal and generated by images from $S(t)$, which is generated by $b'u$ for $b'\in \mathrm{Im}(A_d\to CH^*(X)/p)$. When $s_i\ne 0$ (i.e., $|u|\ge 2$), we see $\mathrm{pr}(b'u)=0$ for the projection $\mathrm{pr}: CH^*(X)/p\to CH^*(R(\mathbb{G}_k))/p$. Hence we have the lemma. □

From the arguments in the proof of preceding lemma, we have

Corollary 5.2. *If $b\in \mathrm{Ker(pr)}$, then we can write $b=\sum b'u'$ with $b'\in A_{2d}$, $0\ne u'\in S(t)/(p,b)$, and $|u'|>0$.*

Corollary 5.3. *If $b_i\ne 0$ in $CH^*(X)/p$, then so in $CH^*(R(\mathbb{G}_k))/p$.*

Proof. Let $\mathrm{pr}(b_i)=0$. Then $b_i=\sum b'u'$ for $|u'|>0$, and hence $b'\in \mathrm{Ideal}(b_1,\dots,b_{i-1})$. This contradicts $(b_1,\dots,b_\ell)$ being regular. □

Let us write

$$y_{top}=\Pi_{i=1}^s y_i^{p^{r_i}-1} \quad (\text{resp. } t_{top})$$

the generator of the highest degree in $P(y)$ (resp. $S(t)/(b)$) so that $f=y_{top}t_{top}$ is the fundamental class in $H^{2d}(G/T)$.

LEMMA 5.4. *The following map is surjective*

$$A_N \twoheadrightarrow CH^*(R(\mathbb{G}_k))/p \quad where \quad N = |y_{top}|.$$

PROOF. In the preceding lemma, $A_N \otimes u$ for $|u| > 0$ maps zero in $CH^*(R(\mathbb{G}_k))/p$. Since each element in $S(t)$ is written by an element in $A_N \otimes S(t)/(b)$, we have the corollary. □

REMARK. In Section 7 in [**Pe-Se**], Petrov and Semenov show

$$CH^*(BB_k)/p \cong CH^*_{G_k}(\mathbb{G}_k/B_k)/p \cong \oplus CH^*_{G_k}(R_{p,G_k}(\mathbb{G}_k))/p$$

where $CH^*_{G_k}(-)$ is the G_k-equivariant Chow ring and $R_{p,G_k}(\mathbb{G}_k)$ is the G_k-equivariant generalized Rost motive. Hence we have

$$CH^*_{G_k}(R_{p,G_k}(\mathbb{G}_k))/p \cong A_\infty = \mathbb{Z}/p[b_1, \dots, b_\ell].$$

Now we consider the torsion index.

LEMMA 5.5. *Let $\tilde{b} = b_{i_1} \cdots b_{i_k}$ in $S(t)$ such that in $H^*(G/T)_{(p)}$*

$$\tilde{b} = p^s \left(y_{top} + \sum yt \right), \quad |t| > 0$$

for some $y \in P(y)$ and $t \in S(t)$. Then the torsion index $t(G)_{(p)} \le p^s$.

PROOF. Suppose $p^s < t(G)_{(p)}$. We can assume $t(G) = p^{s+1}$ multiplying p^i if necessary. Since $tt_{top} = 0 \in S(t)/(b)$, we see

$$tt_{top} \in \mathrm{Ideal}(b_1, \dots, b_\ell) \subset \mathrm{Ideal}(p).$$

Therefore $p^s \sum ytt_{top} \in \mathrm{Ideal}(p^{s+1})$. So it is in $S(t)$, by Karpenko's theorem. Hence $p^s y_{top} t_{top} \in S(t)$. So $t(G) \le p^s$ and this is a contradiction. □

COROLLARY 5.6. *In the preceding lemma, assume $p^s = t(G)_{(p)}$. Then for each subset $(i'_1, \dots, i'_{k'}) \subset (i_1, \dots, i_k)$, the element $b'_{i'_1} \cdots b'_{i_{k'}} \neq 0 \in CH^*(X)/p$.*

PROOF. Let us write $I' = (i'_1, \dots, i'_{k'}) \subset I = (i_1, \dots, i_k)$, $I' \cup I'' = I$, and $b_I = b_{i_1} \cdots b_{i_k}$. It is immediate $b_{I'} \neq 0 \in CH^*(X)/p$ since $b_I = b_{I'} b_{I''} \neq 0 \in CH^*(X)/p$. □

From the above corollary, when $t(G)_{(p)}$ is big enough and there is $\tilde{b}$ in the preceding lemma, we can find many nonzero elements in $CH^*(X)/p$ whose restriction images are zero in $CH^*(\bar{X})/p$.

6. The groups $GL(n)$, $Sp(n)$ and $PU(p)$.

Some results in this section are known. However we write them down since results and arguments are used in other sections. We consider the Lie group $G = U(\ell)$ at first. Note that its cohomology has no torsion. Recall that

$$H^*(U(\ell)) \cong \Lambda(x_1, \dots, x_\ell) \quad \text{with} \quad |x_i| = 2i - 1.$$

So $P(y)/p \cong \mathbb{Z}/p$, and $CH^*(R(\mathbb{G}_k)/p \cong CH^*(\bar{R}(\mathbb{G}_k))/p \cong \mathbb{Z}/p$, that is, there is no twisted form of G_k/B_k. Moreover $CH^*(X)/p \cong S(t)/(p, b_1, \dots, b_\ell)$ for $d_{|x_i|+1}(x_i) = b_i$. It is well known that we can take $b_i = c_i$ the i-th elementary symmetric function on $S(t) \cong \mathbb{Z}[t_1, \dots, t_\ell]$.

Proposition 6.1. *Let $G = U(\ell)$ (i.e., $G_k = GL_\ell$) and p is a prime number. Let $X = G_k/B_k$. Then*

$$CH^*(X)/p \cong S(t)/(p, c_1, \dots, c_\ell)$$

where c_i is the Chern class in $H^*(BT) \cong S(t)$ by the map $T \subset U(\ell)$.

Proof. We consider the fibering $G/T \to BT \to BG$. The composition of the induced maps $H^*(BG) \to H^*(BT) \to H^*(G/T)$ is zero. The first map induces the isomorphism

$$H^*(BG) \cong H^*(BT)^{W_G(T)} \cong \mathbb{Z}[c_1, \dots, c_\ell].$$

Thus $(b_1, \dots, b_\ell) \supset (c_1, \dots, c_\ell)$. By dimensional reason, we have the proposition. □

Next consider in the case $G' = Sp(\ell)$ and recall that

$$H^*(Sp(\ell)) \cong \Lambda(x'_1, \dots, x'_\ell) \quad \text{with} \quad |x'_i| = 4i - 1.$$

So $P(y)'/p \cong \mathbb{Z}/p$, and there is no twisted form of G'_k/B_k. Moreover we have $d_{|x'_i|+1}(x'_i) = p_i$ the Pontryagin class. Hence we have

Proposition 6.2. *Let $G' = Sp(\ell)$ and $X' = G'_k/B_k$. Then for each prime number p, we have*

$$CH^*(X')/p \cong S(t)/(p, p_1, \dots, p_\ell).$$

In particular, when $p = 2$, we have $CH^*(X')/2 \cong S(t)/(2, c_1^2, \dots, c_\ell^2)$.

Now we consider in the case $(G, p) = (PU(p), p)$, which has p-torsion in cohomology, but it is not simply connected. Its mod (p) cohomology is

$$H^*(G; \mathbb{Z}/p) \cong \mathbb{Z}/p[y]/(y^p) \otimes \Lambda(x_1, \dots, x_{p-1}) \quad |y| = 2,\ |x_i| = 2i - 1.$$

So $P(y)/p \cong \mathbb{Z}/p[y]/(y^p)$ with $|y| = 2$. This fact is given by the fibering $U(p) \to PU(p) \to BS^1$ and the induced spectral sequence

$$E_2^{*,*'} \cong H^*(BS^1; H^{*'}(U(p); \mathbb{Z}/p)) \Longrightarrow H^*(PU(p); \mathbb{Z}/p).$$

Here we use that $H^*(BS^1; \mathbb{Z}/p) \cong \mathbb{Z}/p[y]$ and $d_{2p}x_p = y^p$.

Since G is not simply connected, G is not of type (I) while $P(y)$ is generated by only one y. (However $CH^*(X)/p$ quite resembles that of type (I). Compare Theorem 6.5 and Theorem 9.4 below.)

We consider the map $U(p-1) \to U(p) \to PU(p)$ where the maximal tori of $U(p-1)$ and $PU(p)$ are isomorphic, i.e., $T_{U(p-1)} \cong T_{PU(p)}$. By using the map $U(p-1) \to PU(p)$, we know $d_{2i}(x_i) = c_i$. Hence we have

$$grH^*(G/T;\mathbb{Z}/p) \cong \mathbb{Z}/p[y]/(y^p) \otimes S(t)/(c_1,\dots,c_{p-1}).$$

LEMMA 6.3. *Let X split over a field k' over k of index $p^t \cdot a$ for $(a,p)=1$. Then for all $y \in CH^*(\bar{X})$, we see $p^t y \in \mathrm{Im}(\mathrm{res}_{\mathrm{CH}})$.*

PROOF. Using the fact that $\mathrm{res}\otimes\mathbb{Q}$ is isomorphic, there is s such that $p^s y = \mathrm{res}(x)$ for some $x \in CH^*(X)$. Then for the trace map tr, we see

$$p^s \mathrm{res}\cdot \mathrm{tr}(y) = \mathrm{res}\cdot\mathrm{tr}\cdot\mathrm{res}(x) = \mathrm{res}(ap^t x) = ap^{s+t}(y).$$

Since $CH^*(\bar{X})$ is torsion free, we have $\mathrm{res}\cdot\mathrm{tr}(a^{-1}y) = p^t y$. □

LEMMA 6.4. *We have $py^i = c_i \in H^*(G/T)_{(p)}$.*

PROOF. By induction on i, we will prove $py^i = c_i$. It is known from [**Pe-Se-Za**] that $R(\mathbb{G}_k) \cong R_1$ (note $\mathbb{G}_k$ is versal). From the preceding lemma, $py \in \mathrm{Im}(\mathrm{res}_{\mathrm{CH}})$. By Karpenko's theorem, py^i is represented by elements in $CH^*(BT)$. Since $py^i \in \mathrm{Ideal}(c_1,\dots,c_i)$, we can write, for $t(j) \in S(t)$, $\lambda \in \mathbb{Z}$,

$$py^i = \sum_{j<i} c_j t(j) + \lambda c_i.$$

If $\lambda = 0 \in \mathbb{Z}/p$, we see $py^i = \sum py^j t(j)$ by inductive assumption, and this is a contradiction, since $CH^*(\bar{X})$ is p-torsion free. □

THEOREM 6.5. *Let $G = PU(p)$ and $X = \mathbb{G}_k/B_k$. Then there are isomorphisms*

$$CH^*(R(\mathbb{G}_k))/p \cong CH^*(R_1)/p \cong \mathbb{Z}/p\{1, c_1, \dots, c_{p-1}\},$$
$$CH^*(X)/p \cong S(t)/(p, c_i c_j | 1 \le i,j \le p-1).$$

PROOF. From [**Pe-Se-Za**], recall $R(\mathbb{G}_k) \cong R_1$. Hence the second isomorphism follows from $py^i = c_i$ and (Example of) Theorem 4.6,

$$CH^*(R_1)/p \cong \mathbb{Z}/p\{1, py, \dots, py^{p-1}\}.$$

From the main theorem of [**Pe-Se-Za**], we have the additive isomorphism

$$CH^*(X)/p \cong \mathbb{Z}/p\{1, py, \dots, py^{p-1}\} \otimes S(t)/(b)$$

where $b_i = c_i$. Note $c_i c_j = p^2 y^{i+j} = pc_{i+j}$ in $\Omega^*(\bar{X})$. Since $\mathrm{res}_\Omega : \Omega^*(X) \to \Omega^*(\bar{X})$ is injective, we see $c_i c_j = 0 \in CH^*(X)/p$.

