J. Math. Soc. Japan
Vol. 66, No. 3 (2014) pp. 805–818
doi: 10.2969/jmsj/06630805

On polynomial mappings from the plane to the plane

By Iwona Krzyżanowska and Zbigniew Szafraniec

(Received July 25, 2012)

Abstract. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a generic polynomial mapping. There are constructed quadratic forms whose signatures determine the number of positive and negative cusps of f.

1. Introduction.

Mappings between 2-manifolds are a natural object of study in the theory of singularities. Let M, N be smooth surfaces, and let $f : M \to N$ be smooth. Whitney [**14**] proved that the singularity types of critical points of generic f are fold and cusps.

If M, N are oriented and p is a cusp point, we define $\mu(p)$ to be the local topological degree of the germ $f : (M, p) \to (N, f(p))$.

There are several results concerning relations between the topology of M, N and the topology of the critical locus of f (see [**7**], [**13**], [**14**]). In particular, there are results concerning $\sum_p \mu(p)$, where p runs through the set of cusp points (see [**3**], [**11**]). Singularities of map germs of the plane into the plane were studied in [**3**], [**4**], [**8**], [**10**]. For a recent account of the subject, and other related results, we refer the reader to [**2**], [**12**].

In this paper we investigate the number of cusps of one-generic polynomial mappings. Let $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial mapping. Denote $J = \partial(f_1, f_2)/\partial(x, y)$, $F_i = \partial(J, f_i)/\partial(x, y)$, $i = 1, 2$. Let I' be the ideal in $\mathbb{R}[x, y]$ generated by J, F_1, F_2, and $\partial(J, F_i)/\partial(x, y)$, $i = 1, 2$. We shall show that f is one-generic if $I' = \mathbb{R}[x, y]$ (see Proposition 2).

Let I be the ideal in $\mathbb{R}[x, y]$ generated by J, F_1, F_2, and let $\mathcal{A} = \mathbb{R}[x, y]/I$. (In the local case, ideals defined by the same three generators were introduced and investigated in [**3**], [**4**].) Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial. Denote $U = \{p \in \mathbb{R}^2 \mid u(p) > 0\}$.

Assume that $\dim_{\mathbb{R}} \mathcal{A} < \infty$. In Section 4 we construct four quadratic forms $\Theta_1, \ldots, \Theta_4$ on $\mathcal{A}$. We shall prove that signatures of these forms determine the number of positive and negative cusps in $\mathbb{R}^2$ and in U (Theorems 3, 4). These signatures may be computed using computer algebra systems. In Section 5 we present examples which were calculated with the help of Singular [**6**].

2. Mappings of the plane into the plane.

In this section we present useful facts about mappings of the plane into the plane. In particular we show that definitions of fold points and cusp points introduced in [**5**]

2010 *Mathematics Subject Classification.* Primary 14P99; Secondary 58K05.

Key Words and Phrases. singularities, cusps.
This research is supported by National Science Centre (grant 6093/B/H03/2011/40).

and in [**14**] coincide (see Theorem 1). In exposition and notation, we follow closely [**5**].

Let M, N be smooth manifolds, and $p \in M$. For smooth mappings $f, g : M \longrightarrow N$ with $f(p) = g(p) = q$, we say that f has first order contact with g at p if $Df(p) = Dg(p)$ as mapping of $T_pM \longrightarrow T_qN$. By $J^1(M,N)_{(p,q)}$ we shall denote the set of equivalence classes of mappings with $f(p) = q$ having the same first order contact at p. Let

$$J^1(M,N) = \bigcup_{(p,q)\in M\times N} J^1(M,N)_{(p,q)}.$$

An element $\sigma \in J^1(M,N)$ is called a 1-jet from M to N.

Denote $\operatorname{corank}\sigma = \min(\dim M, \dim N) - \operatorname{rank} Df(p)$. Put $S_r = \{\sigma \in J^1(M,N) \mid \operatorname{corank}\sigma = r\}$. According to [**5**, Theorem 5.4], S_r is a submanifold of $J^1(M,N)$ with $\operatorname{codim} S_r = r(|\dim M - \dim N| + r)$. Given a smooth mapping $f : M \longrightarrow N$ there is a canonically defined mapping $j^1 f : M \longrightarrow J^1(M,N)$. Let $S_r(f) = \{p \in M \mid \operatorname{corank} Df(p) = r\} = (j^1 f)^{-1}(S_r)$.

Definition 1. We say that $f : M \longrightarrow N$ is one-generic if $j^1 f$ intersects S_r transversely (denoted by $j^1 f \pitchfork S_r$) for all r.

According to [**5**, Theorem 4.4], if $j^1 f \pitchfork S_r$ then either $S_r(f) = \emptyset$ or $S_r(f)$ is a submanifold of M, with $\operatorname{codim} S_r(f) = \operatorname{codim} S_r$.

Assume that $M = N = \mathbb{R}^2$. In that case $J^1(\mathbb{R}^2, \mathbb{R}^2) \simeq \mathbb{R}^2 \times \mathbb{R}^2 \times M(2,2)$, where $M(2,2) = \{[a_{ij}] \mid 1 \le i,\ j \le 2\}$ is the set of 2×2-matrices.

Let $\phi = a_{11}a_{22} - a_{12}a_{21} : J^1(\mathbb{R}^2, \mathbb{R}^2) \longrightarrow \mathbb{R}$. It is easy to see that $S_0 = \{\phi \neq 0\}$, $S_1 = \{\phi = 0,\ d\phi \neq 0\}$ and $S_2 = \{\phi = 0,\ d\phi = 0\}$. In particular ϕ is locally a submersion at points of S_1. Moreover $\phi \circ j^1 f = J$, where J is the determinant of the Jacobian matrix Df, and $J^{-1}(0) = S_1(f) \cup S_2(f)$.

