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Geometric properties of the Riemann surfaces associated with the Noumi-Yamada systems with a large parameter

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Abstract. The system of algebraic equations for the leading terms of formal solutions to the Noumi-Yamada systems with a large parameter is studied. A formula which gives the number of solutions outside of turning points is established. The number of turning points of the first kind is also given.

1. Introduction.

The aim of this article is to study the structure of the one-dimensional algebraic varieties associated with the Noumi-Yamada systems of differential equations with a large parameter. The Noumi-Yamada systems, which will be denoted by $(NY)_l$ ($l = 2, 3, 4, \dots$) in this article, are discovered by M. Noumi and Y. Yamada as higher-order generalizations of the Painlevé equations from the viewpoint of the affine Weyl group symmetry of type $A_l^{(1)}$. In the pioneering work [**T**], Y. Takei introduces a large parameter η for $(NY)_l$ to analyze them by using the exact WKB analysis, namely, WKB analysis based on the Borel resummation. To be more specific, Takei defines the notions of turning points and Stokes curves for $(NY)_l$ and finds a relation of Stokes geometries between $(NY)_l$ and its underlying Lax pair, which is similar to the case of traditional Painlevé equations. This result suggests that the exact WKB analysis is effective in studying analytic properties of solutions of $(NY)_l$. The research of $(NY)_l$ in this direction is, however, only in the early stage. Starting point of the research is to construct the so-called 0-parameter solutions which are formal solutions expanded in power series in η^{-1}. Such solutions can be easily constructed once the leading terms are specified (see Section 2.3). But the existence of the leading terms of the solutions is highly nontrivial because we have to solve systems of algebraic equations. In fact, the existence for general l is assumed in [**T**] but later the authors proved it for the even l in [**AH**]. This article concerns not only the existence for general l but also the number of the leading terms outside of the set of turning points and the number of turning points of the first kind, which is introduced in [**T**]. Our main theorems

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(Theorems 6 and 7, Section 5) establish formulas which give these numbers.

One of the advantages of our analysis is to treat the system of algebraic equations for the leading terms in a unified manner, although the appearance of the system looks quite different for even l and for odd l. We consider the algebraic variety defined by the system. Then we introduce a hyperbolic system (Section 3) and we show that, for general l, the algebraic variety is isomorphic to the variety defined by the hyperbolic system with some normalization. Since the hyperbolic system has a simple form, we can analyze it easily (Section 4). The main tool of counting the number of solutions for a fixed independent variable or that of turning points of the first kind is the classical theorem of Bézout. We use one of the most sophisticated forms of the theorem due to T. Suwa [**S**]. To use the theorem, we have to consider our problems in the projective space. Hence it is necessary to exclude the solutions at infinity to obtain the number of finite solutions. This part requires careful analysis but our discussion is elementary (Section 5).

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2. Preliminaries.

2.1. The Noumi-Yamada system $(NY)_l$ with a large parameter.

The Noumi-Yamada system $(NY)_l$ $(l = 2, 3, \dots)$ introduced in [**NY1**] and [**NY2**] is a system of non-linear differential equations of $(l+1)$-unknown functions $u_0(t), \dots, u_l(t)$ of the variable t. We consider, in this paper, the system $(NY)_l$ with a large parameter η according to [**T**], for which we can recover the original one if we put $\eta = 1$. Let $\hat{\alpha}_j \in \boldsymbol{C}[\eta^{-1}]$ $(0 \le j \le l)$ be a parameter satisfying

$$\hat{\alpha}_0 + \hat{\alpha}_1 + \cdots + \hat{\alpha}_l = \eta^{-1}, \tag{1}$$

where $\boldsymbol{C}[\eta^{-1}]$ designates the set of polynomials of η^{-1} over $\boldsymbol{C}$. We also denote by $\alpha_j \in \boldsymbol{C}$ the leading term of $\hat{\alpha}_j$ $(j = 0, 1, \dots, l)$, i.e. $\hat{\alpha}_j = \alpha_j + O(\eta^{-1})$. The appearance of the system depends on the parity of l. If $l = 2m$ is even, the system $(NY)_{2m}$ with a large parameter η consists of $(2m+1)$-differential equations

$$\eta^{-1}\frac{du_j}{dt} = u_j(u_{j+1} - u_{j+2} + \cdots - u_{j+2m}) + \hat{\alpha}_j \quad (j = 0, 1, \dots, 2m), \tag{2}$$

and the normalization condition defined by one equation

$$u_0 + u_1 + \cdots + u_{2m} = t. \tag{3}$$

If $l = 2m+1$ is odd, the system $(NY)_{2m+1}$ with a large parameter η consists of $(2m+2)$-differential equations

$$\begin{cases} \eta^{-1}\dfrac{t}{2}\dfrac{du_j}{dt} = u_j\left(\Pi_j + \dfrac{1}{2}\eta^{-1} - \hat{\alpha}_{\text{even}}\right) + \dfrac{1}{2}\hat{\alpha}_j t \quad (j = 0, 2, \ldots, 2m), \\ \eta^{-1}\dfrac{t}{2}\dfrac{du_j}{dt} = u_j\left(\Pi_j + \dfrac{1}{2}\eta^{-1} - \hat{\alpha}_{\text{odd}}\right) + \dfrac{1}{2}\hat{\alpha}_j t \quad (j = 1, 3, \ldots, 2m+1) \end{cases} \tag{4}$$