Of course we have an additive isomorphism

$$S(t)/(p, c_ic_j) \cong \mathbb{Z}/p\{1, c_1, \dots, c_{p-1}\} \otimes S(t)/(c_1, \dots, c_{p-1}).$$

Moreover we have a surjective ring map $S(t)/(p, c_ic_j) \twoheadrightarrow CH^*(X)/p$. From the additive isomorphism, its kernel is zero, which induces the ring isomorphism of the theorem. □

Since $CH^*(X)$ is torsion free, we also get the above theorem by considering the restriction map $CH^*(X) \to CH^*(\bar{X})$.

We note here the following lemma for a (general) split algebraic group G_k and a G_k-torsor $\mathbb{G}_k$.

Lemma 6.6. *The composition of the following maps is zero for* $* > 0$

$$CH^*(BG_k)/p \to CH^*(BB_k)/p \to CH^*(\mathbb{G}_k/B_k)/p.$$

Proof. Take U (e.g., GL_N for a large N) such that U/G_k approximates the classifying space BG_k [**To3**]. Namely, we can take $\mathbb{G}_k = f^*U$ for the classifying map $f : \mathbb{G}_k/G_k \to U/G_k$. Hence we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{G}_k/B_k & \longrightarrow & U/B_k \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) \cong \mathbb{G}_k/G_k & \longrightarrow & U/G_k \end{array}$$

where U/B_k (resp. U/G_k) approximates BB_k (resp. BG_k). Since $CH^*(\mathrm{Spec}(k))/p = 0$ for $* > 0$, we have the lemma. □

7. The orthogonal group $SO(m)$ and $p = 2$.

We consider the orthogonal groups $G = SO(m)$ and $p = 2$ in this section. The mod 2-cohomology is written as (see for example [**Mi-Tod**], [**Ni**])

$$grH^*(SO(m); \mathbb{Z}/2) \cong \Lambda(x_1, x_2, \dots, x_{m-1})$$

where $|x_i| = i$, and the multiplications are given by $x_s^2 = x_{2s}$. We write $y_{2(odd)} = x_{odd}^2$. Hence we can write

$$H^*(SO(m); \mathbb{Z}/2) \cong P(y) \otimes \Lambda(x_1, x_3, \dots, x_{\bar{m}}),$$
$$\text{with} \quad P(y) = \otimes_{i=0}^{s} \mathbb{Z}/2[y_{4i+2}]/(y_{4i+2}^{2^{r_i}}), \quad grP(y) \cong \Lambda(x_2, x_4, \dots, x_{m'})$$

for adequate integers $\bar{m}, m', s, r_i$. For ease of argument, at first, we only consider the case $m = 2\ell + 1$ so that

$$H^*(G; \mathbb{Z}/2) \cong P(y) \otimes \Lambda(x_1, x_3, \dots, x_{2\ell-1})$$
$$grP(y)/2 \cong \Lambda(y_2, \dots, y_{2\ell}),$$

letting $y_{2i} = x_{2i}$ hence $y_{4i} = y_{2i}^2$. Here the suffix means its degree in this section.

The Steenrod operation is given as $Sq^k(x_i) = \binom{i}{k}(x_{i+k})$. The Q_i-operations are given by Nishimoto [**Ni**]

$$Q_n x_{2i-1} = y_{2i-2^{n+1}-2}, \qquad Q_n y_{2i} = 0.$$

Considering the maps $U(\ell) \to SO(2\ell) \to SO(2\ell+1)$, we see that $b_i = c_i \mod (2)$ for the transgression $d_{2i}(x_{2i-1}) = b_i$ and c_i which is the i-th elementary symmetric function on $S(t)$, from Proposition 6.1 in the preceding section. Moreover we see $Q_0(x_{2i-1}) = y_{2i}$ in $H^*(G;\mathbb{Z}/2)$. From Lemma 3.1 or Corollary 3.2, we have

$$2y_{2i} = c_i \mod (4)$$

in $H^*(G/T)$. Indeed, the cohomology $H^*(G/T)$ is computed completely by Toda–Watanabe [**Tod-Wa**].

THEOREM 7.1 ([**Tod-Wa**]). *There are $y_{2i} \in H^*(G/T)$ for $1 \leq i \leq \ell$ such that $\pi^*(y_{2i}) = y_{2i}$ for $\pi : G \to G/T$, and that we have an isomorphism*

$$H^*(G/T) \cong \mathbb{Z}[t_i, y_{2i}]/(c_i - 2y_{2i}, J_{2i})$$

where $J_{2i} = 1/4(\sum_{j=0}^{2i}(-1)^j c_j c_{2i-j}) = y_{4i} - \sum_{0<j<2i}(-1)^j y_{2j}y_{4i-2j}$ letting $y_{2j} = 0$ for $j > \ell$.

By using Nishimoto's result for Q_i-operation, from Corollary 3.2, we have

COROLLARY 7.2. *In $BP^*(G/T)/I_\infty^2$, we have*

$$c_i = 2y_{2i} + \sum_{n\geq 1} v_n y(2i + 2^{n+1} - 2)$$

for some $y(j)$ with $\pi^*(y(i)) = y_i$.

It is known by Marlin and Merkurjev (see [**To2**] for details) that the torsion index of $SO(2\ell+1)$ (and $SO(2\ell+2)$) is 2^ℓ. Here we give an another proof.

THEOREM 7.3. $t(G) = t(SO(2\ell+1)) = 2^\ell$.

PROOF. We consider in $H^*(G/T)$

$$c_1 \cdots c_\ell = (2y_2)(2y_4)\cdots(2y_{2\ell}) = 2^\ell y_{top}$$

where $y_{top} = y_2 \cdots y_{2\ell}$. Hence $t(G) \leq 2^\ell$.

Conversely, let $2^{\ell-1}y_{top} = t$ in $S(t)$. Then $t = 0 \in H^*(G/T;\mathbb{Z}/2)$ and hence t is in the ideal $(c_1,\dots,c_\ell)$ in $S(t)$. So we can write $t = \sum c_i t(i)$. Then we have

$$2^{\ell-1}y_{top} = 2\sum y_{2i}t(i)$$

which implies $2^{\ell-2}y_{top} = \sum y_{2i}t(i)$ since $H^*(G/T)$ has no torsion.

Continue this argument. Then we have a relation $y_{top} = \sum yt$ with $t \in S(t)$ where the number of y_{2s} in each monomial in y is less or equal to $\ell-1$, while the number for y_{top} is ℓ. This is a contradiction. □

Let $W = W_{SO(2\ell+1)}(T)$ be the Weyl group. Then $W \cong S_\ell^{\pm}$ is generated by permutations and change of signs so that $|S_\ell^{\pm}| = 2^\ell \ell!$. Hence we have

$$H^*(BT)^W \cong \mathbb{Z}_{(2)}[p_1,\dots,p_\ell] \subset H^*(BT) \cong \mathbb{Z}_{(2)}[t_1,\dots,t_\ell],\ |t_i| = 2$$

where the Pontryagin class p_i is defined by $\Pi_i(1+t_i^2) = \sum_i p_i$. Consider the maps

$$\eta : T \overset{\eta_1}{\subset} U(\ell) \to SO(2\ell+1) \overset{\eta_2}{\to} U(2\ell+1).$$

Then $c_{2i}(\eta) = p_i \in CH^*(BT)^W$ which is also the image of $c_{2i}(\eta_2)$ in $CH^*(BSO(2\ell+1))$.

On the other hand, $p_i = c_i(\eta_1)^2 \mod (2)$, where $c_i(\eta_1) = \sigma_i$ is the elementary symmetric function in $S(t)$. Now we consider a versal torsor $\mathbb{G}_k$ and the versal flag $X = \mathbb{G}_k/B_k$. From Lemma 6.6, the composition of the following maps

$$CH^*(BG_k)/2 \to CH^*(BB_k)/2 \to CH^*(X)/2$$

is zero for $* > 0$, we get $c_i(\eta_1)^2 = \sigma_i^2 = 0$ in $CH^*(X)/2$.

This fact is also seen directly from considering the natural inclusion $SO(2\ell+1) \to Sp(2\ell+1)$ and Proposition 6.2.

Lemma 7.4. *We have $c_i^2 = 0$ in $CH^*(X)/2$.*

Lemma 7.5. *There is an additive injection*

$$\mathbb{Z}/2[c_1,\dots,c_\ell]/(c_1^2,\dots,c_\ell^2) = \Lambda(c_1,\dots,c_\ell) \subset CH^*(R(\mathbb{G}_k))/2.$$

Proof. At first we note that $c_1 \cdots c_\ell \neq 0$ in $CH^*(X)/2$. Otherwise, it is represented by $2S(t)$ since $CH^*(X)$ is generated by elements from $S(t)$. It means that $2^{\ell-1}y_{top} = 1/2(c_1\cdots c_\ell) \in S(t)$. Hence $t(G) < 2^\ell$ and it is a contradiction.

For $I = (i_1,\dots,i_k) \subset (1,\dots,\ell)$, let $c_I = c_{i_1}\cdots c_{i_k}$ and $y_I = y_{2i_1}\cdots y_{2i_k}$ and $|I| = k$. Suppose $c_I \in \mathrm{Ker}(pr)$ for $\mathrm{pr} : \mathrm{CH^*(X)/2 \to CH^*(R(\mathbb{G}_k))/2}$. Then from Corollary 5.2, we can write

$$c_I = \sum_J c_J u(J)$$

with $u(J) \in S(t)$ and $|u(J)| > 0$ for some J, since c_I is not zero in $CH^*(X)/2$.