Lemma 1. *A mapping $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is one-generic if and only if $dJ(p) \neq 0$ for all $p \in J^{-1}(0)$. If that is the case then $S_1(f) = J^{-1}(0)$.*

Proof. As $\operatorname{codim} S_2 = 4$, then $j^1 f \pitchfork S_2$ if and only if $S_2(f) = \emptyset$, i.e. $Df(p) \neq 0$ for $p \in J^{-1}(0)$. By [**5**, Lemma 4.3], $j^1 f \pitchfork S_1$ if and only if $\phi \circ j^1 f = J$ is locally a submersion at every $p \in S_1(f)$, i.e. $dJ(p) \neq 0$ for $p \in S_1(f)$.

If f is one-generic, then $S_2(f) = \emptyset$. Hence $J^{-1}(0) = S_1(f)$ and $dJ(p) \neq 0$ for $p \in J^{-1}(0)$.

If $dJ(p) \neq 0$ for all $p \in J^{-1}(0)$, then $Df(p) \neq 0$ for all $p \in \mathbb{R}^2$. Hence $S_2(f) = \emptyset$ and $j^1 f \pitchfork S_1$. $\square$

Take a one-generic mapping $f = (f_1, f_2) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. For $p \in S_1(f)$ one of the following two conditions can occur.

$$T_pS_1(f) + \ker Df(p) = \mathbb{R}^2, \tag{1}$$

$$T_pS_1(f) = \ker Df(p). \tag{2}$$

If $p \in S_1(f)$ satisfies (1), then p is called a fold point.

Put $\mathbf{0} = (0,0) \in \mathbb{R}^2$. Notice that the space $T_pS_1(f)$ is spanned by a vector $(-\partial J/\partial y(p), \partial J/\partial x(p))$, so we get

Lemma 2. *A point $p \in \mathbb{R}^2$ is a fold point if and only if $J(p) = 0$ and*

$$Df(p) \cdot \begin{bmatrix} -\dfrac{\partial J}{\partial y}(p) \\ \dfrac{\partial J}{\partial x}(p) \end{bmatrix} \neq \mathbf{0},$$

i.e. p is a regular point of $f|_{S_1(f)}$.

So $p \in S_1(f)$ satisfies condition (2) *if and only if $J(p) = 0$ and*

$$Df(p) \cdot \begin{bmatrix} -\dfrac{\partial J}{\partial y}(p) \\ \dfrac{\partial J}{\partial x}(p) \end{bmatrix} = \mathbf{0}.$$

Assume that condition (2) holds at $p \in S_1(f)$. Take a smooth function k on a neighbourhood U of p such that $k \equiv 0$ on $S_1(f) \cap U$ and $dk(p) \neq 0$. (By Lemma 1, J satisfies both these conditions.) Let ξ be a nonvanishing vector field along $S_1(f)$ such that ξ is in the kernel of Df at each point of $S_1(f) \cap U$. Then $dk(\xi)$ is a function on $S_1(f)$ having a zero at p. The order of this zero does not depend on the choice of k and ξ (see [**5**, p. 146]), so it equals the order of $dJ(\xi)$ at p.

Definition 2. We say that p is a simple cusp if p is a simple zero of $dJ(\xi)$.

Let $F = (F_1, F_2) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be given by

$$F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = [Df] \cdot \begin{bmatrix} -\dfrac{\partial J}{\partial y} \\ \dfrac{\partial J}{\partial x} \end{bmatrix}.$$

So

$$F_1 = -\frac{\partial f_1}{\partial x}\frac{\partial J}{\partial y} + \frac{\partial f_1}{\partial y}\frac{\partial J}{\partial x} = \frac{\partial(J, f_1)}{\partial(x,y)},$$

$$F_2 = -\frac{\partial f_2}{\partial x}\frac{\partial J}{\partial y} + \frac{\partial f_2}{\partial y}\frac{\partial J}{\partial x} = \frac{\partial(J, f_2)}{\partial(x,y)}.$$

According to Lemma 2, $p \in S_1(f)$ is a fold point if and only if $F(p) \neq \mathbf{0}$.

Lemma 3. *A point $p \in S_1(f)$ is a simple cusp if and only if $F(p) = \mathbf{0}$ and*

$$[DF(p)] \cdot \begin{bmatrix} -\dfrac{\partial J}{\partial y}(p) \\ \dfrac{\partial J}{\partial x}(p) \end{bmatrix} \neq \mathbf{0},$$

i.e. if $J(p) = 0$, $F_1(p) = 0$, $F_2(p) = 0$, *and either* $\partial(J, F_1)/\partial(x, y)(p) \neq 0$ *or* $\partial(J, F_2)/\partial(x, y)(p) \neq 0$.

Proof. Put $\xi_i = (\partial f_i/\partial y, -\partial f_i/\partial x)$ on $S_1(f)$. We have $dJ(\xi_i) = F_i$. Then both $dJ(\xi_i)(p) = 0$ if and only if $F(p) = \mathbf{0}$. If that is the case then p is a simple zero of at least one $dJ(\xi_i)$ if and only if p is a regular point of at least one $F_{i|S_1(f)}$, i.e.

$$\left(\frac{\partial J}{\partial x}\frac{\partial F_i}{\partial y} - \frac{\partial J}{\partial y}\frac{\partial F_i}{\partial x}\right)(p) \neq 0.$$

The last condition holds if and only if

$$[DF(p)] \cdot \begin{bmatrix} -\dfrac{\partial J}{\partial y}(p) \\ \dfrac{\partial J}{\partial x}(p) \end{bmatrix} \neq \mathbf{0}.$$

Of course the fields ξ_1, ξ_2 are linearly dependent along $S_1(f)$, and at least one does not vanish at p. Without loss of generality we may assume that $\xi_1(p) \neq 0$, so that $\xi_2 = s \cdot \xi_1$, where s is a smooth function on $S_1(f)$. In particular, $dJ(\xi_2) = s \cdot dJ(\xi_1)$. A short computation shows that

$$Df \cdot \xi_1 \equiv 0 \quad \text{on} \quad S_1(f),$$

so that ξ_1 is in the kernel of Df along $S_1(f)$. Of course, both $dJ(\xi_i)(p) = 0$ if and only if $dJ(\xi_1)(p) = 0$, and in this case p is a simple zero of at least one $dJ(\xi_i)$ if and only if p is a simple zero of $dJ(\xi_1)$, i.e. p is a simple cusp. $\square$