and the normalization condition defined by two equations

$$u_0 + u_2 + \cdots + u_{2m} = \frac{t}{2}, \quad u_1 + u_3 + \cdots + u_{2m+1} = \frac{t}{2}, \tag{5}$$

where we set

$$\begin{aligned} \Pi_j &= \sum_{1 \le r \le s \le m} u_{j+2r-1}u_{j+2s} - \sum_{1 \le s \le q \le m} u_{j+2s}u_{j+2q+1}, \\ \hat{\alpha}_{\text{even}} &= \hat{\alpha}_0 + \hat{\alpha}_2 + \cdots + \hat{\alpha}_{2m}, \quad \hat{\alpha}_{\text{odd}} = \hat{\alpha}_1 + \hat{\alpha}_3 + \cdots + \hat{\alpha}_{2m+1}. \end{aligned} \tag{6}$$

Note that, throughout this paper, the indices of $\hat{\alpha}_j$'s and u_j's are considered to be cyclic mod $l+1$.

REMARK 1. In [**T**], the extra condition $\hat{\alpha}_{\text{even}} = \hat{\alpha}_{\text{odd}} = 0$ is assumed for the sake of simplicity. Our discussion does not use this assumption.

2.2. A leading variety associated with $(NY)_l$.

We introduce, in this subsection, the system of algebraic equations which the leading term of a 0-parameter solution of $(NY)_l$ satisfies (see the next subsection for the definition of 0-parameter solution). We also introduce the leading variety associated with $(NY)_l$, on which the Stokes geometry for $(NY)_l$ is defined.

Recall that $\alpha_j \in \boldsymbol{C}$ $(0 \le j \le l)$ denotes the leading term of $\hat{\alpha}_j$, for which

$$\alpha_{\text{total}} := \alpha_0 + \alpha_1 + \cdots + \alpha_l = 0 \tag{7}$$

is satisfied by (1). We define the polynomials Π_j and f_j $(j = 0, 1, \ldots, l)$ of the variables $u = (u_0, \ldots, u_l)$ by, for even $l = 2m$,

$$\begin{aligned} \Pi_j &:= u_{j+1} - u_{j+2} + \cdots - u_{j+2m} \quad (j = 0, 1, \ldots, l), \\ f_j &:= u_j \Pi_j + \alpha_j \qquad (j = 0, 1, \ldots, l) \end{aligned} \tag{8}$$

and, for odd $l = 2m+1$,

$$\Pi_j := \sum_{1 \le r \le s \le m} u_{j+2r-1} u_{j+2s} - \sum_{1 \le s \le q \le m} u_{j+2s} u_{j+2q+1} \qquad (j = 0, 1, \dots, 2m+1), \tag{9}$$

$$\begin{cases} f_j := u_j(\Pi_j - \alpha_{\mathrm{even}}) + \dfrac{1}{2}\alpha_j t & (j = 0, 2, \dots, 2m), \\ f_j := u_j(\Pi_j - \alpha_{\mathrm{odd}}) + \dfrac{1}{2}\alpha_j t & (j = 1, 3, \dots, 2m+1), \end{cases} \tag{10}$$

where we set $\alpha_{\mathrm{even}} := \alpha_0 + \alpha_2 + \cdots + \alpha_{2m}$ and $\alpha_{\mathrm{odd}} := \alpha_1 + \alpha_3 + \cdots + \alpha_{2m+1}$. We also set

$$u_{\mathrm{total}} := \sum_{k=0}^{l} u_k, \quad u_{\mathrm{even}} := \sum_{k=0}^{m} u_{2k}, \quad u_{\mathrm{odd}} := \sum_{k=0}^{m} u_{2k+1}.$$

Let X be a complex affine space $\boldsymbol{C}_t \times \boldsymbol{C}_u^{l+1}$ with a system of coordinates $(t, u) = (t, u_0, \dots, u_l)$, and let $\mathscr{O}_X$ denote the sheaf of holomorphic functions on X. We define the $\mathscr{O}_X$ ideal

$$\mathscr{I} := \begin{cases} \mathscr{O}_X(f_0, f_1, \dots, f_l, u_{\mathrm{total}} - t) & \text{if } l \text{ is even}, \\ \mathscr{O}_X\left(f_0, f_1, \dots, f_l, u_{\mathrm{even}} - \dfrac{t}{2}, u_{\mathrm{odd}} - \dfrac{t}{2}\right) & \text{if } l \text{ is odd}, \end{cases} \tag{11}$$

and the $\mathscr{O}_X$ module

$$\mathscr{N} := \frac{\mathscr{O}_X}{\mathscr{I}}. \tag{12}$$

Then *the leading variety* V *associated with* $(NY)_l$ is, by definition, the support of $\mathscr{N}$, that is, the common zero set of functions belonging to the ideal $\mathscr{I}$.