Then we have $2^{|I|}y_I = \sum_J 2^{|J|}y_j u(j)$. Since $H^*(G/T)$ has no 2-torsion, dividing by $\min(2^{|I|}, 2^{|J|})$, we have a contradiction since $H^*(G/T;\mathbb{Z}/2) \cong P(y) \otimes S(t)/(b)$. Thus $c_I \neq 0$ in also $CH^*(R(\mathbb{G}_k))/2$. □

Theorem 7.6. *Let $(G,p) = (SO(2\ell+1),2)$ and $X = \mathbb{G}_k/B_k$. Then there are isomorphisms*

$$CH^*(X)/2 \cong S(t)/(2, c_1^2,\dots,c_\ell^2), \quad CH^*(R(\mathbb{G}_k))/2 \cong \Lambda(c_1,\dots,c_\ell).$$

PROOF. We have the additive and surjective map

$$gr(S(t)/(2,c_1^2,\dots,c_\ell^2)) \cong \Lambda(c_1,\dots,c_\ell)\otimes S(t)/(c_1,\dots,c_\ell)$$
$$\twoheadrightarrow CH^*(X)/2 \cong CH^*(R(\mathbb{G}_k))\otimes S(t)/(2,c_1,\dots,c_\ell).$$

Therefore we see $CH^*(R(\mathbb{G}_k))/2 \cong \Lambda(c_1,\dots,c_\ell)$ from the preceding lemma. From Lemma 7.4, we have the ring homomorphism

$$S(t)/(2,c_1^2,\dots,c_\ell^2)\to CH^*(X)/2,$$

which induces the ring isomorphism from the additive isomorphism. □

COROLLARY 7.7. *In the above theorem, $CH^*(X)$ is torsion free.*

PROOF. Let us write $\Lambda_{\mathbb{Z}}(a_1,\dots,a_m)=\mathbb{Z}\{a_{i_1}\cdots a_{i_s}|1\le i_1<\cdots<i_s\le m\}$. We consider the restriction maps

$$\begin{array}{ccc} CH^*(R(\mathbb{G}_k))\cong\Lambda_{\mathbb{Z}}(c_1,\dots,c_\ell)/J & \xrightarrow{(1)} & CH^*(\bar{R}(\mathbb{G}_k))\cong\Lambda_{\mathbb{Z}}(y_2,\dots,y_{2\ell}) \\ {\scriptstyle (2)}\downarrow & & {\scriptstyle (3)}\downarrow{\scriptstyle inj.} \\ CH^*(X) & \xrightarrow{(4)} & CH^*(\bar{X}). \end{array}$$

for some module J. The map (1) (and (4)) is given by $c_i\mapsto 2y_{2i}$, and since the last map (4) is a ring map, we see that (4)(2) maps $c_{i_1}\cdots c_{i_s}\mapsto 2^s y_{i_1}\cdots y_{i_s}$, which is injective. Hence the first map (1) is (additively) injective and $J=0$. Thus $CH^*(R(\mathbb{G}_k))$ is torsion free, and so is $CH^*(X)$ from the theorem by Petrov, Semenov and Zainoulline such that $M(X)_{(2)}\cong\oplus_i R(\mathbb{G}_k)\otimes\mathbb{T}^{i\otimes}$. □

REMARK. The above lemmas, theorem and corollary are also given from a result by Petrov (Theorem 1 in [**Pe**], see also Theorem 7.13 below).

COROLLARY 7.8. *Let $(G',p)=(SO(2\ell),2)$ and $X'=\mathbb{G}_k'/B_k$ so that $G'\subset G=SO(2\ell+1)$. Then $t(G')=2^{\ell-1}$, and*

$$CH^*(R(\mathbb{G}_k'))/2\cong CH^*(R(\mathbb{G}_k))/(2,c_\ell)\cong\Lambda(c_1,\dots,c_{\ell-1}),$$
$$CH^*(X')/2\cong CH^*(X)/(2,c_\ell)\cong S(t)/(2,c_1^2,\dots,c_{\ell-1}^2,c_\ell).$$

PROOF. This corollary is easily shown from $H^*(G';\mathbb{Z}/2)\cong H^*(G;\mathbb{Z}/2)/(y_{2\ell})$. For example, $grP(y)'\cong\Lambda(y_2,\dots,y_{2\ell-2})$ and $t(G')=2^{\ell-1}$. □

From Proposition 6.2 and Theorem 7.6, we note

COROLLARY 7.9. *Let $G''=Sp(2\ell+1)$ and $X''=\mathbb{G}_k''/B_k''$. Then the natural maps $G\to G''\supset Sp(\ell)$ induce the isomorphisms*

$$CH^*(X)/2\cong CH^*(X'')/(2,t_i|i>\ell)\cong H^*(Sp(\ell)/T;\mathbb{Z}/2).$$

We now study $CH^*(X|_K)/2$ for some interesting extension K over k. Let K be an extension of k such that X does not split over K but splits over an extension over K of degree $2a$, $(a,2)=1$. Suppose that

$$(*) \quad y_{2i} \in \mathrm{Res_K}, \quad \text{for} \quad 1 \le i \le \ell-1$$

where $\mathrm{Res_K} = \mathrm{Im}(\mathrm{res} : CH^*(X|_K)/2 \to CH^*(\bar{X})/2)$. We want to consider the case $y_{2\ell} \notin \mathrm{Res_K}$.

LEMMA 7.10. *Suppose* $(*)$ *and* $\ell \neq 2^n-1$ *for* $n>0$. *Then* $y_{2\ell} \in \mathrm{Res_K}$.

PROOF. We see that if $\ell \neq 2^n-1$, then each $y_{2\ell}$ is a target of the Steenrod operation Sq^{2k}. Recall $Sq^{2k}(y_{2i}) = \binom{i}{k} y_{2(i+k)}$. It is well known that if $i = \sum i_s 2^s$ and $k = \sum k_s 2^s$ for $i_s, k_s = 0$ or 1, then (in mod (2))

$$\binom{i}{k} = \binom{i_m}{k_m} \cdots \binom{i_s}{k_s} \cdots \binom{i_0}{k_0}.$$

Note that if $i = 2^n-1$, then all $i_s = 1$ (for $s<n$). Otherwise there is s such that $i_s = 1$ but $i_{s-1} = 0$. Take $k = 2^{s-1}$ and $i' = i - 2^{s-1}$. Then $i'+k = i$ and

$$\binom{i'}{k} = \binom{i_m = 1}{0} \cdots \binom{i'_s = 0}{k_s = 0} \binom{1}{1} \cdots \binom{i_0}{0} = 1.$$

This means $Sq^{2k}(y_{2i'}) = y_{2i}$ if $i \neq 2^n-1$. □

LEMMA 7.11. *Suppose* $(*)$ *and* $\ell = 2^n-1$. *Then elements* $py_{2\ell}, v_1 y_{2\ell}, \ldots, v_{n-1} y_{2\ell}$ *are all in* $\mathrm{Im}(\mathrm{res}_\Omega)$ *where* $\mathrm{res}_\Omega : \Omega^*(X)/2 \to \Omega^*(\bar{X})/2$.

PROOF. From Corollary 7.2, we see

$$\begin{aligned} c_{\ell-2^j+1} &= 2y(2(\ell-2^j+2^0)) + v_1 y(2(\ell-2^j+2^1)) + \cdots + v_j(y(2\ell)) \\ &= v_j(y_{2\ell}) \mod (y_2, y_4, \ldots, y_{2\ell-2}). \end{aligned}$$

Hence we have $\mathrm{res}_\Omega(c_{\ell-(2^j-1)}) = v_j(y_{2\ell}) \mod (y_2, y_4, \ldots, y_{2\ell-2})$. □

Thus we have

THEOREM 7.12. *Suppose* $(*)$ *and* $\ell = 2^n-1$. *Then*

$$CH^*(R(\mathbb{G}_k)|_K)/2 \cong \Lambda(y_2, \ldots, y_{2\ell-2}) \otimes CH^*(R_n)/2,$$

with $CH^*(R_n)/2 \cong \mathbb{Z}/2\{1, c_0(y_{2\ell}), \ldots, c_{n-1}(y_{2\ell})\} \cong \mathbb{Z}/2\{1, py_{2\ell}, \ldots, v_{n-1}y_{2\ell}\}$. *Moreover we have*

$$\mathrm{res}_{\mathrm{k}}^{\mathrm{K}}(CH^*(R(\mathbb{G}_k))/2) \cong CH^*(R_n)/2 \subset CH^*(R(\mathbb{G}_k)|_K)/2.$$

The restriction maps are given $c_j \mapsto c_s(y_{2\ell}) = v_s y_{2\ell}$ *if* $j = \ell - (p^s-1)$, *and* $c_j \mapsto 0$ *otherwise.*

At last of this section, we consider the case $X(\mathbb{C}) = G/P$ with

$$G = SO(2\ell+1) \quad \text{and} \quad P = U(\ell).$$

Let us write this X by Y, i.e., $Y = \mathbb{G}_k/P_k$. From the fibering $SO(2\ell+1) \to Y(\mathbb{C}) \to BU(\ell)$, we have the spectral sequence

$$\begin{aligned} E_2^{*,*'} &\cong H^*(SO(2\ell+1);\mathbb{Z}/2) \otimes H^{*'}(BU(\ell)) \\ &\cong P(y) \otimes \Lambda(x_1,\dots,x_{2\ell-1}) \otimes \mathbb{Z}/2[c_1,\dots,c_\ell] \Longrightarrow H^*(Y(\mathbb{C});\mathbb{Z}/2). \end{aligned}$$

Here the differential is given as $d_{2i}(x_{2i-1}) = c_i$. Hence

$$CH^*(\bar{Y};\mathbb{Z}/2) \cong H^*(Y(\mathbb{C});\mathbb{Z}/2) \cong P(y)/2.$$

This case is studied by Vishik [**Vi**] and Petrov [**Pe**] as maximal orthogonal (or quadratic) grassmannian. (see Theorem 5.1 in [**Vi**]). From Theorem 7.6, we have

THEOREM 7.13 ([**Pe**], [**Vi**]). *Let $G = SO(2\ell+1)$ and $\mathbb{G}_k$ be a versal G_k-torsor. Let $Y = \mathbb{G}_k/U(\ell)_k$. Then*

$$CH^*(Y)/2 \cong CH^*(R(\mathbb{G}_k))/2 \cong \Lambda(c_1,\dots,c_\ell).$$

REMARK. Petrov computes the integral Chow ring for more general situations [**Pe**]. From the above theorem, we note that $CH^*(R(\mathbb{G}_k))/2$ has the ring structure in this case.

In [**Vi**], Vishik originally defined the J-invariant $J(q)$ of a quadratic form q which corresponds to the quadratic grassmannian (see Definition 5.11, Corollary 5.10 in [**Vi**]) by

$$J(q) = \{i_k | y_{2i_k} \in \mathrm{Res}_{\mathrm{CH}}\} \subset \{0,\dots,\ell\}.$$

Let I be the fundamental ideal of the Witt ring $W(k)$ so that $grW(k) = \oplus_n I^n/I^{n+1} \cong K_*^M(k)/2$ where $K_*^M(k)$ is the Milnor K-theory of k. Smirnov and Vishik (Proposition 3.2.31 in [**Sm-Vi**]) prove that

$$q \in I^n \quad \text{if and only if} \quad \{0,\dots,2^{n-1}-2\} \subset J(q).$$

Hence the condition $(*)$ in Theorem 7.12 is equivalent to $q \in I^n$ for the quadratic form q corresponding to $Y|_K$. We also note that $G = \mathrm{Spin}(m)$ cases correspond to $q \in I^3$ from $1, y_2, y_4 \in \mathrm{Res}_{\mathrm{CH}}$ (see (8.1) below). This fact is of course, well known.

8. The spin group $\mathbf{Spin}(2\ell+1)$ and $p = 2$.

Throughout this section, let $p = 2$, $G = SO(2\ell+1)$ and $G' = \mathrm{Spin}(2\ell+1)$. By definition, we have the 2 covering $\pi : G' \to G$. It is well known that $\pi^* : H^*(G/T) \cong H^*(G'/T')$ where T' is a maximal torus of G'. However the twisted flag varieties are not isomorphic.