Recall that $J^{-1}(0) = S_1(f)$ is a smooth 1-manifold, and $dJ \neq 0$ on $S_1(f)$. Take $p \in J^{-1}(0)$. We can find a smooth parametrization $\psi : (\mathbb{R}, 0) \longrightarrow (J^{-1}(0), p)$. There exists a smooth nowhere zero function ρ such that

$$\frac{d\psi}{dt}(t) = \rho(t)\biggl(-\frac{\partial J}{\partial y}(\psi(t)), \frac{\partial J}{\partial x}(\psi(t))\biggr).$$

It is easy to check that

$$\frac{d(f \circ \psi)}{dt}(t) = \rho(t)F(\psi(t)),$$

$$\frac{d^2(f\circ\psi)}{dt^2}(t) = \rho'(t)F(\psi(t)) + \rho^2(t)[DF(\psi(t))]\begin{bmatrix} -\dfrac{\partial J}{\partial y}(\psi(t)) \\ \dfrac{\partial J}{\partial x}(\psi(t)) \end{bmatrix}.$$

So we get

THEOREM 1. *Suppose that $f:\mathbb{R}^2\to\mathbb{R}^2$ is one-generic. Then,*

(i) *a point $p\in\mathbb{R}^2$ is a fold if and only if $J(p)=0$ and*

$$\frac{d(f\circ\psi)}{dt}(0)\neq\mathbf{0},$$

(ii) *a point $p\in\mathbb{R}^2$ is a simple cusp if and only if $J(p)=0$,*

$$\frac{d(f\circ\psi)}{dt}(0)=\mathbf{0}\quad\text{and}\quad\frac{d^2(f\circ\psi)}{dt^2}(0)\neq\mathbf{0},$$

i.e. if p is a non-degenerate critical point of $f_{|S_1(f)}$.

THEOREM 2 ([**14**, Theorem 16A], [**5**, Theorem 2.4]). *A point p is a simple cusp if and only if the germ $f:(\mathbb{R}^2,p)\to(\mathbb{R}^2,f(p))$ is differentiably equivalent to the germ $(u,v)\mapsto(u,uv+v^3)$.*

Denote by $\mu(p)$ the local topological degree of the germ $f:(\mathbb{R}^2,p)\to(\mathbb{R}^2,f(p))$. Hence we have

COROLLARY 1. *If p is a cusp point of f then $\mu(p)=\pm1$.*

3. Degree at a cusp point.

In this section we shall show how to interpret the sign of $\det DF(p)$ at a cusp point p.

LEMMA 4. *A translation in the domain does not change the determinant of $DF(p)$.*

PROOF. Take $p=(x_0,y_0)$ and the translation $T(x,y)=(x+x_0,y+y_0)$. The determinant of the Jacobian matrix associated with $g=f\circ T$ equals $J\circ T$. With g we can also associate a mapping G the same way as in Section 1, i.e.

$$G=[Dg]\cdot\begin{bmatrix} -\dfrac{\partial(J\circ T)}{\partial y} \\ \dfrac{\partial(J\circ T)}{\partial x} \end{bmatrix}.$$

Now it is easy to see that $G=F\circ T$, so $\det DG(\mathbf{0})=\det DF(p)$. □

A translation in the target obviously does not change $\det DF(p)$. From now on we shall assume that $f : \mathbb{R}^2 \to \mathbb{R}^2$ has a cups point at the origin and $f(\mathbf{0}) = \mathbf{0}$.

Lemma 5. *An orthogonal change of coordinates in the target does not change the determinant of* $DF(\mathbf{0})$.

Proof. Take an orthogonal isomorphism $L(x,y) = (ax - by, bx + ay)$, where $a^2 + b^2 = 1$. Then $\mathbf{0}$ is a cusp point of $L \circ f$, and the determinant of the Jacobian matrix associated with this mapping equals $(a^2+b^2)J = J$. With $g = L \circ f$ we can also associate

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = D(L \circ f) \cdot \begin{bmatrix} -\dfrac{\partial J}{\partial y} \\ \dfrac{\partial J}{\partial x} \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \cdot [Df] \cdot \begin{bmatrix} -\dfrac{\partial J}{\partial y} \\ \dfrac{\partial J}{\partial x} \end{bmatrix}$$
$$= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \cdot \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \cdot F = L \circ F.$$

Now it is easy to see that $\det DG(\mathbf{0}) = \det DF(\mathbf{0})$. □

Lemma 6. *An orthogonal change of coordinates in the domain does not change the determinant of* $DF(\mathbf{0})$.

Proof. Take an orthogonal isomorphism $R(x,y) = (cx - dy, dx + cy)$, where $c^2 + d^2 = 1$. Then $\mathbf{0}$ is a cusp point of $f \circ R$ and the determinant of the Jacobian matrix associated with this mapping equals $J \circ R$. With $g = f \circ R$ we can also associate

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = D(f \circ R) \cdot \begin{bmatrix} -\dfrac{\partial (J \circ R)}{\partial y} \\ \dfrac{\partial (J \circ R)}{\partial x} \end{bmatrix}$$
$$= [Df \circ R] \cdot \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \cdot \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \cdot \begin{bmatrix} -\dfrac{\partial J}{\partial y} \circ R \\ \dfrac{\partial J}{\partial x} \circ R \end{bmatrix}$$
$$= \begin{bmatrix} F_1 \circ R \\ F_2 \circ R \end{bmatrix} = F \circ R.$$

Now it is easy to see that $\det DG(\mathbf{0}) = (c^2 + d^2) \det DF(\mathbf{0}) = \det DF(\mathbf{0})$. □