Note that, if l is odd, the hypersurface $\{t = 0\} \subset X$ is exceptional in the sense that the system $(NY)_l$ has regular singularity along $t = 0$ and the canonical projection $\pi|_V \colon V \to \boldsymbol{C}_t$ is neither proper nor finite over $t = 0$. Hence we usually consider the problem outside of $\{t = 0\}$ if l is odd, and we introduce the following notation for convenience:

$$\check{V} := \begin{cases} V & \text{if } l \text{ is even,} \\ V \setminus \{t = 0\} & \text{if } l \text{ is odd,} \end{cases}$$
$$\check{\boldsymbol{C}}_t := \begin{cases} \boldsymbol{C}_t & \text{if } l \text{ is even,} \\ \boldsymbol{C}_t \setminus \{0\} & \text{if } l \text{ is odd} \end{cases} \tag{13}$$

and

$$\check{\pi}_t : \check{\boldsymbol{C}}_t \times \boldsymbol{C}^{l+1} \to \check{\boldsymbol{C}}_t \tag{14}$$

being the canonical projection with respect to the variable t.

Since parameters satisfy $\alpha_{\text{total}} = 0$, we can easily see, for an even $l = 2m$,

$$f_0 + f_1 + \cdots + f_l = 0 \tag{15}$$

and, for an odd $l = 2m + 1$,

$$f_0 + f_2 + \cdots + f_{2m} + \alpha_{\text{even}}\left(u_{\text{even}} - \frac{t}{2}\right) = 0,$$
$$f_1 + f_3 + \cdots + f_{2m+1} + \alpha_{\text{odd}}\left(u_{\text{odd}} - \frac{t}{2}\right) = 0. \tag{16}$$

Hence, if we define $(l+1)$-polynomials $h_0, \ldots, h_l$ by

$$\{h_0, \ldots, h_l\} := \begin{cases} \{f_0, \ldots, f_{l-1}, u_{\text{total}} - t\} & \text{if } l \text{ is even,} \\ \left\{f_0, \ldots, f_{l-2}, u_{\text{even}} - \dfrac{t}{2}, u_{\text{odd}} - \dfrac{t}{2}\right\} & \text{if } l \text{ is odd,} \end{cases} \tag{17}$$

then we have $\mathscr{I} = \mathscr{O}_X(h_0, h_1, \ldots, h_l)$. It follows from Theorem 5 that $\{h_0, \ldots, h_l\}$ forms a regular sequence over $\mathscr{O}_{X,p}$ for any $p \in \check{V}$.

2.3. A construction of a 0-parameter solution for $(\boldsymbol{NY})_l$.

Let $u(t) = (u_0(t), \ldots, u_l(t))$ be a formal power series of η^{-1} in the form

$$u(t) = u^{(0)}(t) + u^{(1)}(t)\eta^{-1} + u^{(2)}(t)\eta^{-2} + u^{(3)}(t)\eta^{-3} + \cdots. \tag{18}$$

Here we set $u^{(k)}(t) = (u_0^{(k)}(t), \ldots, u_l^{(k)}(t))$ and $u_j^{(k)}(t)$ $(k = 0, 1, 2, \ldots;\ j = 0, 1, \ldots, l)$ are multi-valued holomorphic functions over $\check{\boldsymbol{C}}_t$ except for a finite number of exceptional points. We say that $u(t)$ is *a 0-parameter formal solution* of $(NY)_l$ if it satisfies $(NY)_l$ as a formal power series of η^{-1}.

We briefly explain how to construct such a 0-parameter formal solution, which does not necessarily exist for an arbitrary parameter of $(NY)_l$. We introduce several subsets of the space of parameters $(\alpha_0, \alpha_1, \dots, \alpha_l)$ to describe a condition which assures the existence of a 0-parameter solution.

Let $A^l \subset \boldsymbol{C}^{l+1}$ denote the space of allowable parameters

$$\bigl\{(\alpha_0, \alpha_1, \dots, \alpha_l) \in \boldsymbol{C}^{l+1}; \alpha_0 + \alpha_1 + \cdots + \alpha_l = 0\bigr\}, \tag{19}$$

and let ${A_{\mathrm{e}}}^l$ denote the set $A^l \cap \{\alpha_{\mathrm{even}} = 0\}$. We define, for $0 \le i \le l$,

$$E_i^l := \bigl\{(\alpha_0, \alpha_1, \dots, \alpha_l) \in A^l; \alpha(i;0)\alpha(i;1)\cdots\alpha(i;l-1) = 0\bigr\}, \tag{20}$$

where $\alpha(i;k)$ designates $\alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+k}$, and we set

$$E_{\mathrm{cup}}^l := \bigcup_{0 \le i \le l} E_i^l, \qquad E_{\mathrm{cap}}^l := \bigcap_{0 \le i \le l} E_i^l. \tag{21}$$

Note that the set E_{cup}^l (resp. $E_{\mathrm{cup}}^l \cap {A_{\mathrm{e}}}^l$) is a proper analytic subset in A^l (resp. ${A_{\mathrm{e}}}^l$). In what follows, we assume that $\alpha := (\alpha_0, \alpha_1, \dots, \alpha_l)$ belongs to $A^l \setminus E_{\mathrm{cup}}^l$.