Let $2^t \le \ell < 2^{t+1}$, i.e., $t = [\log_2 \ell]$. The mod 2 cohomology is

$$H^*(G';\mathbb{Z}/2)\cong H^*(G;\mathbb{Z}/2)/(x_1,y_2)\otimes\Lambda(z)$$
$$\cong P(y)'\otimes\Lambda(x_3,x_5,\dots,x_{2\ell-1})\otimes\Lambda(z),\quad |z|=2^{t+2}-1$$

where $P(y)\cong\mathbb{Z}/2[y_2]/(y_2^{2^{t+1}})\otimes P(y)'$. (Here the element z is defined by $d_{2^{t+2}}(z)=y_2^{2^{t+1}}$ for $0\neq y_2\in H^2(B\mathbb{Z}/2;\mathbb{Z}/2)$ in the spectral sequence induced from the fibering $G'\to G\to B\mathbb{Z}/2$.) Hence

$$grP(y)'\cong\otimes_{2i\neq 2^j}\Lambda(y_{2i})\cong\Lambda(y_6,y_{10},y_{12},\dots,y_{2\bar{\ell}})\tag{8.1}$$

where $\bar{\ell}=\ell-1$ if $\ell=2^j$, and $\bar{\ell}=\ell$ otherwise. The Q_i operation for z is given by Nishimoto [**Ni**]

$$Q_0(z)=\sum_{i+j=2^{t+1},i<j}y_{2i}y_{2j},\quad Q_n(z)=\sum_{i+j=2^{t+1}+2^{n+1}-2,i<j}y_{2i}y_{2j}$$

for $n\geq 1$.

We know that

$$grH^*(G/T)/2\cong P(y)'\otimes\mathbb{Z}[y_2]/(y_2^{2^{t+1}})\otimes S(t)/(2,c_1,c_2,\dots,c_\ell)$$
$$grH^*(G'/T')/2\cong P(y)'\otimes S(t')/(2,c_2',\dots,c_\ell',c_1^{2^{t+1}}).$$

Here $c_i'=\pi^*(c_i)$ and $d_{2^{t+2}}(z)=c_1^{2^{t+1}}$ in the spectral sequence converging $H^*(G'/T')$. These are additively isomorphic. In particular, we have

LEMMA 8.1. *The element $\pi^*(y_2)=c_1\in S(t')$ and $\pi^*(t_j)=c_1+t_j$ for $1\leq j\leq\ell$.*

Take k such that $\mathbb{G}_k$ is a versal G_k-torsor so that $\mathbb{G}_k'$ is also a versal G_k'-torsor. Let us write $X=\mathbb{G}_k/B_k$ and $X'=\mathbb{G}_k'/B_k'$. Then

$$CH^*(\bar{R}(\mathbb{G}_k'))/2\cong P(y)'/2,\quad\text{and}\quad CH^*(\bar{R}(\mathbb{G}_k))/2\cong P(y)/2.$$

THEOREM 8.2. *Let $(G,p)=(SO(2\ell+1),2)$, $(G',p)=(\mathrm{Spin}(2\ell+1),2)$, and $\pi: G'\to G$ be the natural projection. Let $c_i'=\pi^*(c_i)$. Then π^* induces maps such that their composition map is surjective*

$$CH^*(R(\mathbb{G}_k))/(2,c_1)\cong\Lambda(c_2,\dots,c_\ell)\xrightarrow{\pi^*}CH^*(R(\mathbb{G}_k'))/2\twoheadrightarrow\mathbb{Z}/2\{1,c_2',\dots,c_{\bar{\ell}}'\}$$

where $\bar{\ell}=\ell-1$ if $\ell=2^j$ for some $j>0$, otherwise $\bar{\ell}=\ell$.

PROOF. From Corollary 5.3, we only need to show $c_i'\neq 0$ in $\Omega^*(G_k'/T_k')/(I_\infty\cdot\mathrm{Im}(\mathrm{res}_\Omega))$. In fact, when $i\neq 2^j$, in $H^*(G'/T')/4$, we have

$$2y_{2i}=c_j'\in S(t)$$

which is nonzero in $BP^*(G/T)/(I_\infty\cdot\mathrm{Im}(\mathrm{res}_\Omega))$. Because $y_{2i}\in P(y)'$ and $y_{2i}\notin\mathrm{Im}(\mathrm{res}_{CH})$ from Lemma 4.5 since X is a versal flag variety.

When $i=2^j$, we note $y_{2i}=y_{2^j}\in S(t')$, in fact $y_{2^j}\notin P(y)'$. But in $BP^*(G'/T')/I_\infty^2$, we have

$$2y_{2i}+v_1(y(2i+2))+\cdots+v_n(y(2i+2^{n+1}-2))+\cdots=c_i'\in BP^*(BT').$$

When $i+1\le\ell$, this element is nonzero in $BP^*(G/T)/I_\infty\cdot\mathrm{Im}(\mathrm{res}_\Omega)$ because

$$c_i'=v_1(y(2i+2))\neq 0\in k(1)^*(G/T)/(v_1\cdot\mathrm{Im}(\mathrm{res}_{k(1)}))$$

where $\mathrm{res}_{k(1)}:k(1)^*(X')\to k(1)^*(\bar{X})$. Otherwise $y(2i+2)\in\mathrm{Im}(\mathrm{res}_{CH})$, and this is a contradiction since $y_{2^j+2}\notin\mathrm{Im}(\mathrm{res}_{CH})$, which follows from $y_{2j+2}\in P(y)'$ and Corollary 4.5.

When $2^j=\ell$, we note

$$CH^*(\bar{R}(\mathbb{G}_k'))/2\cong CH^*(\bar{R}(\mathbb{G}_k''))/2\quad\text{for } G''=\mathrm{Spin}(2\ell-1),$$

in fact $y_{2\ell}=y_{2^j}\notin CH^*(\bar{R}(\mathbb{G}_k'))$. From a theorem by Vishik–Zainoulline (Corollary 6 in [**Vi-Za**]), we get $CH^*(R(\mathbb{G}_k'))/2\cong CH^*(R(\mathbb{G}_k''))/2$. Hence we can take $c_\ell'=0$. □

COROLLARY 8.3. *The elements* $c_{2^j}''=c_{2^j}'-c_1^{2^j}$, $j>0$ *are torsion elements in* $CH^*(X)_{(2)}$.

PROOF. Note that $\mathrm{res}_\Omega(c_{2^j}'')\in BP^{<0}\cdot\Omega^*(\bar{X})$, and $\mathrm{res}_{CH}(c_{2^j}'')=0\in CH^*(\bar{X})$. It is well known that $\mathrm{res}_{CH}\otimes\mathbb{Q}$ is isomorphic. Hence c_{2^j}'' must be torsion. □

EXAMPLE. Let $G=SO(7)$ and $G'=\mathrm{Spin}(7)$, i.e., $\ell=3$. Their cohomologies are

$$H^*(G;\mathbb{Z}/2)\cong\mathbb{Z}/2[y_2,y_6]/(y_2^4,y_6^2)\otimes\Lambda(x_1,x_3,x_5),$$
$$H^*(G';\mathbb{Z}/2)\cong\mathbb{Z}/2[y_6]/(y_6^2)\otimes\Lambda(x_3,x_5,z_7).$$

The cohomologies of flag manifolds are

$$H^*(G/T;\mathbb{Z}/2)\cong\mathbb{Z}/2[y_2,y_6]/(y_2^4,y_6^2)\otimes S(t)/(c_1,c_2,c_3),$$
$$H^*(G'/T';\mathbb{Z}/2)\cong\mathbb{Z}/2[y_6]/(y_6^2)\otimes S(t)/(c_2',c_3',c_1^4).$$

These cohomologies are isomorphic by $\pi^*(y_2)=c_1$. The torsion indexes are $t(G)=2^3$ and $t(G')=2$. The Chow rings of versal flag varieties are

$$CH^*(X)/2\cong S(t)/(2,c_1^2,c_2^2,c_3^2),\quad CH^*(R(\mathbb{G}_k))/2\cong\Lambda(c_1,c_2,c_3),$$
$$CH^*(X')/2\cong S(t)/(2,(c_2')^2,c_2'c_3',(c_3')^2,c_1^4),\quad CH^*(R(\mathbb{G}_k'))/2\cong\mathbb{Z}/2\{1,c_2',c_3'\}.$$

Here $\pi^*(t_i)=c_1+t_i$ so that $\pi^*(c_1)=0 \mod (2)$. For the third and the last isomorphisms, see Corollary 9.5 below. In fact G' is a group of type (I).

LEMMA 8.4 (Marlin's bound). *The torsion index* $t(G')$ *divides* $2^{\ell-[\log_2\ell]-1}$.

PROOF. It follows from

$$\Pi_{i\neq 2^j}c_i=\Pi_{i\neq 2^j}(2y_{2i})=2^{\ell-t-1}y_{top}'$$

where y'_{top} is the generator of top degree elements in $P(y)'$. $\square$

The exact value of $t(G')$ is determined by Totaro, namely $t(G') = \ell - [\log_2(\binom{\ell+1}{2}+1)]$ or that expression plus 1. (It is known $t(\mathrm{Spin}(2\ell+1)) = t(\mathrm{Spin}(2\ell+2))$).

Marlin's bound fails first for Spin(11). This fact was first found by using a property of 12-dimensional quadratic forms [**To2**]. However we show it using the Q_0-operation.

Lemma 8.5. *For* $(G',p) = (\mathrm{Spin}(11), 2)$*, we have* $t(G') = 2$ *and the surjection*

$$CH^*(R(\mathbb{G}'_k))/2 \twoheadrightarrow \mathbb{Z}/2\{1, c'_2, c'_3, c'_4, c'_5, c'_2c'_4, c_1^8\}.$$

Proof. Recall the cohomology

$$H^*(G';\mathbb{Z}/2) \cong \mathbb{Z}/2[y_6, y_{10}]/(y_6^2, y_{10}^2) \otimes \Lambda(x_3, x_5, x_7, x_9, z_{15}).$$

By Nishimoto, we know $Q_0(z_{15}) = y_6y_{10}$. It implies $2y_6y_{10} = d_{16}(z_{15}) = c_1^8$. Since $y'_{top} = y_6y_{10}$, we have $t(G') = 2$. (Note that $c_1^8 \neq 0 \in CH^*(R(\mathbb{G}))/2$, otherwise $t(G') = 1$.)

We will show $c'_2c'_4 \neq 0 \in CH^*(X)/2$. The elements c'_2, c'_3, c'_4 in $CH^*(R(\mathbb{G}'_k))/2$ correspond to $v_1y_6, 2y_6, v_1y_{10}$ in $\Omega^*(\bar{R}(\mathbb{G}'_k))$ respectively. In particular $c'_2c'_4$ corresponds to $v_1^2y_6y_{10}$. If $c'_2c'_4 = 0 \in CH^*(R(\mathbb{G}'_k))/2$, then $v_1y_6y_{10}$ must be in Res_Ω. This means $v_1y_6y_{10} = b''$ for some $b'' \in BP^*(BT')$. However there is no $x \in H^{13}(G';\mathbb{Z}/2)$ such that $Q_1(x) = y_6y_{10}$ with $d_{12}(x) = b''$. $\square$

Remark. Quite recently, Karpenko ([**Ka2**]) showed that the above surjection is an isomorphism.