Lemma 7. *Assume that* $\operatorname{rank} Df(\mathbf{0}) = 1$, $dJ(\mathbf{0}) \neq 0$ *and* $F(\mathbf{0}) = 0$. *Then after an orthogonal change of coordinates in the domain and in the target we may have*

$$\frac{\partial f_1}{\partial x}(\mathbf{0}) = \frac{\partial f_2}{\partial x}(\mathbf{0}) = \frac{\partial f_2}{\partial y}(\mathbf{0}) = 0, \quad \frac{\partial f_1}{\partial y}(\mathbf{0}) \neq 0,$$
$$\frac{\partial J}{\partial x}(\mathbf{0}) = 0, \quad \frac{\partial J}{\partial y}(\mathbf{0}) \neq 0.$$

PROOF. There exist $a, b, c, d \in \mathbb{R}$ such that $a^2 + b^2 = 1$, $c^2 + d^2 = 1$ and

$$b\left(\frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial y}\right) + a\left(\frac{\partial f_2}{\partial x}, \frac{\partial f_2}{\partial y}\right) = \mathbf{0}, \quad c\,\frac{\partial J}{\partial x} + d\,\frac{\partial J}{\partial y} = 0$$

at the origin. Let us consider two orthogonal isomorphisms: $L(x, y) = (ax - by, bx + ay)$ and $R(x, y) = (cx - dy, dx + cy)$. Put $g = (g_1, g_2) = L \circ f \circ R$. Then $\operatorname{rank} Dg(\mathbf{0}) = \operatorname{rank} Df(\mathbf{0}) = 1$ and

$$\frac{\partial g_2}{\partial x}(\mathbf{0}) = 0, \quad \frac{\partial g_2}{\partial y}(\mathbf{0}) = 0. \tag{3}$$

Let J' denote the determinant of the Jacobian matrix Dg. Of course $J' = (a^2 + b^2)(c^2 + d^2)J \circ R = J \circ R$, so that $dJ'(\mathbf{0}) \neq 0$. One may check that

$$\frac{\partial J'}{\partial x}(\mathbf{0}) = 0, \quad \text{so } \frac{\partial J'}{\partial y}(\mathbf{0}) \neq 0. \tag{4}$$

Then $(J')^{-1}(0)$ is locally a smooth 1-manifold near $\mathbf{0}$, and there is $\varphi : (\mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ such that $J'(t, \varphi(t)) \equiv 0$. Hence $0 \equiv (d/dt)J'(t, \varphi(t)) = (\partial J'/\partial x)(t, \varphi(t)) + (\partial J'/\partial y)(t, \varphi(t))\varphi'(t)$. By (4), $\varphi'(0) = 0$. Because $F(\mathbf{0}) = 0$, then $\mathbf{0}$ is a critical point of both $f|S_1(f)$ and $g|S_1(g)$. So $g(t, \varphi(t))$ has a critical point at 0. By (3)

$$\mathbf{0} = \frac{d}{dt}\begin{bmatrix} g_1(t, \varphi(t)) \\ g_2(t, \varphi(t)) \end{bmatrix}_{|t=0} = \begin{bmatrix} \frac{\partial g_1}{\partial x}(\mathbf{0}) + \frac{\partial g_1}{\partial y}(\mathbf{0})\varphi'(0) \\ \frac{\partial g_2}{\partial x}(\mathbf{0}) + \frac{\partial g_2}{\partial y}(\mathbf{0})\varphi'(0) \end{bmatrix} = \begin{bmatrix} \frac{\partial g_1}{\partial x}(\mathbf{0}) \\ 0 \end{bmatrix},$$

and so $(\partial g_1/\partial x)(\mathbf{0}) = 0$. Because $\operatorname{rank} Dg(\mathbf{0}) = 1$, so $(\partial g_1/\partial y)(\mathbf{0}) \neq 0$. □

Assume that $\operatorname{rank} Df(\mathbf{0}) = 1$, $dJ(\mathbf{0}) \neq 0$ and $F(\mathbf{0}) = \mathbf{0}$. Choose coordinates satisfying Lemma 7. If that is the case then $(\partial J/\partial y)(\mathbf{0}) = -(\partial f_1/\partial y)(\mathbf{0})(\partial^2 f_2/\partial x \partial y)(\mathbf{0})$, hence

$$\frac{\partial^2 f_2}{\partial x \partial y}(\mathbf{0}) \neq 0. \tag{5}$$

Moreover $0 = (\partial J/\partial x)(\mathbf{0}) = -(\partial f_1/\partial y)(\mathbf{0})(\partial^2 f_2/\partial x^2)(\mathbf{0})$, so that

$$\frac{\partial^2 f_2}{\partial x^2}(\mathbf{0}) = 0. \tag{6}$$

As above there is smooth $\varphi : (\mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ such that $J(t, \varphi(t)) \equiv 0$ and $\varphi'(0) = 0$. Then

$$0 = \frac{d^2}{dt^2}J(t, \varphi(t))_{|t=0} = \frac{\partial^2 J}{\partial x^2}(\mathbf{0}) + \frac{\partial J}{\partial y}(\mathbf{0})\varphi''(0),$$

and $\varphi''(0) = -(\partial^2 J/\partial x^2)(\mathbf{0})/(\partial J/\partial y)(\mathbf{0})$. As $\mathbf{0}$ is a cusp point then, according to Theorem 1,

$$\frac{d^2}{dt^2}\begin{bmatrix} f_1(t,\varphi(t)) \\ f_2(t,\varphi(t)) \end{bmatrix}_{|t=0} \neq \mathbf{0}.$$