By putting (18) into (2) and (3) (resp. (4) and (5)) when l is even (resp. odd), we find that the leading term $u^{(0)}(t)$ of (18) is a common zero point of functions belonging to the ideal $\mathscr{I}$ defined by (11). Hence we see that $u^{(0)}(t)$ is the leading term of a 0-parameter solution if and only if

$$\bigl(t, u^{(0)}(t)\bigr) = \bigl(t, u_0^{(0)}(t), \dots, u_l^{(0)}(t)\bigr) \in \check{V}.$$

Existence of the leading term of a 0-parameter solution is established by [**AH**]. As we will see later (the first statement of Theorem 6 in Section 5.2), $u^{(0)}(t)$ can be solved as a multi-valued holomorphic function of the variable t with branching points of finite degree, moreover, it is still bounded near the branching points. We also note that the leading term $u^{(0)}(t)$ should be unique as a multi-valued holomorphic function because we will prove that $\check{V}$ is connected (the second statement of Theorem 5 in Section 5.1).

Let H' denote the Jacobian matrix $\partial(h_0, \dots, h_l)/\partial(u_0, \dots, u_l)$ of the polynomials $h_0, \dots, h_l$ given in (17), and $D := \det(H')$. Then the set of branching points of $u^{(0)}(t)$ is clearly contained in $\check{\pi}_t(\check{V} \cap \{D = 0\})$. The function D is not identically zero on $\check{V}$, in particular, the set $\check{\pi}_t(\check{V} \cap \{D = 0\})$ is finite. This follows from Theorem 7 which will be proved in Section 5.3.

Now we construct the lower order term $u^{(k)}(t)$ $(k \geq 1)$. By the normalization condition (3) (resp. (5)) and the differential equations except for one corresponding to $j = 2m$ in (2) (resp. ones corresponding to $j = 2m$ and $j = 2m+1$ in (4)) when l is even (resp. odd), we can obtain the following recursive relations:

$$H'(u^{(0)})u^{(k+1)} = R^{(k)}\left(t, u^{(0)}, \ldots, u^{(k)}, \frac{du^{(k)}}{dt}\right) \quad (k = 0, 1, 2, \ldots). \tag{22}$$

Here $R^{(k)}$ consists of polynomials of the variables t, $u^{(0)}$, $\ldots$, $u^{(k)}$ and $du^{(k)}/dt$. Since $D = \det(H')$ does not vanish on $\check{V}$ except for finite points as we have already noted, we can successively determine $u^{(k)}(t)$ by (22).

Hence, outside of $\check{\pi}_t(\check{V} \cap \{D = 0\})$, we can construct a 0-parameter formal solution of $(NY)_l$ for a parameter whose leading term with respect to η^{-1} belongs to $A^l \setminus E^l_{\mathrm{cup}}$. Each branch of the 0-parameter solution $u(t)$ at a non-branching point $t = t_0$ is called a germ of the 0-parameter solution at $t = t_0$. The number of germs of the 0-parameter solution will be given by the second statement of Theorem 6.

A point in $\check{\pi}_t(\check{V} \cap \{D = 0\})$ is called *a turning point of the first kind* of $(NY)_l$. Note that a point in $\check{V} \cap \{D = 0\}$ itself is also called a turning point of the first kind in this paper, which is, geometrically, nothing but a ramification point for the projection $\check{\pi}_t \mid_{\check{V}} \colon \check{V} \to \check{\boldsymbol{C}}$. We finally note that a formula for the number of turning points of the first kind will be also given in Theorem 7.

2.4. Some notes for a turning point of the first kind.

In the previous subsection, a turning point of the first kind is defined as a ramification point for the projection $\check{\pi}_t \mid_{\check{V}} \colon \check{V} \to \check{\boldsymbol{C}}$. Our definition is slightly different from the original definition by [**T**], which we briefly explain from now on.

Let us consider the characteristic polynomial on the leading variety $\check{V}$ defined by

$$\Lambda(\lambda, t, u) := \det\left(\lambda I - \frac{\partial(f_0, \ldots, f_l)}{\partial(u_0, \ldots, u_l)}\right)\Bigg|_{\check{V}}, \quad (t, u) \in \check{V}. \tag{23}$$

Note that the polynomials $f_0, \ldots, f_l$ (not $h_0, \ldots, h_l$) are used here. The explicit forms of $\Lambda(\lambda, t, u)$ is given in [**T**]:

$$\Lambda(\lambda, t, u) = \begin{cases} \lambda\tilde{\Lambda}(\lambda, t, u) & \text{if } l = 2m, \\ \left(\lambda^2 - \alpha_{\mathrm{even}}^2\right)\tilde{\Lambda}(\lambda, t, u) & \text{if } l = 2m+1, \end{cases} \tag{24}$$

where $\tilde{\Lambda}$ is an even polynomial of λ, which is a key feature of the characteristic polynomial of $(NY)_l$. Since [**T**] assumes $\alpha_{\mathrm{even}} = 0$, we will give the proof of (24) in Section 4 (Theorem 3). It follows from (24) that the equation $\tilde{\Lambda}(\lambda, t, u) = 0$ of the unknown variable λ has m-pairs of roots $\lambda = \pm\lambda_1(t, u), \dots, \pm\lambda_m(t, u)$ on $\check{V}$.