In most cases, from the result of Totaro, we see $\Pi_{i\neq 2^j} c'_i = 0$. However from [**To2**] when $\ell = 8$, we know that $2^{\ell-[\log_2(\ell)]-1} = 2^4 = t(\mathrm{Spin}(17))$. (Note $y_{16} - 2y_6y_{10} \in S(t)$ but $y_{16} \notin S(t)$ when $\ell = 8$.) Hence we have

Lemma 8.6. *Let* $\ell \geq 8$ *and* $G' = \mathrm{Spin}(2\ell+1)$*, and* $X' = \mathbb{G}'_k/B'_k$*. Then we have*

$$c'_3c'_5c'_6c'_7,\ c'_3c'_4c'_6c'_7 \neq 0 \in CH^*(X')/2.$$

Proof. When $\ell = 8$, we see that elements

$$c'_3c'_5c'_6c'_7 = 2^4y_6y_{10}y_{12}y_{14} = 2^4y'_{top} \quad \text{and} \quad c'_3c'_4c'_6c'_7 = 2^3v_1y'_{top}$$

are BP^*-module generators in $BP^*(G/T)/(I_\infty \cdot \mathrm{Im}(\mathrm{res}_\Omega))$. Hence these elements are nonzero in $CH^*(X')/2$ for $\ell = 8$. We get cases $\ell \geq 8$ from the map $\mathrm{Spin}(17) \to \mathrm{Spin}(2\ell+1)$. $\square$

9. The exceptional group E_8 and $p = 5$.

In this section, we consider the case $(G,p) = (E_8, 5)$. The similar arguments also hold for $(G,p) = (G_2, 2), (F_4, 3)$. The mod (5) cohomology of $G = E_8$ ([**Mi-Tod**]) is given by

Theorem 9.1. *The* mod (5) *cohomology* $H^*(E_8;\mathbb{Z}/5)$ *is isomorphic to*

$$\mathbb{Z}/5[y_{12}]/(y_{12}^5)\otimes\Lambda(z_3,z_{11},z_{15},z_{23},z_{27},z_{35},z_{39},z_{47})$$

where suffix means its degree. The cohomology operations are given

$$\beta(z_{11})=y_{12},\quad \beta(z_{23})=y_{12}^2,\quad \beta(z_{35})=y_{12}^3,\quad \beta(z_{47})=y_{12}^4,$$
$$P^1z_3=z_{11},\quad P^1z_{15}=z_{23},\quad P^1z_{27}=z_{35},\quad P^1z_{39}=z_{47}.$$

We use the notation such that $y=y_{12}$ and $x_1=z_3,\dots,x_8=z_{47}$ as used in Section 2. Hence we can rewrite the cohomology as

$$H^*(G;\mathbb{Z}/p)\cong\mathbb{Z}/p[y]/(y^p)\otimes\Lambda(x_1,\dots,x_{2p-2})$$

for $(G,p)=(E_8,5)$. The above isomorphism also holds for $(G,p)=(G_2,2)$, $(F_4,3)$. So hereafter in this section, we assume (G,p) is one of $(G_2,2)$, $(F_4,3)$ or $(E_8,5)$. The cohomology operations are given as

$$\beta: x_{2i}\mapsto y^i,\quad P^1: x_{2i-1}\mapsto x_{2i}\qquad \text{for}\quad 1\le i\le p-1.$$

Hence the Q_i operations are given

$$Q_1(x_{2i-1})=Q_0(x_{2i})=y^i\quad \text{for}\quad 1\le i\le p-1.$$

Therefore we have the following lemma, by using Lemma 3.1 or Corollary 3.2.

Lemma 9.2. *In $BP^*(G/T)/I_\infty^2$, we have*

$$py^i=b_{2i}\qquad \mathrm{mod}\ (b_2,b_4,\dots,b_{2i-2}),$$
$$v_1y^i=b_{2i-1}\qquad \mathrm{mod}\ (b_1,b_2,\dots,b_{2i-2}).$$

Proof. First note that $Q_0x_{2i}=y^i$ and $d_r(x_{2i})=b_{2i}$. From Corollary 3.2, there is $y(2i)\in BP^*(G/T)/I_\infty^2$ such that $py(2i)=b_{2i}$ and $\pi^*(y(2i))=y^i$, that is

$$y(2i)=y^i+\sum_{j<i}y^jt(j)$$

where $t(j)\in S(t)$ $|t(j)|\ge 2$. By induction on i, we get the first equation.

From $Q_1(x_{2i-1})=y^i$ and $d_{r'}(x_{2i-1})=b_{2i-1}$, there is $y(2i-1)$ such that $v_1y(2i-1)=b_{2i-1}$ and $\pi^*(y(2i-1))=y^i$. Hence we get the second equation similarly. □

The fundamental class is written $y^{p-1}t_{top}\in H^*(G/T)$, i.e., $y_{top}=y^{p-1}$. Since $py^{p-1}=b_{2p-2}\in S(t)$, we see $t(G)_{(p)}=p$.

By Petrov–Semenov–Zainoulline, it is known when G is one of $(G_2,2)$, $(F_4,3)$ or $(E_8,5)$, the motive $R(\mathbb{G}_k)$ in Theorem 4.2 is just the original Rost motive R_2 defined by Rost and Voevodsky. (Recall Theorem 4.6.) The restriction $res_{\Omega|R}:\Omega^*(R(\mathbb{G}_k))\to\Omega^*(\bar{R}(\mathbb{G}_k))$ is injective. Hence the following restriction is also injective

$$res_\Omega:\Omega^*(X)\to\Omega^*(\bar{X})\cong BP^*(G/T).$$

COROLLARY 9.3. *We see*

$$CH^*(R_2)/p \cong CH^*(R(\mathbb{G}_k))/p \cong \mathbb{Z}/p\{1, b_1, \dots, b_{2p-2}\}.$$

In particular, $b_s \neq 0 \in CH^*(X)/p$. *Moreover for* $1 \leq s, r \leq 2p-2$, *we see* $b_s b_r = 0$ *in* $CH^*(X)/p$.

PROOF. Recall Corollary 5.3. We will prove $b_1 \neq 0 \in CH^*(X)$. Other cases are proved similarly. Note $b_1 = v_1 y \in \Omega^*(\bar{X})$. If $b_1 \in BP^{<0} \cdot \mathrm{Im}(\mathrm{res}_\Omega)$, then $y \in \mathrm{Im}(\mathrm{res}_\Omega)$ and this is a contradiction. So $b_1 \neq 0$ in

$$CH^*(X) \cong \Omega^*(X)/(BP^{<0} \cdot \Omega^*(X)) \cong \mathrm{Im}(\mathrm{res}_\Omega)/(BP^{<0} \cdot \mathrm{Im}(\mathrm{res}_\Omega)).$$

For the last isomorphism, we used the injectivity of res_Ω. We prove $b_1^2 = 0 \in CH^*(X)$. We see

$$b_1^2 = (v_1 y)^2 = v_1^2 y^2 = v_1 b_3 \in BP^*(G/T).$$

This element is contained in $BP^{<0} \cdot \mathrm{Im}(\mathrm{res}_\Omega)$. Hence b_1^2 is zero in $CH^*(X)$ as above. The other cases can be proved similarly. □

THEOREM 9.4. *Let* $(G, p) = (G_2, 2)$, $(F_4, 3)$ *or* $(E_8, 5)$, *and let* $X = \mathbb{G}_k/T_k$. *Then there is an isomorphism*

$$CH^*(X)/p \cong S(t)/(p, b_i b_j | 1 \leq i, j \leq 2p-2).$$

PROOF. From the preceding corollary we have the surjection

$$S(t)/(p, b_i b_j) \to CH^*(X)/p.$$

On the other hand, it is immediate that there is an additive isomorphism

$$S(t)/(p, b_i b_j) \cong \mathbb{Z}/p\{1, b_1, \dots, b_{2p-2}\} \otimes S(t)/(p, b).$$

There is an injection from the above right hand side module into $\Omega^*(\bar{X})/(BP^{<0} \cdot \mathrm{Im}(\mathrm{res}_\Omega))$. Hence we have the theorem. □

EXAMPLE. Let $G = F_4$ and $p = 3$. We note $G'' = \mathrm{Spin}(9) \subset G$ and

$$H^*(BG'')/3 \cong H^*(BT'')^{W''}/3 \cong \mathbb{Z}/3[p_1, \dots, p_4]$$

for the Pontryagin classes p_i [**Tod1**]. So $H^*(G''/T'')/3 \cong S(t)/(3, p_1, \dots, p_4)$. By using the induced map from $G'' \subset G$, we can see $b_i = p_i$ in $CH^*(X)/3$. Hence

$$CH^*(X)/3 \cong S(t)/(3, p_i p_j | 0 \leq i, j \leq 4).$$

Let G' be of type (I). Then it is well known ([**Mi-Tod**]) that there is a natural embedding $i : G \subset G'$ where $(G, p) = (G_2, 2), (F_4, 3)$ or $(E_8, 5)$ such that $i^* : H^*(G'; \mathbb{Z}/p) \to H^*(G; \mathbb{Z}/p)$ is surjective. Moreover the polynomial rings $P(y)$ and

$P(y)'$ are isomorphic by this map i^*. This means $CH^*(\bar{R}(\mathbb{G}_k)) \cong CH^*(\bar{R}(\mathbb{G}'_k))$. This fact implies

$$CH^*(R(\mathbb{G}_k)) \cong CH^*(R(\mathbb{G}'_k))$$

from a theorem by Vishik and Zainoulline (Corollary 6 in [**Vi-Za**]). Thus we have

COROLLARY 9.5. *Let G' be of type (I). Then there are isomorphisms*

$$CH^*(R(\mathbb{G}'_k))/p \cong \mathbb{Z}/p\{1, b_1, \dots, b_{2p-2}\},$$
$$CH^*(X')/p \cong S(t)/(p, b_ib_j, b_k | 1 \le i, j \le 2p-2,\ 2p-1 \le k \le \ell).$$

PROOF. We only need to show that for $2p-1 \le k$, we can take b_k such that $b_k = 0 \in CH^*(X')/p$. Since $b_k = 0$ in $BP^*(G/T)/I_\infty \cong H^*(G/T)/p$, in $BP^*(G/T)/I_\infty^2$, we can write

$$b_k = \sum py^it(i) + \sum v_1y^it(i)'$$

where $t(i), t(i)' \in BP^* \otimes S(t)$. Take new b_k by $b_k - \sum b_{2i}t(i) - \sum b_{2i-1}t(i)'$. Then $b_k = 0$ in $BP^*(G/T)/I_\infty^2$. □

EXAMPLE. Recall the case $(G', p) = (\mathrm{Spin}(7), 2)$ and $(G, p) = (G_2, 2)$. Then we can take $b_1 = c'_2$, $b_2 = c'_3$, and $b_3 = c_1^4$, in fact

$$CH^*(X')/2 \cong S(t)/((c'_2)^2, c'_2c'_3, (c'_3)^2, c_1^4), \quad CH^*(X)/2 \cong CH^*(X')/(c_1).$$

10. The case $G = E_8$ and $p = 3$.

In this section, we study the case $(G, p) = (E_8, p = 3)$. The cohomology $H^*(E_8; \mathbb{Z}/3)$ is isomorphic to ([**Mi-Tod**])

$$\mathbb{Z}/3[y_8, y_{20}]/(y_8^3, y_{20}^3) \otimes \Lambda(z_3, z_7, z_{15}, z_{19}, z_{27}, z_{35}, z_{39}, z_{47}).$$

Here the suffix means its degree, e.g., $|z_i| = i$. By Kono–Mimura [**Ko-Mi**] the actions of cohomology operations are also known.