It is easy to see that $(d^2/dt^2)f_2(t,\varphi(t))_{|t=0} = 0$, so

$$\begin{aligned} 0 \neq \frac{d^2}{dt^2} f_1(t,\varphi(t))_{|t=0} &= \frac{\partial^2 f_1}{\partial x^2}(\mathbf{0}) + \frac{\partial f_1}{\partial y}(\mathbf{0})\varphi''(0) \\ &= \frac{\partial^2 f_1}{\partial x^2}(\mathbf{0}) - \left(\frac{\partial f_1}{\partial y}(\mathbf{0})\frac{\partial^2 J}{\partial x^2}(\mathbf{0})\right)\Big/\frac{\partial J}{\partial y}(\mathbf{0}). \end{aligned} \tag{7}$$

Two non-zero vectors

$$v_1 = \frac{d^2}{dt^2}\begin{bmatrix} f_1(t,\varphi(t)) \\ f_2(t,\varphi(t)) \end{bmatrix}_{|t=0} = \begin{bmatrix} \dfrac{\partial^2 f_1}{\partial x^2}(\mathbf{0}) - \dfrac{\partial f_1}{\partial y}(\mathbf{0})\dfrac{\partial^2 J}{\partial x^2}(\mathbf{0})\Big/\dfrac{\partial J}{\partial y}(\mathbf{0}) \\ 0 \end{bmatrix},$$

$$v_2 = Df(\mathbf{0}) \cdot \begin{bmatrix} \dfrac{\partial J}{\partial x}(\mathbf{0}) \\ \dfrac{\partial J}{\partial y}(\mathbf{0}) \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f_1}{\partial y}(\mathbf{0})\dfrac{\partial J}{\partial y}(\mathbf{0}) \\ 0 \end{bmatrix}$$

point in the same direction if and only if

$$\frac{\partial f_1}{\partial y}(\mathbf{0})\left(\frac{\partial^2 f_1}{\partial x^2}(\mathbf{0})\frac{\partial J}{\partial y}(\mathbf{0}) - \frac{\partial f_1}{\partial y}(\mathbf{0})\frac{\partial^2 J}{\partial x^2}(\mathbf{0})\right) > 0.$$

Lemma 8. *We have*

(i) $\det DF(\mathbf{0}) \neq 0$,
(ii) *vectors v_1 and v_2 point in the same* (*resp. opposite*) *direction if and only if* $\det DF(\mathbf{0}) < 0$ (*resp.* $\det DF(\mathbf{0}) > 0$).

Proof. We have

$$F = (F_1, F_2) = \left(-\frac{\partial f_1}{\partial x}\frac{\partial J}{\partial y} + \frac{\partial f_1}{\partial y}\frac{\partial J}{\partial x}, -\frac{\partial f_2}{\partial x}\frac{\partial J}{\partial y} + \frac{\partial f_2}{\partial y}\frac{\partial J}{\partial x}\right).$$

By (5), (6), (7),

$$\det DF(\mathbf{0}) = \det\begin{bmatrix} -\dfrac{\partial^2 f_1}{\partial x^2}(\mathbf{0})\dfrac{\partial J}{\partial y}(\mathbf{0}) + \dfrac{\partial f_1}{\partial y}(\mathbf{0})\dfrac{\partial^2 J}{\partial x^2}(\mathbf{0}) & \dfrac{\partial F_1}{\partial y}(\mathbf{0}) \\ 0 & -\dfrac{\partial^2 f_2}{\partial x \partial y}(\mathbf{0})\dfrac{\partial J}{\partial y}(\mathbf{0}) \end{bmatrix}$$

$$= \frac{\partial J}{\partial y}(\mathbf{0})\frac{\partial^2 f_2}{\partial x \partial y}(\mathbf{0})\left(\frac{\partial^2 f_1}{\partial x^2}(\mathbf{0})\frac{\partial J}{\partial y}(\mathbf{0}) - \frac{\partial f_1}{\partial y}(\mathbf{0})\frac{\partial^2 J}{\partial x^2}(\mathbf{0})\right) \neq 0,$$

$$\operatorname{sgn}\det DF(\mathbf{0}) = \operatorname{sgn}\left[-\frac{\partial f_1}{\partial y}(\mathbf{0})\left(\frac{\partial^2 f_2}{\partial x \partial y}(\mathbf{0})\right)^2\left(\frac{\partial^2 f_1}{\partial x^2}(\mathbf{0})\frac{\partial J}{\partial y}(\mathbf{0}) - \frac{\partial f_1}{\partial y}(\mathbf{0})\frac{\partial^2 J}{\partial x^2}(\mathbf{0})\right)\right]$$

$$= (-1)\operatorname{sgn}\left[\frac{\partial f_1}{\partial y}(\mathbf{0})\left(\frac{\partial^2 f_1}{\partial x^2}(\mathbf{0})\frac{\partial J}{\partial y}(\mathbf{0}) - \frac{\partial f_1}{\partial y}(\mathbf{0})\frac{\partial^2 J}{\partial x^2}(\mathbf{0})\right)\right]. \qquad \square$$

Lemma 9. *The local topological degree of the germ $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ equals -1 (resp. $+1$) if vectors v_1, v_2 point in the same (resp. opposite) direction.*

Proof. By Theorem 2 one may conclude that there exists open neighbourhoods U, V of the origin in the plane such that

1. $f(U) = V$,
2. the set of regular values of $f : U \to V$, i.e. $V \setminus f(S_1(f) \cap U)$, consists of two connected components V_1, V_2 such that $q \in V_1$ if and only if $f^{-1}(q) \cap U$ has one element, and $q \in V_2$ if and only if $f^{-1}(q) \cap U$ has three element,
3. there exists a unit vector v such that for any sequence $(q_n) \subset \bar{V}_2$ with $\lim q_n = \mathbf{0}$ and $q_n \neq \mathbf{0}$ we have

$$\lim \frac{q_n}{|q_n|} = v.$$

Hence, if q lies close to the origin and the scalar product $\langle q, v\rangle$ is negative then $q \in V_1$.