Then a turning point of the first kind is, by the original definition, a point $t = t_0$ at which some pair of roots merges, that is, $\lambda_k(t_0, u(t_0)) = -\lambda_k(t_0, u(t_0))$ holds for some k. Clearly this is equivalent to saying that it is a zero point of $\tilde{\Lambda}(0, t, u(t))$.

The equivalence of our definition and the original one immediately comes from the following lemma which is easily proved (see the proof of Theorem 3):

Lemma 1. *We have the equality*

$$\tilde{\Lambda}(0, t, u) = D \tag{25}$$

for $(t, u) \in \check{V}$. *Here* D *was given in the previous subsection, which is the Jacobian for the polynomials* $h_0, \dots, h_l$.

2.5. An intersection multiplicity number.

We briefly recall the definition of an intersection multiplicity number and its properties that are frequently used in this paper. Let X be an n-dimensional complex manifold and $\mathscr{O}_X$ the sheaf of holomorphic functions on X, and let $\mathscr{M}$ (resp. $\mathscr{N}$) be a coherent $\mathscr{O}_X$ module on X. We denote by $[\mathscr{M}]$ (resp. $[\mathscr{N}]$) the analytic cycle defined by $\mathscr{M}$ (resp. $\mathscr{N}$).

Let p be a point in $\mathrm{supp}(\mathscr{M}) \cap \mathrm{supp}(\mathscr{N})$. We assume that the supports of $\mathscr{M}$ and $\mathscr{N}$ intersect properly at p, i.e., the point p is isolated in $\mathrm{supp}(M) \cap \mathrm{supp}(N)$. Then the intersection multiplicity number of $[\mathscr{M}]$ and $[\mathscr{N}]$ at p is defined by

$$\mathrm{mul}([\mathscr{M}], [\mathscr{N}]; p) := \sum_{k \geq 0} (-1)^k \dim_{\boldsymbol{C}} \left(\mathrm{Tor}^k_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})_p \right).$$

In particular, for analytic subsets V and W that intersect properly at p, we define the intersection multiplicity number of V and W at p by

$$\mathrm{mul}(V, W; p) := \mathrm{mul}\left(\left[\frac{\mathscr{O}_X}{\mathscr{I}_V} \right], \left[\frac{\mathscr{O}_X}{\mathscr{I}_W} \right]; p \right),$$

where $\mathscr{I}_V$ (resp. $\mathscr{I}_W$) denotes the defining ideal of V (resp. W).

Let Y be a complex curve in X and $p \in Y$, and let $f \in \mathscr{O}_{X,p}$ that does not vanish identically on Y. Then the intersection multiplicity number $\mathrm{mul}(Y, [\mathscr{O}_X / \mathscr{O}_X(f)]; p)$ coincides with the degree of zero point of the holomorphic function $f\,|_Y$ at p. This is a special case of the following general result.

Let $f_1, f_2, \ldots, f_n \in \mathscr{O}_{X,p}$ and set, for $1 \le l < n$,

$$\mathscr{M} = \frac{\mathscr{O}_X}{\mathscr{O}_X(f_1, f_2, \ldots, f_l)}, \qquad \mathscr{N} = \frac{\mathscr{O}_X}{\mathscr{O}_X(f_{l+1}, f_{l+2}, \ldots, f_n)}.$$

If $f_1, f_2, \ldots, f_n$ form a regular sequence over $\mathscr{O}_{X,p}$, then we have

$$\mathrm{mul}([\mathscr{M}], [\mathscr{N}]; p) = \dim_{\boldsymbol{C}} \left(\mathscr{M} \underset{\mathscr{O}_X}{\otimes} \mathscr{N} \right)_p = \dim_{\boldsymbol{C}} \left(\frac{\mathscr{O}_X}{\mathscr{O}_X(f_1, \ldots, f_n)} \right)_p.$$

3. A hyperbolic system with a normalization condition.

The appearance of the system of algebraic equations associated with the leading terms of $(NY)_{2m}$ seems completely different from that of $(NY)_{2m+1}$. Indeed, the former is defined by the second order algebraic equations with one normalization condition while the latter is defined by the third order algebraic equations with two normalization conditions. We will show in Theorems 1 and 2 that the both systems of algebraic equations can be reduced to the same hyperbolic system with a normalization condition. By these theorems, we can reduce problems of the Stokes geometry for $(NY)_l$ to those for the hyperbolic system. Moreover, as our hyperbolic system has a very simple form, we can easily investigate their properties. As such applications, in subsequent sections, we give several important results for the Stokes geometry of $(NY)_l$ where Theorems 1 and 2 are effectively used.