THEOREM 10.1 ([**Ko-Mi**]). *We have $P^3y_8 = y_{20}$, and*

$$\beta : z_7 \mapsto y_8, \quad z_{15} \mapsto y_8^2, \quad z_{19} \mapsto y_{20}, \quad z_{27} \mapsto y_8y_{20},$$
$$z_{35} \mapsto y_8^2y_{20}, \quad z_{39} \mapsto y_{20}^2, \quad z_{47} \mapsto y_8y_{20}^2,$$
$$P^1 : \ z_3 \mapsto z_7, \quad z_{15} \mapsto z_{19}, \quad z_{35} \mapsto z_{39},$$
$$P^3 : \ z_7 \mapsto z_{19}, \quad z_{15} \mapsto z_{27} \mapsto -z_{39}, \quad z_{35} \mapsto z_{47}.$$

We use notations $y = y_8, y' = y_{20}$, and $x_1 = z_3, \dots, x_8 = z_{47}$. Then we can rewrite the isomorphisms

$$H^*(G; \mathbb{Z}/3) \cong \mathbb{Z}/3[y, y']/(y^3, (y')^3) \otimes \Lambda(x_1, \dots, x_8).$$

$$grH^*(G/T;\mathbb{Z}/3) \cong \mathbb{Z}/3[y,y']/(y^3,(y')^3) \otimes S(t)/(b_1,\dots,b_8).$$

From Lemma 3.4, we have

COROLLARY 10.2. *We can take $b_1 \in BP^*(BT)$ such that in $BP^*(G/T)/I_\infty^2$,*

$$v_1y + v_2y' = b_1.$$

From the preceding theorem, we know that all $y^i(y')^j$ except for $(i,j)=(2,2)$ are β-image. Hence we have

COROLLARY 10.3. *For all nonzero monomials $u \in P(y)/3$ except for $(yy')^2$, it holds $3u \in S(t)$. That is, for $2 \le k = i+3j+1 \le 8$, $0 \le i \le 2$, we can take in $H^*(G/T)/(3^2)$*

$$b_k = b_{i+3j+1} = 3y^i(y')^j.$$

LEMMA 10.4. *Let $(G,p)=(E_8,3)$ and $X = \mathbb{G}_k/T_k$. In $BP^*(X)$, there are $b_i \in S(t)$ such that $b_i \neq 0 \in CH^*(X)/3$ and in $BP^*(G/T)/I_\infty^2$*

$$b_k = b_{i+3j+1} = \begin{cases} v_1y+v_2y' & \text{if } k=1 \\ 3y^i(y')^j & \text{if } 0\le i\le 1,\ 2\le k \\ 3y^2(y')^j + v_1(y')^{j+1} & \text{if } i=2. \end{cases}$$

PROOF. Applying r_{Δ_1} to the equation $v_1y+v_2y'=b_1$ in $BP^*(X)/I_\infty^2$, we have

$$3y + v_1r_{\Delta_1}(y) + v_2r_{\Delta_1}(y') = r_{\Delta_1}(b_1).$$

Note $P^1(y), P^1(y') \in S(t)/3$ in $H^*(G/T;\mathbb{Z}/3)$ since they are primitive. Hence $v_1r_{\Delta_1}(y), v_2r_{\Delta_1}(y') \in BP^*\otimes S(t) \mod (I_\infty^2)$. So we have $3y=b_2$ in $BP^*(G/T)/I_\infty^2$. Applying $r_{3\Delta_1}$ to the equation $3y=b_2 \in BP^*(X)/I_\infty^2$, we have $3y' = r_{3\Delta_1}(b_2)$, which is written by b_3.

Next we study the element $3y^2$ in $BP^*(X)/I_\infty^2$. Since $3y^2 = b_3$ in $H^*(X)/(9)$, we have in $BP^*(X)/I_\infty^2$

$$3y^2 + v_1(a_1) + v_2(a_2) = b_3.$$

We can take $a_1 = y'$ by using $Q_1(x_3)=y'$ and the relation $v_1y+v_2y'=b_1$. (For example, when $a_1 = y'+yb$, we use $v_1yb = -v_2y'b$.) Since v_2a_2 is primitive in $k(2)^*(G/T)/(I_\infty^2)$ (Recall the proof of Lemma 3.4), we can take $a_2=0$. Otherwise if $a_2 = \sum y^i(y')^jb$, for $i=1,2$, then

$$v_2y^i \otimes (y')^jb \neq 0 \in k(2)^*(G)\otimes_{k(2)^*} k(2)^*(G/T).$$

Hence we get $3y^2+v_1y' = b_3$ in $BP^*(X)/I_\infty^2$.

Applying $r_{3\Delta_1}$ and $r_{6\Delta_1}$ to the above equation, we have the formulas for yy' and $(y')^2$. Here we used that $r_{3\Delta_1}(y) = y'$, and $r_{n\Delta_1}(y') \in BP^*(BT)/(I_\infty^2)$ since it is primitive. Similar arguments work for the element y^2y', and we can get the formula for $y(y')^2$. □

COROLLARY 10.5. *The torsion index $t(E_8)_{(3)} = 3^2$.*

PROOF. The fundamental class f (localized at 3) is given by $f = y_{top}t = y^2(y')^2t$ for $t = t_{top} \in S(t)$. Since $b_2b_8 = (3y)(3y(y')^2) = 3^2y_{top} \in S(t)$, we see $t(E_8)_{(3)} = 3$ or 3^2.

Suppose $t(E_8)_{(3)} = 3$, namely, $3y^2(y')^2 = b' \in S(t)$. From Lemma 3.1, this implies that there is $x \in H^*(G;\mathbb{Z}/3)$ such that $Q_0(x) = y^2(y')^2$ and $d_r(x) = b'$. But such x does not exist from Theorem 10.2. □

Recall that $A_N = \mathbb{Z}/3\{b_{i_1}\cdots b_{i_s} | |b_{i_1}| + \cdots + |b_{i_s}| \le N\}$. From Lemma 5.4, we have the surjection $A_M \otimes S(t)/(b) \twoheadrightarrow CH^*(X)/3$ for $M = |(yy')^2| = 56$.

THEOREM 10.6. *Let $(G,p) = (E_8, 3)$ and $\mathbb{G}_k$ is a versal G_k-torsor. Then we have surjective maps*

$$A_{56} \twoheadrightarrow CH^*(R(\mathbb{G}_k))/3 \twoheadrightarrow \mathbb{Z}/3\{1, b_1, \ldots, b_8, b_1b_6, b_1b_8, b_2b_8\},$$

PROOF. Since $t(E_8)_{(3)} = 3^2$ and X is a versal flag variety, we see $3(yy')^2f \notin \mathrm{res}_{\mathrm{CH}}$. It follows $3(yy')^2 \notin \mathrm{res}_{\mathrm{CH}}$. Therefore $9(yy')^2$, $3v_1(yy')^2$, $3v_2(yy')^2$ are BP^*-module generators in $\mathrm{Res}_\Omega = \mathrm{Im}(\mathrm{res}_\Omega)$. Since the restriction res_Ω is written as

$$(b_2b_8) \mapsto 9(yy')^2, \quad (b_1b_8) \mapsto 3v_1(yy')^2, \quad (b_1b_6) \mapsto 3v_2(yy')^2,$$

we have the theorem. □

COROLLARY 10.7. *Let* Tor $\subset CH^*(R(\mathbb{G}_k))$ *be the module of torsion elements. Then we have the isomorphism*

$$(CH^*(R(\mathbb{G}_k))/\mathrm{Tor}) \otimes \mathbb{Z}/3 \cong \mathbb{Z}/3\{1, b_2, \ldots, b_8, b_2b_8\}.$$

PROOF. Let us write by $b_i = py_{(i)}$ for $i \ge 2$. Let $y_{(i)}y_{(j)} \ne y^2(y')^2$. Then there is k such that $y_{(i)}y_{(j)} = y_{(k)}$. Hence $b_ib_j = 3b_k$ in $CH^*(\bar{X})$. So $b_ib_j - 3b_k$ is a torsion element because $\mathrm{res}_{\mathrm{CH}} \otimes \mathbb{Q}$ is isomorphic. □

We recall that there is an embedding $F_4 \subset E_8$. Let K/k be a field extension of degree $3a$ with $(3,a) = 1$ such that the flag variety $X|_K = (\mathbb{G}_k/T_k)|_K$ is still twisted but $X|_{K'}$ is split for an extension K'/K of degree $3a'$ with $(3,a') = 1$. Note $P^3y = y'$ and if $y \in \mathrm{res}_{\mathrm{K}}^{\bar{\mathrm{K}}}$, then so is y'. Since $X|_K$ is twisted, we see $y' \in \mathrm{res}_{\mathrm{K}}^{\bar{\mathrm{K}}}$ but y is not. Hence the J-invariants are

$$J(\mathbb{G}_K) = (1,0) \quad \text{but} \quad J(\mathbb{G}_k) = (1,1).$$

(See also 4.1.3 in [**Pe-Se-Za**], [**Se**] for E_8, $1 \ge j_1 \ge j_2$).

We know that the generalized Rost motive for F_4 and $p = 3$ is just the original Rost motive R_2. Hence the natural map $i : F_4 \to E_8$ induces the isomorphism of Chow groups over $\bar{K}$ of R_2 and $R(\mathbb{G}_K)$. By Vishik–Zainoulline ([**Vi-Za**]), we have the isomorphism

$$CH^*(R_2)/3 \cong CH^*(R(\mathbb{G}_K))/3.$$

PROPOSITION 10.8. *Let us write the restriction map* res_k^K

$$CH^*(R(\mathbb{G}_k))/3 \to CH^*(R(\mathbb{G}_k)|_K)/3 \cong CH^*(R_2)\otimes \mathbb{Z}/3[y']/((y')^3).$$

Then we have $\mathrm{Im}(\mathrm{res}_k^K)\cong \mathbb{Z}/3\{1,b_1,b_2,b_3,b_5,b_6,b_8\}$.

PROOF. This proposition is proved by considering the restriction on $\Omega^*(\bar{X})$. For example, $b_8 = 3y(y')^2 \neq 0$ in $CH^*(X|_K)/3$, but $b_2b_8 = 3\cdot(3y)(y')^2 = 0$. In particular, we use the fact that $b_4 = 3y', b_7 = 3(y')^2$ are in $\mathrm{Ker}(\mathrm{res_k^K})$. □

11. The case $G = E_8$ and $p = 2$.

In this section, we consider the case $(G,p) = (E_8,2)$. The mod (2) cohomology $H^*(E_8;\mathbb{Z}/2)$ is given [**Mi-Tod**] as

$$\mathbb{Z}/2[z_3,z_5,z_9,x_{15}]/(z_3^{16},z_5^8,z_9^4,z_{15}^4)\otimes\Lambda(z_{17},z_{23},z_{27},z_{29}).$$

Here we consider a graded algebra $grH^*(E_8;Z/2)$ identifying $y_{2i} = z_i^2$ for $i = 3,5,9,15$.