The curve $(t, \varphi(t))$ is a smooth parametrization of $S_1(f)$ near the origin. There is $\epsilon > 0$ such that $\gamma(t) = f(t, \varphi(t)) \in \bar{V}_2 \setminus \{\mathbf{0}\}$ for $0 < |t| < \epsilon$. By Theorem 1,

$$\frac{d\gamma}{dt}(0) = \mathbf{0}, \quad v_1 = \frac{d^2\gamma}{dt^2}(0) \neq \mathbf{0}.$$

There exists smooth $\alpha : \mathbb{R}, 0 \to \mathbb{R}^2$ such that $\alpha(0) = v_1/2$ and $\gamma(t) = t^2 \cdot \alpha(t)$. Then

$$v = \lim \frac{\gamma(t)}{|\gamma(t)|} = \lim \frac{\alpha(t)}{|\alpha(t)|} = \frac{v_1}{|v_1|},$$

so that vectors v, v_1 point in the same direction.

Let $\eta(t) = f((\partial J/\partial x)(\mathbf{0})t, (\partial J/\partial y)(\mathbf{0})t)$. Then $\eta(0) = \mathbf{0}$, and

$$\frac{d\eta}{dt}(0) = Df(\mathbf{0}) \cdot \begin{bmatrix} \dfrac{\partial J}{\partial x}(\mathbf{0}) \\ \dfrac{\partial J}{\partial y}(\mathbf{0}) \end{bmatrix} = v_2 \neq \mathbf{0}.$$

There exists a smooth function $\beta : \mathbb{R}, 0 \to \mathbb{R}^2$ such that $\beta(0) = v_2$ and $\eta(t) = t \cdot \beta(t)$.

If vectors v, v_2 point in the same direction then the scalar product $\langle \eta(t), v\rangle =$

$t\langle\beta(t), v\rangle$ is negative for all $t < 0$ sufficiently close to the origin. Then $\eta(t) \in V_1$, so that $\eta(t)$ is a regular value and $f^{-1}(\eta(t)) \cap U = ((\partial J/\partial x)(\mathbf{0})t, (\partial J/\partial y)(\mathbf{0})t)$.

In this case $J((\partial J/\partial x)(\mathbf{0})t, (\partial J/\partial y)(\mathbf{0})t)$ is negative, hence the local topological degree of the mapping $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ equals -1.

If vectors v, v_2 point in the opposite direction then the scalar product $\langle\eta(t), v\rangle = t\langle\beta(t), v\rangle$ is negative for all $t > 0$ sufficiently close to the origin. As before, $f^{-1}(\eta(t)) \cap U = ((\partial J/\partial x)(\mathbf{0})t, (\partial J/\partial y)(\mathbf{0})t)$.

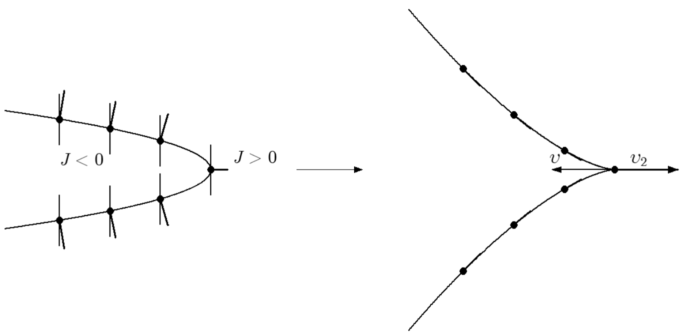
In that case $J((\partial J/\partial x)(\mathbf{0})t, (\partial J/\partial y)(\mathbf{0})t)$ is positive, hence the local topological degree of the mapping $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ equals $+1$. $\square$

Proposition 1. *Assume that p is a cusp point of a mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$. Then* $\det DF(p) \neq 0$, *and the local topological degree $\mu(p)$ of the germ $f : (\mathbb{R}^2, p) \to (\mathbb{R}^2, f(p))$ equals* $\operatorname{sgn} \det DF(p)$.

Proof. A translation, as well as an orthogonal isomorphism, does not change the local topological degree. So we may assume that $p = f(p) = \mathbf{0}$, and choose coordinates satisfying Lemma 7. The assertion of the proposition is a consequence of Lemmas 8 and 9. $\square$

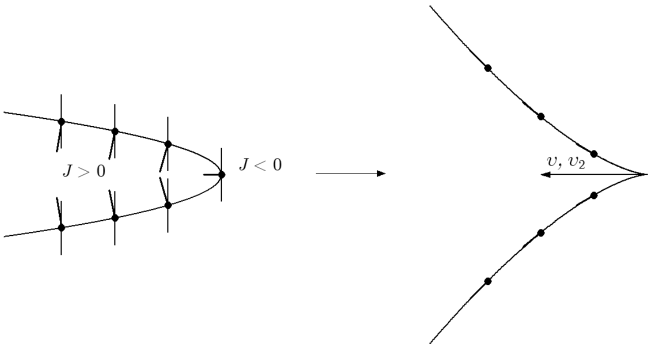
In order to help the reader to understand a geometric meaning of Proposition 1 and the previous computation, we present two figures.

The mapping $(x, y) \mapsto (x, y^3 + xy)$ has a positive cusp point at $\mathbf{0}$.



The vertical lines in the source show the kernel fields. The notations •— show the gradients of J along the critical locus and their images. In this case $v = (-1, 0)$, $v_2 = (1, 0)$, and they point in the opposite direction.

The mapping $(x, y) \mapsto (x, -y^3 - xy)$ has a negative cusp point at $\mathbf{0}$.



In this case $v=(-1,0)$, $v_2=(-1,0)$, and they point in the same direction.

4. Polynomial mappings.

This section is devoted to the problem of determining the number of cusps of a polynomial mapping $f=(f_1,f_2):\mathbb{R}^2\to\mathbb{R}^2$. Denote by Σ the set of cusp points of f. Let I be the ideal in $\mathbb{R}[x,y]$ generated by J, F_1, F_2. Let I' be the one generated by J, F_1, F_2, $\partial(J,F_1)/\partial(x,y)$, $\partial(J,F_2)/\partial(x,y)$.