We first define our hyperbolic system with a normalization condition. Let $l \ge 2$ be a natural number and let $X_{t,\xi}$ designate the complex affine space $\boldsymbol{C}_t \times \boldsymbol{C}_\xi^{l+1}$ with a system of coordinates $(t; \xi_0, \xi_1, \ldots, \xi_l)$. We set

$$\xi_{\mathrm{total}} := \xi_0 + \xi_1 + \cdots + \xi_l$$

and

$$g_k := \frac{1}{4}\left(\xi_{k+1}^2 - \xi_k^2\right) + \alpha_k, \quad k = 0, 1, \ldots, l$$

with $\xi_{l+1} := \xi_0$ for convenience. Let $\mathscr{O}_{X_{t,\xi}}$ denote the sheaf of holomorphic functions on $X_{t,\xi}$. Then we define the $\mathscr{O}_{X_{t,\xi}}$ ideal $\mathscr{I}_{t,\xi}$ of the hyperbolic system and the $\mathscr{O}_{X_{t,\xi}}$ module $\mathscr{N}_{t,\xi}$ by

$$\mathscr{I}_{t,\xi} := \mathscr{O}_{X_{t,\xi}}(g_0, g_1, \dots, g_l, \xi_{\mathrm{total}} - t) \text{ and } \mathscr{N}_{t,\xi} := \frac{\mathscr{O}_{X_{t,\xi}}}{\mathscr{I}_{t,\xi}},$$

respectively. We also set $V_{t,\xi} := \mathrm{supp}(\mathscr{N}_{t,\xi})$.

3.1. The system associated with the leading terms of $(\boldsymbol{NY})_{2m}$.

Throughout the subsection l is assumed to be an even number $2m$ ($m \geq 1$). Let us consider the linear maps $\Psi : X_{t,\xi} := \boldsymbol{C}_t \times \boldsymbol{C}_\xi^{l+1} \to X_{t,u} := \boldsymbol{C}_t \times \boldsymbol{C}_u^{l+1}$ and $\Phi : X_{t,u} \to X_{t,\xi}$ defined respectively by

$$\Psi(t;\xi) = \left(t; \frac{\xi_0 + \xi_1}{2}, \frac{\xi_1 + \xi_2}{2}, \dots, \frac{\xi_k + \xi_{k+1}}{2}, \dots, \frac{\xi_l + \xi_0}{2}\right) \tag{26}$$

and

$$\Phi(t;u) = \big(t; u_l + \Pi_l, u_0 + \Pi_0, \dots, u_{k-1} + \Pi_{k-1}, \dots, u_{l-1} + \Pi_{l-1}\big),$$

where Π_k ($k = 0, 1, \dots, l$) was given in (8). As the equalities

$$\Pi_i + \Pi_{i+1} = u_{i+1} - u_i, \quad i = 0, 1, \dots, l \tag{27}$$

always hold, we have the following lemma:

Lemma 2. *We have $\Psi \circ \Phi = \mathrm{id}_{X_{t,u}}$. In particular, Ψ and Φ are linear isomorphisms.*

Remark 2. When l is odd, the linear map Ψ does not give an isomorphism.

For a function $\varphi(t,u)$ on $X_{t,u}$, we denote by $\Psi^*(\varphi)$ the function $\varphi(\Psi(t,\xi))$ on $X_{t,\xi}$. As $\Phi \circ \Psi = \mathrm{id}_{X_{t,\xi}}$ holds, we have

$$\xi_{k+1} = \Psi^*(\Pi_k + u_k) = \Psi^*(\Pi_k) + \Psi^*(u_k).$$

This implies

$$\Psi^*(\Pi_k) = \xi_{k+1} - \Psi^*(u_k) = \xi_{k+1} - \frac{\xi_{k+1} + \xi_k}{2} = \frac{1}{2}(\xi_{k+1} - \xi_k),$$

from which we have

$$\Psi^*(f_k) = \Psi^*(u_k)\Psi^*(\Pi_k) + \alpha_k = \frac{1}{4}\big(\xi_{k+1}^2 - \xi_k^2\big) + \alpha_k = g_k.$$

We also see $\Psi^*(u_{\mathrm{total}} - t) = \xi_{\mathrm{total}} - t$. Hence we have obtained the following theorem:

THEOREM 1. *The map Ψ gives an isomorphism between the analytic sets V and $V_{t,\xi}$. Moreover $\Psi^{-1}\mathscr{N}$ and $\mathscr{N}_{t,\xi}$ are isomorphic as $\pi^{-1}\mathscr{O}_{\boldsymbol{C}_t}$ modules. Here $\pi : X_{t,\xi} \to \boldsymbol{C}_t$ is the canonical projection with respect to the variable t.*

3.2. The system associated with the leading terms of $(\boldsymbol{NY})_{2m+1}$. Throughout this subsection we assume l to be an odd number $2m+1$ ($m \geq 1$). Set

$$U_{t,u} := \{(t,u) \in X_{t,u}; t \neq 0\} \subset X_{t,u} \tag{28}$$

and

$$W_{t,u} := \left\{(t,u) \in U_{t,u}; u_{\mathrm{even}} - \frac{t}{2} = 0, u_{\mathrm{odd}} - \frac{t}{2} = 0\right\}. \tag{29}$$