THEOREM 11.1. *The cohomology* $grH^*(E_8;\mathbb{Z}/2)$ *is given*

$$\mathbb{Z}/2[y_6,y_{10},y_{18},y_{30}]/(y_6^8,y_{10}^4,y_{18}^2,y_{30}^2)\otimes\Lambda(z_3,z_5,z_9,z_{15},z_{17},z_{23},z_{27},z_{29}).$$

Let us write $y_1 = y_6,\dots,y_4 = y_{30}$ and $x_1 = z_3, x_2 = z_5,\dots,x_8 = z_{29}$. For ease of argument, let $x_4 = z_{17}$ and $x_5 = z_{15}$. Hence we can write

$$grH^*(E_8;\mathbb{Z}/2)\cong\mathbb{Z}/2[y_1,y_2,y_3,y_4]/(y_1^8,y_2^4,y_3^2,y_4^2)\otimes\Lambda(x_1,\dots,x_8).$$

LEMMA 11.2. *The cohomology operations acts as*

$$\begin{array}{ccccccc}
x_1 = z_3 & \xrightarrow{Sq^2} & x_2 = z_5 & \xrightarrow{Sq^4} & x_3 = z_9 & \xrightarrow{Sq^8} & x_4 = z_{17}\\
x_5 = z_{15} & \xrightarrow{Sq^8} & x_6 = z_{23} & \xrightarrow{Sq^4} & x_7 = z_{27} & \xrightarrow{Sq^2} & x_8 = z_{29}\\
x_5 = z_{15} & \xrightarrow{Sq^2} & x_4 = z_{17}. & & & &
\end{array}$$

The Bockstein acts $Sq^1(x_{i+1}) = y_i$ *for* $1\le i\le 3$, $Sq^1(x_8) = y_4$ *and*

$$Sq^1\ :\ x_5 = z_{15}\mapsto y_1y_2,\quad x_6 = z_{23}\mapsto y_1y_3+y_1^4,\quad x_7 = z_{27}\mapsto y_2y_3.$$

Then we see from Lemma 3.4

COROLLARY 11.3. *In* $BP^*(X)/I_\infty^2$, *we can take* y_1 *such that for* $r_{2\Delta_1}(y_1) = y_2$ *and* $r_{4\Delta_1}(y_2) = y_3$, *we have for* $b_1\in BP^*(BT)$

$$v_1y_1+v_2y_2+v_3y_3 = b_1.$$

From Lemma 3.1 and the Sq^1 action in Lemma 11.2, it is immediate that

LEMMA 11.4. *Let* $(G,p) = (E_8,2)$ *and* $X = \mathbb{G}_k/T_k$. *In* $H^*(X)/(4)$, *there are*

$b_i \in S(t)$ such that

$$b_i = \begin{cases} 2y(1)\ (\textit{resp. } 2y(2),\ 2y(3)) & \textit{if } \ k=2\ (\textit{resp. } k=3,4) \\ 2y(1,2)\ (\textit{resp. } 2y(1,3),\ 2y(2,3)) & \textit{if } \ k=5\ (\textit{resp. } k=6,7) \\ 2y(4) & \textit{if } \ k=8, \end{cases}$$

where $\pi^*y(i) = y_i$, $\pi^*y(i,j) = y_iy_j$ for the map $\pi : G \to G/T$.

We will study b_i by using the Quillen operation r_α. In particular recall $\rho r_\alpha(x) = \chi P^\alpha(\rho(x))$ for $\rho : BP^*(X) \to H^*(X;\mathbb{Z}/2)$. The anti-automorphism χ is defined by

$$\chi(Sq^0) = Sq^0, \quad \sum_i Sq^i\chi(Sq^{n-i}) = 0 \quad \text{for} \ \ n > 0.$$

For example, (when $Sq^1 = 0$) $\chi(P^i) = P^i$ for $i = 1,2$ and $\chi(P^3) = P^2P^1$, ($P^3 = P^1P^2$), and $\chi(P^4) = P^4 + P^2P^2$.

LEMMA 11.5. *In $BP^*(X)/I_\infty^2$, we have*

$$b_i = \begin{cases} 2y_1 + v_2(y_1^2) + v_3(y_2^2) & \textit{if } \ i=2 \\ 2y_2 + v_1(y_1^2) + v_3(y_1^4) & \textit{if } \ i=3 \\ 2y_3 + v_1(y_2^2) & \textit{if } \ i=4. \end{cases}$$

PROOF. Applying the operation r_{Δ_1} to the equation $v_1y_1 + v_2y_2 + v_3y_3 = b_1$, we get

$$2y_1 + v_1r_{\Delta_1}(y_1) + v_2r_{\Delta_1}(y_2) + v_3r_{\Delta_1}(y_3) = r_{\Delta_1}(b_1).$$

Recall that $P^1(y_i)$ are primitive in $H^*(G/T;\mathbb{Z}/2)$. In fact, by Kono–Ishitoya, we know (Theorem 5.9 in [**Ko-Is2**])

$$P^1(y_1) \in S(t), \quad P^1(y_2) = y_1^2, \quad P^1(y_3) = y_2^2.$$

Thus we have $2y_1 + v_2(y_1^2) + v_3(y_2^2) = b_2$. (Note also $Q_2(x_2) = y_1^2$, $Q_3(x_2) = y_2^2$.)

Applying $r_{2\Delta_1}$ to this formula, we have

$$\begin{aligned} b_3 &= 2y_2 + v_1(y_1^2) + v_2(r_{\Delta_1}(y_1)^2) + v_3(r_{\Delta_1}(y_2)^2) \\ &= 2y_2 + v_1(y_1^2) + v_3(y_1^4). \end{aligned}$$

Applying $r_{4\Delta_1}$, we have $b_4 = 2y_3 + v_1y_2^2$ where we used $P^2(y_1) = y_2$. □

From Lemma 3.1, we see that

$$b_5 = 2y(1,2) = 2(y_1y_2 + \lambda y_1^2b')$$

where $\lambda \in \mathbb{Z}/2$, $b, b' \in S(t)$. However, stronger results are known by Nakagawa [**Na**] and Totaro [**To1**].

LEMMA 11.6 ([**Na**], [**To1**]). *In $H^*(G/T)/4$, we see $2y_1y_2 \in S(t)$. Indeed, in the notation in* [**To1**] $d_8 = 1/9d_4^2 - 2/3g_3g_5$ *where* $g_3 = y_1$, $g_5 = y_2$ *and* $d_i \in S(t)$.

LEMMA 11.7. *In $BP^*(G/T)/(I_\infty^2)$, we have, for some $b', b'' \in S(t)$*

$$b_5 = 2y_1y_2 + v_1(y_3) + v_2(y_2b' + y_2^2b'') + v_3(y_4).$$

PROOF. From the preceding lemma, we can write in $BP^*(G/T)/(v_3, I_\infty^2)$,

$$b_5 = 2y_1y_2 + v_1(a_1) + v_2(a_2) + v_3(a_3).$$

We may assume that a_1 does not contain y_1 by using the relation $b_1 = v_1y_1 + \cdots$. Note that in $k(i)^*(G/T)/I_\infty^2$, each v_ia_i is primitive. Since y_2 is not in Q_1-image in $H^*(G;\mathbb{Z}/2)$, we see y_2 is v_1-torsion free in $k(1)^*(G)$. So if a_1 contains y_2, then v_1a_1 is not primitive in $k(1)^*(G)$, which is a contradiction. (E.g., if $a_1 = y_2y$, then $\mu^*(v_1a_1) = v_1y_2 \otimes y + \cdots$.) So a_1 contains only y_3, indeed $Q_1x_5 = y_3$ implies $a_1 = y_3$.

For a_2, we know that y_1, y_3 are not v_2-torsion. Therefore a_2 only contains y_2, that is,

$$a_2 = y_2b' + y_2^2b'' \mod (y_2^2) \quad \text{for } b', b'' \in S(t).$$

By the primitivity in $k(3)^*(G/T)$, the element a_3 only contains y_3, y_4. We know $Q_3(x_5) = y_4$. If $a_3 = v_1(y_4 + y_3b'')$, then let new y_4 be the element $y_4 + y_3b''$. Thus we have the result. □

LEMMA 11.8. *In $BP^*(G/T)/(I_\infty^2, v_2, v_3)$, we have*

$$b_6 = 2(y_1y_3 + y_1^4 + y_1^2b''') \quad \textit{for } b''' \in S(t).$$

PROOF. We apply $r_{4\Delta_1}$ on b_5. By Cartan formula, we see

$$r_{4\Delta_1}(y_1y_2) = \sum_i r_{i\Delta_1}(y_1)r_{(4-i)\Delta_1}(y_2).$$

Here $r_{3\Delta_1} = \chi(P^3) = P^2P^1 \mod (2)$. Hence we have with mod (2)

$$r_{3\Delta_1}(y_1)r_{\Delta_1}(y_2) = P^2P^1(y_1)P^1(y_2) = by_1^2,$$
$$r_{2\Delta_1}(y_1)r_{2\Delta_1}(y_2) = y_2b'', \text{ and } 2y_2b'' \in S(t),$$
$$r_{\Delta_1}(y_1)r_{3\Delta}(y_2) = b''b''' \in S(t), \text{ and } y_1r_{4\Delta_1}(y_2) = y_1y_3.$$

Hence $r_{4\Delta_1}(y_1y_2) = y_1y_3 + by_1^2 \mod (BP^* \otimes S(t))$.

Next consider

$$r_{4\Delta_1}(v_1y_3) = 2r_{3\Delta}(y_3) + v_1(r_{4\Delta_1}(y_3)).$$

Here with mod (2) we see

$$r_{3\Delta}(y_3) = P^2P^1(y_3) = P^2(y_2^2) = y_1^4.$$

We also see $r_{4\Delta_1}(y_3) = P^4 y_3 \in S(t)$ from the primitivity in $H^*(G/T;\mathbb{Z}/2)$.

At last we can see

$$r_{4\Delta_1} v_2(b' y_2 + b'' y_2^2) = v_1 r_{2\Delta_1}(b' y_2 + b'' y_2^2) = 0 \mod (v_2).$$

Because if it contains $v_1 y_2$ or $v_1 y_2^2$, then it is not primitive in $k(1)^*(G/T)$, and this is a contradiction. If it contains $v_1 y_1$, then it is in Ideal(v_2) by the relation b_1. Thus we have the result (with mod (v_2, v_3)) of this lemma. □

Similarly considering $r_{2\Delta}(b_6)$ and $Q_1 x_7 = y_4$, we have

LEMMA 11.9. *In $BP^*(G/T)/(I_\infty^2, v_2, v_3)$, we have $b_7 = 2(y_2 y_3 + b y_1^2 + b' y_2^2) + v_1 y_4$.*

REMARK. For the preceding two lemmas, Totaro gets stronger and explicit results with mod $(v_1, v_2, \ldots)$. Totaro (Lemma 4.4 in [**To1**]) shows in $H^*(G/T)_{(2)}$

$$d_6^2 - 25/81 d_4^3 + 2(15 g_9 g_3 + 1/3 g_3^4 - 5/3 g_5 d_7 - 125/9 g_5 g_3 d_4) + 2^2(-23/3 g_3^2 d_6) = 0$$

where $g_3 = y_1, g_5 = y_2, g_9 = y_3$ and $d_i \in S(t)$. This implies $2(y_1 y_3 + y_1^4) \in S(t)$. Therefore we can take $b''' = 0$ in Lemma 11.8. Totaro also gives explicit formula d_7, d_8 in $H^*(G/T)$. In particular, in Lemma 4.4 in [**To1**], he shows $b = b' = 0$ in the above lemma.