Proposition 2. *If $I'=\mathbb{R}[x,y]$ then f is one-generic and the set of critical points of f consists of fold points and cusp points. Moreover, $\Sigma=\{J=0,F_1=0,F_2=0\}$ is finite.*

Proof. One may observe that I' is contained in the ideal generated by J, $\partial J/\partial x$, $\partial J/\partial y$. Therefore the last ideal also equals $\mathbb{R}[x,y]$, and then its set of zeroes is empty. Hence, if $J(p)=0$ then either $\partial J/\partial x(p)\neq 0$ or $\partial J/\partial y(p)\neq 0$. By Lemma 1, f is one-generic.

Let p be a critical point, so that $J(p)=0$. Because the set of zeroes of I' is empty, then either $F_i(p)\neq 0$ or $\partial(J,F_i)/\partial(x,y)(p)\neq 0$ for some i. By Lemma 2, if $F_i(p)\neq 0$ then p is a fold point.

If both $F_1(p)=0$, $F_2(p)=0$ then some $\partial(J,F_i)/\partial(x,y)(p)\neq 0$, and then p is a cusp point by Lemma 3. Thus Σ is an algebraic set given by three equations $J=0$, $F_1=0$, $F_2=0$. On the other hand Σ is always discrete, and then finite. □

From now on we shall assume that $I'=\mathbb{R}[x,y]$. By the previous proposition, Σ equals the set of zeroes of I. Let $\mathcal{A}$ denote the $\mathbb{R}$-algebra $\mathbb{R}[x,y]/I$. Assume that $\dim_{\mathbb{R}}\mathcal{A}<\infty$.

For $h\in\mathcal{A}$, we denote by $T(h)$ the trace of the $\mathbb{R}$-linear endomorphism $\mathcal{A}\ni a\mapsto h\cdot a\in\mathcal{A}$. Then $T:\mathcal{A}\to\mathbb{R}$ is a linear functional. Take $\delta\in\mathbb{R}[x,y]$. Let $\Theta:\mathcal{A}\to\mathbb{R}$ be the quadratic form given by $\Theta(a)=T(\delta\cdot a^2)$.

According to [**1**], [**9**], the signature $\sigma(\Theta)$ of Θ equals

$$\sigma(\Theta)=\sum \operatorname{sgn}\delta(p),\quad \text{where } p\in\Sigma, \tag{8}$$

and if Θ is non-degenerate then $\delta(p)\neq 0$ for each $p\in\Sigma$.

Define quadratic forms $\Theta_1(a) = T(a^2)$, $\Theta_2(a) = T(\det DF \cdot a^2)$.

THEOREM 3. *Suppose that* $I' = \mathbb{R}[x,y]$ *and* $\dim_R \mathcal{A} < \infty$. *Then*

(i) $\#\Sigma = \sigma(\Theta_1)$,
(ii) $\sum_{p\in\Sigma} \mu(p) = \sigma(\Theta_2)$,
(iii) $\#\{p \in \Sigma \mid \mu(p) > 0\} = (\sigma(\Theta_1) + \sigma(\Theta_2))/2$,
$\#\{p \in \Sigma \mid \mu(p) < 0\} = (\sigma(\Theta_1) - \sigma(\Theta_2))/2$.

PROOF. By Propositions 1, 2, if p is a zero of the ideal I then $p \in \Sigma$ and $DF(p) \neq 0$. Since $\Theta_1(a) = T(1 \cdot a^2)$, by (8) its signature equals $\sum_{p\in\Sigma} 1 = \#\Sigma$. By (8) and Theorem 1, the signature of Θ_2 equals $\sum_{p\in\Sigma} \operatorname{sgn} DF(p) = \sum_{p\in\Sigma} \mu(p)$. Assertion (iii) is now obvious. □

Take $u \in \mathbb{R}[x,y]$. Put $U = \{p \in \mathbb{R}^2 \mid u(p) > 0\}$. The remainder of this section is devoted to the problem of determining the number of cusps in U. Define quadratic forms $\Theta_3(a) = T(u \cdot a^2)$, $\Theta_4(a) = T(u \cdot \det DF \cdot a^2)$.

THEOREM 4. *Suppose that* $I' = \mathbb{R}[x,y]$ *and* $\dim_R \mathcal{A} < \infty$. *If* Θ_3 *is non-degenerate then*

(i) $\Sigma \cap u^{-1}(0) = \emptyset$,
(ii) $\#\{p \in \Sigma \cap U \mid \mu(p) = +1\} = (\sigma(\Theta_1) + \sigma(\Theta_2) + \sigma(\Theta_3) + \sigma(\Theta_4))/4$,
(iii) $\#\{p \in \Sigma \cap U \mid \mu(p) = -1\} = (\sigma(\Theta_1) - \sigma(\Theta_2) + \sigma(\Theta_3) - \sigma(\Theta_4))/4$.

PROOF. As in the previous proof, $\det DF(p) \neq 0$ at each $p \in \Sigma$. Since Θ_3 is non-degenerate, by (8) $u(p) \neq 0$ at each $p \in \Sigma$.