Let $\mathscr{O}_{W_{t,u}}$ be the sheaf of holomorphic functions on the submanifold $W_{t,u}$, that is,

$$\mathscr{O}_{W_{t,u}} = \frac{\mathscr{O}_{U_{t,u}}}{\mathscr{O}_{U_{t,u}}\left(u_{\mathrm{even}} - \frac{t}{2}, u_{\mathrm{odd}} - \frac{t}{2}\right)}.$$

We also define the corresponding sets in $X_{t,\xi}$ respectively by

$$U_{t,\xi} := \left\{(t,\xi) \in \boldsymbol{C}_t \times \boldsymbol{C}_\xi^{l+1}; t \neq 0\right\} \subset X_{t,\xi} \tag{30}$$

and

$$W_{t,\xi} := \left\{(t,\xi) \in U_{t,\xi}; \xi_{\mathrm{total}} - t = 0, \tau(\xi) - 4\alpha_{\mathrm{even}} = 0\right\}, \tag{31}$$

where we set

$$\tau(\xi) := \sum_{0 \leq k \leq m} \xi_{2k}^2 - \sum_{0 \leq k \leq m} \xi_{2k+1}^2. \tag{32}$$

Note that $W_{t,\xi}$ is smooth in $U_{t,\xi}$. Let $\mathscr{O}_{W_{t,\xi}}$ denote the sheaf of holomorphic functions on the submanifold $W_{t,\xi}$. By the definition, we have

$$\mathscr{O}_{W_{t,\xi}} = \frac{\mathscr{O}_{U_{t,\xi}}}{\mathscr{O}_{U_{t,\xi}}(\xi_{\mathrm{total}} - t, \tau(\xi) - 4\alpha_{\mathrm{even}})}.$$

Let us recall the linear map $\Psi : U_{t,\xi} \to U_{t,u}$ defined by

$$\Psi(t; \xi_0, \xi_1, \dots, \xi_l) = \left(t; \frac{\xi_0 + \xi_1}{2}, \frac{\xi_1 + \xi_2}{2}, \dots, \frac{\xi_k + \xi_{k+1}}{2}, \dots, \frac{\xi_l + \xi_0}{2}\right).$$

The map Ψ is not an isomorphism, however, we have the following:

Proposition 1. *The analytic map $\Psi \mid_{W_{t,\xi}}$ gives an isomorphism between $W_{t,\xi}$ and $W_{t,u}$.*

Proof. Clearly we have $\Psi(W_{t,\xi}) \subset W_{t,u}$. We introduce the analytic map $\Phi : U_{t,u} \to U_{t,\xi}$ by

$$\begin{cases} t = t, \\ \xi_{2k} = u_{2k} - \dfrac{2}{t}(\Pi_{2k} - \alpha_{\mathrm{even}}) & (k = 0, 1, \dots, m), \\ \xi_{2k+1} = u_{2k} + \dfrac{2}{t}(\Pi_{2k} - \alpha_{\mathrm{even}}) & (k = 0, 1, \dots, m), \end{cases} \tag{33}$$

where Π_k was given in (9). It follows from the equality $\sum_{k=0}^{m} u_{2k}\Pi_{2k} = 0$ that $\Phi(W_{t,u}) \subset W_{t,\xi}$ holds. Hence $\Phi \mid_{W_{t,u}}$ gives a morphism from $W_{t,u}$ to $W_{t,\xi}$.

Let us show that $\Phi \mid_{W_{t,u}}$ and $\Psi \mid_{W_{t,\xi}}$ are mutually reciprocal. We first prove the identity $\Psi \mid_{W_{t,\xi}} \circ \Phi \mid_{W_{t,u}} = \mathrm{Id}_{W_{t,u}}$. Set

$$\Pi_j^{(1)} = \sum_{1 \le r \le s \le m} u_{j+2r-1} u_{j+2s}, \quad \Pi_j^{(2)} = \sum_{1 \le s \le q \le m} u_{j+2s} u_{j+2q+1}.$$

Then we have

$$\begin{aligned} &\Pi_{2k+2}^{(1)} - \Pi_{2k}^{(1)} \\ &\quad = \sum_{1 \le r \le s \le m} u_{2(k+r)+1} u_{2(k+s+1)} - \sum_{1 \le r \le s \le m} u_{2(k+r)-1} u_{2(k+s)} \\ &\quad = \sum_{2 \le r \le s \le m+1} u_{2(k+r)-1} u_{2(k+s)} - \sum_{1 \le r \le s \le m} u_{2(k+r)-1} u_{2(k+s)} \end{aligned}$$

$$
\begin{aligned}
&= \Bigg(\sum_{1 \le r \le s \le m+1} u_{2(k+r)-1} u_{2(k+s)} - u_{2k+1} \sum_{1 \le s \le m+1} u_{2(k+s)} \Bigg) \\
&\quad - \Bigg(\sum_{1 \le r \le s \le m+1} u_{2(k+r)-1} u_{2(k+s)} - u_{2k} \sum_{1 \le r \le m+1} u_{2(k+r)-1} \Bigg) \\
&= u_{\mathrm{odd}} u_{2k} - u_{\mathrm{even}} u_{2k+1}.
\end{aligned}
$$