At last, from $\beta(x_8) = y_4$, we note

LEMMA 11.10. *In $H^*(G/T)/4$, we see $2y_4 = b_8$.*

Now we study the torsion index. Recall

$$y_{top} = \Pi_{i=1}^4 y_i^{2^{r_i}-1} = y_1^7 y_2^3 y_3 y_4 \in P(y)$$

and t_{top} are top degree elements in $P(y)$ and $S(t)/(b)$ so that $f = y_{top} t_{top}$ for the fundamental class f of $H^*(G/T)_{(2)}$.

LEMMA 11.11 (Totaro [**To1**]). *We have $t(E_8)_{(2)} = 2^6$.*

PROOF. We consider the element

$$\tilde{b} = b_5^3 b_6 b_4 b_8 = 2^6 (y_1 y_2)^3 (y_1 y_3 + y_1^4 + y_1^2 b'')(y_3)(y_4).$$

Here using $y_3^2 = b' \in S(t) \mod (2)$, we have

$$(y_1 y_3 + y_1^4 + y_1^2 b'') y_3 = y_1^4 y_3 + (y_1 b' + y_1^2 b'') y_3.$$

Hence we can write

$$\tilde{b} = 2^6 \left(y_{top} + \sum yt \right) \quad \text{for} \quad |t| > 0.$$

From Lemma 5.5, we see $t(E_8)_{(2)} \leq 2^6$.

Suppose $t(E_8)_{(2)} \leq 2^5$, that is, $2^5 f = 2^5 y_{top} t_{top} \in S(t)$. Then $2^5 f$ must be in the ideal $I = (b_1, \ldots, b_8)$, and we can write for $b_i = 2y_{(i)}$ (note $y_{(1)} = 0$, and $y_{(i)}$ is not a monomial, in general)

$$(*) \quad 2^5 f = \sum b_i t(i) = 2\sum y_{(i)} t(i) \quad \text{for} \quad t(i) \in S(t).$$

Since $H^*(G/T)$ has no torsion, we have $2^4 f = \sum y_{(i)} t(i)$.

Let us rewrite $s = \sum y_{(i)} t(i) = \sum_I y^I t(I)$ for a monomial $y^I = y_1^{i_1} \cdots y_4^{i_4} \in P(y)$ for $I = (i_1, \ldots, i_4)$, and $t(I) \in S(t)$. Then $s \in \mathrm{Ideal}(2)$ implies each $t(I) \in \mathrm{Ideal}(b_1, \ldots, b_8) \subset S(t)$, since $H^*(G/T)/2 \cong P(y) \otimes S(t)/(b)$. Continue this argument, and then we have, in $H^*(G/T)$,

$$(**) \quad f = \sum y^I t(I).$$

Consider this equation in $H^*(G/T)/2$, and we see $f = \sum y^I t(I)$, that is $y^I = y_{top}$ and $t(I) = t_{top}$.

To get $(**)$ from $(*)$, we change b_i to $2y_{(i)}$ at most five times.

Let us write by $\sharp_y(s)$ the number of y_i's in s, namely, the largest number of $(i_1 + \cdots + i_4)$ for monomials y_I in $s = \sum y_I t(I)$. For example,

$$\sharp_y(y_{top}) = \sharp_y(y_1^7 y_2^3 y_3 y_4) = 7 + 3 + 1 + 1 = 12.$$

On the other hand, we note that $\sharp_y(y_{(j)})$ is 1 or 2 except for

$$\sharp_y(y_{(6)}) = \sharp_y((y_1 y_3 + y_1^4 + y_1^2 b)) = 4.$$

We easily see that $y_{(6)}$ appears as $y_{(i)}$ just one time in the process $(*)$ to $(**)$. We also see that $y_{(i)} = y_{(8)}$ just one time for the existence of y_4. Hence

$$\sharp_y(y_{(i_1)} \cdots y_{(i_5)}) \le 2 \times 3 + 4 + 1 = 11.$$

This is a contradiction. Thus $t(E_8)_2 \ge 2^6$. □

Lemma 11.12. *Let $(i_1, \ldots, i_k) \subset (4, 5, 5, 5, 6, 8)$. Then $\tilde{b} = b_{i_1} \cdots b_{i_k} \neq 0$ in $CH^*(X)/2$ since $b_5^3 b_4 b_6 b_8 \neq 0$.*

Let K be an extension of k such that X does not split over K but splits over an extension over K of degree $2a$, $(2, a) = 1$. Suppose that

$$(*) \quad y_1, y_2, y_3 \in \mathrm{Res_K}, \quad \text{but} \quad y_4 \notin \mathrm{Res_K}$$

where $\mathrm{Res_K} = \mathrm{Im}(\mathrm{res} : CH^*(X|_K)/2 \to CH(\bar{X})/2)$. (Compare the above condition $(*)$ with the condition $(*)$ in Section 7.) That is, $J(\mathbb{G}_K) = (0, 0, 0, 1)$ and such K exists (see [**Pe-Se-Za**], [**Se**]). Then we have the following theorem by arguments similar to those to get Theorem 7.12. (The motive $R(\mathbb{G}_k)|_K$ in the theorem is an example of motives given in Lemma 8.4 in [**Se**].)

Theorem 11.13. *There is an isomorphism*

$$CH^*(R(\mathbb{G}_k)|_K)/2 \cong \mathbb{Z}/2[y_1, y_2, y_3]/(y_1^8, y_2^4, y_3^2) \otimes CH^*(R_4)/2,$$

where $CH^*(R_4)/2 \cong \mathbb{Z}/2\{1, 2y_4, v_1y_4, v_2y_4, v_3y_4\}$. *We have*

$$\mathrm{Res}_k^K(CH^*(R(\mathbb{G}_k))/2) \cong CH^*(R_4)/2 \subset CH^*(R(\mathbb{G}_k)|_K)/2.$$

The restriction map is given as $b_j \mapsto v_{8-j}y_4$ *if* $5 \le j \le 8$, *and* $b_j \mapsto 0$ *if* $1 \le j \le 4$.

12. The exceptional group E_7 and $p = 2$.

The mod (2) cohomology of E_7 is given

$$H^*(E_7; \mathbb{Z}/2) \cong H^*(E_8; \mathbb{Z}/2)/(z_3^4, z_5^4, z_{15}^2, z_{29}).$$

We use the notations in the preceding sections.

Theorem 12.1. *We have an isomorphism*

$$grH^*(E_7; \mathbb{Z}/2) \cong \mathbb{Z}/2[y_1, y_2, y_3]/(y_1^2, y_2^2, y_3^2) \otimes \Lambda(x_1, \dots, x_7),$$

where $i^*(y_j) = y_j$ *for* $1 \le j \le 3$ *and* $i^*(x_i) = x_i$ *for* $1 \le i \le 7$ *and* $i^*(y_4) = i^*(x_8) = 0$ *for the natural embedding* $i : E_7 \subset E_8$.

Corollary 12.2. *In* $BP^*(X)/I_\infty^2$, *we can take* y_1 *such that for* $r_{2\Delta_1}(y_1) = y_2$ *and* $r_{4\Delta_1}(y_2) = y_3$, *it holds* $v_1y_1 + v_2y_2 + v_3y_3 = b_1$ *for* $b_1 \in BP^*(BT)$.

From Lemma 3.1 and the Sq^1 action in Lemma 11.2, it is immediate.

Lemma 12.3. *Let* $(G, p) = (E_7, 2)$. *In* $H^*(G/T)/(4)$, *for all monomials* $u \in P(y)/2$, *except for* $y_{top} = y_1y_2y_3$, *the elements* $2u$ *are written as elements in* $H^*(BT)$. *Namely, in* $H^*(G/T)/(4)$, *there are* $b_i \in S(t)$ *such that*

$$b_k = \begin{cases} 2y_1 \ (\textit{resp. } 2y_2,\ 2y_3) & \textit{if} \ \ k = 2 \ (\textit{resp. } k = 3, 4) \\ 2y_1y_2 \ (\textit{resp. } 2y_1y_3,\ 2y_2y_3) & \textit{if} \ \ k = 5 \ (\textit{resp. } k = 6, 7). \end{cases}$$

From Lemma 11.5, it is immediate.

Lemma 12.4. *In* $BP^*(X)/I_\infty^2$, *we have* $2y_1 = b_2$, $2y_2 = b_3$, $2y_3 = b_4$.

Lemma 12.5. *We have* $t(E_7)_{(2)} = 2^2$.

Proof. We get the result from $b_2b_7 = (2y_1)(2y_2y_3) = 2^2y_{top}$. □

Corollary 12.6. *There are surjective maps*

$$A_{34} \twoheadrightarrow CH^*(R(\mathbb{G}_k))/2 \twoheadrightarrow \mathbb{Z}/2\{1, b_1, \dots, b_7, b_1b_5, b_1b_6, b_1b_7, b_2b_7\}.$$

Proof. Note that $|y_1y_2y_3| = 34$. In $\Omega^*(\bar{X})$, we see

$$b_1b_5 = 2v_3y_{top}, \quad b_1b_6 = 2v_2y_{top}, \quad b_1b_7 = 2v_1y_{top}.$$

These elements are Ω^*-module generators in $\mathrm{Im}(res_k^{\bar{k}}(\Omega^*(X) \to \Omega^*(\bar{X}))$ because $2y_1y_2y_3 \notin \mathrm{Im}(res_k^{\bar{k}})$ from the fact $t(\mathbb{G}_k) = 2^2$. □

By the arguments similar to Corollary 10.7, we have

COROLLARY 12.7. *Let* Tor $\subset CH^*(R(\mathbb{G}))$ *be the module of torsion elements. Then we have an isomorphism*

$$CH^*(R(\mathbb{G}_k))/(2, \mathrm{Tor}) \cong \mathbb{Z}/2\{1, b_2, \dots, b_7, b_2b_7\}.$$

Let us write $G' = E_8$ and $G'' = G_2$ so that $G'' \subset G = E_7 \subset G'$. Take fields $k \subset K \subset K'$ such that

$$(**) \quad y_1^2, y_2^2, y_4 \in Res_K, \quad y_1, y_2, y_3 \notin Res_K,$$
$$(***) \quad y_1^2, y_2, y_3, y_4 \in Res_{K'}, \quad y_1 \notin Res_{K'}.$$

Then the following proposition is almost immediate

PROPOSITION 12.8. *Suppose* $(**)$ *and* $(***)$*. We have isomorphisms,*

$$CH^*(R(\mathbb{G}_k')|_K)/2 \cong \mathbb{Z}/2[y_1^2, y_2^2, y_4]/(y_1^8, y_2^4, y_4^2) \otimes CH^*(R(\mathbb{G}_K))/2,$$

the restriction is given by $b_i \mapsto b_i$ *for* $1 \le i \le 7$ *and* $b_8 \mapsto 0$*, and*

$$CH^*(R(\mathbb{G}_K)|_{K'})/2 \cong \mathbb{Z}/2[y_2, y_3]/(y_2^2, y_3^2) \otimes CH^*(R_2)/2,$$

the restriction is given by $b_i \mapsto b_i$ *for* $i = 1, 2$*, and* $b_i \mapsto 0$ *for* $3 \le i \le 7$*.*

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