For $0 \leq i,\ j \leq 1$ denote

$$a_{ij} = \#\{p \in \Sigma \mid \operatorname{sgn}\det DF(p) = (-1)^i,\ \operatorname{sgn} u(p) = (-1)^j\}.$$

These numbers satisfy the equations:

$$a_{00} + a_{10} + a_{01} + a_{11} = \sigma(\Theta_1),$$
$$a_{00} - a_{10} + a_{01} - a_{11} = \sigma(\Theta_2),$$
$$a_{00} + a_{10} - a_{01} - a_{11} = \sigma(\Theta_3),$$
$$a_{00} - a_{10} - a_{01} + a_{11} = \sigma(\Theta_4).$$

Now it is easy to verify that $a_{00} = (\sigma(\Theta_1) + \cdots + \sigma(\Theta_4))/4$, $a_{10} = (\sigma(\Theta_1) - \sigma(\Theta_2) + \sigma(\Theta_3) - \sigma(\Theta_4))/4$. □

5. Examples.

EXAMPLE 1. Let $f = (f_1, f_2) = (xy^2 - x^2 + y^2 + x - y, x - y) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. It is easy to check that

$$J = -2xy - y^2 + 2x - 2y, \quad F_1 = -2xy^2 + 2y^3 - 4x^2 - 2y^2 - 2x + 8y, \quad F_2 = 2x + 4y.$$

Using SINGULAR one may verify that $I' = \mathbb{R}[x,y]$. According to Proposition 2 the mapping f is one-generic having only folds and cusps as critical points. Moreover the set of cusps Σ is finite. The algebra $\mathcal{A} = \mathbb{R}[x,y]/I$ is two-dimensional, and has a basis $e_1 = y$, $e_2 = 1$. Put $u = 1 - x^2 - y^2$. The matrices of quadratic forms $\Theta_1, \Theta_2, \Theta_3, \Theta_4$ are

$$\begin{bmatrix} +4 & +2 \\ +2 & +2 \end{bmatrix}, \quad \begin{bmatrix} -96 & -48 \\ -48 & -48 \end{bmatrix}, \quad \begin{bmatrix} -76 & -38 \\ -38 & -18 \end{bmatrix}, \quad 24 \cdot \begin{bmatrix} +76 & +38 \\ +38 & +18 \end{bmatrix}.$$

So the quadratic form Θ_3 is non-degenerate and $\sigma(\Theta_1) = 2$, $\sigma(\Theta_2) = -2$, $\sigma(\Theta_3) = \sigma(\Theta_4) = 0$. According to Theorems 3 and 4 the mapping f has two cusps, both of negative sign, one of them lies in $U = \{u > 0\}$. In this example it is easy to verify that the cusp points are $(0,0)$ and $(-4,2)$.

EXAMPLE 2. Put $f = (x^2y^3 - x^2y + xy^2 - x, x^3y - x^2y + y^3 + x - y) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $u = x^2 + y^2 - 1$. Using the same method as before with the help of SINGULAR, one can check that f is one-generic and the dimension of $\mathcal{A} = \mathbb{R}[x,y]/I$ equals 38. Moreover f has eight cusps, six of them are positive and two are negative. All negative and three positive ones lie in $U = \{u > 0\}$.

EXAMPLE 3. Let $f = (10x^2y^3 + 4x^2y^2 - 2xy^3 - 6x^2y + 8xy^2 - 5xy, 5x^4y + 10x^4 - y^4 + 5x^2 - 3xy - 9y)$ and $u = x - 1$. In this case f is one-generic and the dimension of $\mathcal{A} = \mathbb{R}[x,y]/I$ equals 56. Moreover f has six cusps, five of them are positive and one is negative. The negative one lies in $U = \{u > 0\}$.

References

[1] E. Becker and T. Wöermann, On the trace formula for quadratic forms, Contemp. Math., **155**, In: Recent Advances in Real Algebraic Geometry and Quadratic Forms, California, 1990/1991, (eds. W. B. Jacob, T. Y. Lam, and R. O. Robson), Amer. Math. Soc., Providence, RI, 1994, pp. 271–291.

[2] N. Dutertre and T. Fukui, On the topology of stable maps, J. Math. Soc. Japan, **66** (2014), 161–203.

[3] T. Fukuda and G. Ishikawa, On the number of cusps of stable perturbations of a plane-to-plane singularity, Tokyo J. Math., **10** (1987), 375–384.

[4] T. Gaffney and D. M. Q. Mond, Cusps and double folds of germs of analytic maps $\mathbb{C}^2 \to \mathbb{C}^2$, J. London Math. Soc. (2), **43** (1991), 185–192.

[5] M. Golubitsky and V. Guillemin, Stable Mappings and Their Singularities, Grad. Texts in Math., **14**, Springer-Verlag, New York, 1973.

[6] G.-M. Greuel, G. Pfister and H. Schönemann, SINGULAR 3.0.2. A Computer Algebra System for Polynomial Computations.

[7] H. I. Levine, Mappings of manifolds into the plane, Amer. J. Math., **88** (1966), 357–365.

[8] J. A. Moya-Pérez and J. J. Nuño-Ballesteros, The link of a finitely determined map germ from $\boldsymbol{R}^2$ to $\boldsymbol{R}^2$, J. Math. Soc. Japan, **62** (2010), 1069–1092.

[9] P. Pedersen, M.-F. Roy and A. Szpirglas, Counting real zeros in the multivariate case, In: Computational Algebraic Geometry, Nice, 1992, (eds. F. Eyssette and A. Galligo), Progr. Math., **109**, Birkhäuser, 1993, pp. 203–224.

[10] J. H. Rieger, Families of maps from the plane to the plane, J. London Math. Soc. (2), **36** (1987), 351–369.

[11] J. R. Quine, A global theorem for singularities of maps between oriented 2-manifolds, Trans. Amer. Math. Soc., **236** (1978), 307–314.

[12] O. Saeki, Topology of Singular Fibers of Differentiable Maps, Lecture Notes in Math., **1854**, Springer-Verlag, 2004, Berlin.

[13] R. Thom, Les singularités des applications différentiables, Ann. Inst. Fourier, Grenoble, **6** (1955–1956), 43–87.

[14] H. Whitney, On singularities of mapping of Euclidean spaces. I. Mappings of the plane into the plane, Ann. of Math. (2), **62** (1955), 374–410.

Iwona Krzyżanowska
Institute of Mathematics
University of Gdańsk
80-952 Gdańsk
Wita Stwosza 57, Poland
E-mail: Iwona.Krzyzanowska@mat.ug.edu.pl

Zbigniew Szafraniec
Institute of Mathematics
University of Gdańsk
80-952 Gdańsk
Wita Stwosza 57, Poland
E-mail: Zbigniew.Szafraniec@mat.ug.edu.pl