In the same way, we have

$$
\Pi^{(2)}_{2(k+1)} - \Pi^{(2)}_{2k} = u_{\mathrm{even}} u_{2k+1} - u_{\mathrm{odd}} u_{2(k+1)}.
$$

Thus we get

$$
\begin{aligned}
\Pi_{2(k+1)} - \Pi_{2k} &= -2u_{\mathrm{even}} u_{2k+1} + u_{\mathrm{odd}} (u_{2k} + u_{2(k+1)}) \\
&= \frac{t}{2} (u_{2k+2} - 2u_{2k+1} + u_{2k}) \\
&\quad - 2u_{2k+1} \left(u_{\mathrm{even}} - \frac{t}{2} \right) + (u_{2k+2} + u_{2k}) \left(u_{\mathrm{odd}} - \frac{t}{2} \right).
\end{aligned}
$$

This implies

$$
\begin{aligned}
&(\Psi \circ \Phi)_{2k+1}(u) \\
&\quad = \frac{1}{2} \left(u_{2k+2} + u_{2k} - \frac{2}{t} (\Pi_{2k+2} - \Pi_{2k}) \right) \\
&\quad = u_{2k+1} + \frac{1}{t} \left(2u_{2k+1} \left(u_{\mathrm{even}} - \frac{t}{2} \right) - (u_{2k+2} + u_{2k}) \left(u_{\mathrm{odd}} - \frac{t}{2} \right) \right).
\end{aligned}
$$

Since $(\Psi \circ \Phi)_{2k}(u) = u_{2k}$ is easily confirmed, we have obtained $\Psi |_{W_{t,\xi}} \circ \Phi |_{W_{t,u}} = \mathrm{Id}_{W_{t,u}}$.

Next we will show $\Phi |_{W_{t,u}} \circ \Psi |_{W_{t,\xi}} = \mathrm{Id}_{W_{t,\xi}}$. We need the following lemma.

Lemma 3. *We have*

$$
\Psi^*(\Pi_k) = \frac{1}{4} \big(\xi_{\mathrm{total}} (\xi_{k+1} - \xi_k) + (-1)^k \tau(\xi) \big).
$$

Proof. First we compute $\Psi^*(\Pi^{(1)}_k)$ directly:

$$
\begin{aligned}
4\Psi^*\big(\Pi_k^{(1)}\big) &= \sum_{1\le r\le m} (\xi_{k+2r-1} + \xi_{k+2r}) \sum_{r\le s\le m} (\xi_{k+2s} + \xi_{k+2s+1}) \\
&= \sum_{1\le r\le m} \xi_{k+2r-1} \sum_{r\le s\le m} (\xi_{k+2s} + \xi_{k+2s+1}) \\
&\quad + \sum_{1\le r\le m} \xi_{k+2r} \sum_{r\le s\le m} (\xi_{k+2s} + \xi_{k+2s+1}) \\
&= \Bigg(\sum_{1\le r\le m} \xi_{k+2r-1} \sum_{2r-1\le s\le 2m+1} \xi_{k+s} - \sum_{1\le r\le m} \xi_{k+2r-1}^2 \Bigg) \\
&\quad + \Bigg(\sum_{1\le r\le m} \xi_{k+2r} \sum_{2r\le s\le 2m+1} \xi_{k+s} \Bigg) \\
&= \Bigg(\sum_{1\le r\le 2m-1, r \text{ odd}} \xi_{k+r} \sum_{r\le s\le 2m+1} \xi_{k+s} - \sum_{1\le r\le m} \xi_{k+2r-1}^2 \Bigg) \\
&\quad + \Bigg(\sum_{2\le r\le 2m, r \text{ even}} \xi_{k+r} \sum_{r\le s\le 2m+1} \xi_{k+s} \Bigg) \\
&= \sum_{1\le r\le 2m} \xi_{k+r} \sum_{r\le s\le 2m+1} \xi_{k+s} - \sum_{1\le r\le m} \xi_{k+2r-1}^2 .
\end{aligned}
$$

In the same way, we have

$$
4\Psi^*\big(\Pi_k^{(2)}\big) = \sum_{3\le r\le 2m+2} \xi_{k+r} \sum_{2\le s\le r} \xi_{k+s} - \sum_{1\le r\le m} \xi_{k+2r+2}^2 .
$$

Therefore we get

$$
\begin{aligned}
4\Psi^*\big(\Pi_k^{(1)}\big) &= \sum_{1\le r\le 2m+2} \xi_{k+r} \sum_{r\le s\le 2m+2} \xi_{k+s} - \xi_{\text{total}}\xi_k - \sum_{0\le r\le m} \xi_{k+2r-1}^2 , \\
4\Psi^*\big(\Pi_k^{(2)}\big) &= \sum_{1\le r\le 2m+2} \xi_{k+r} \sum_{1\le s\le r} \xi_{k+s} - \xi_{\text{total}}\xi_{k+1} - \sum_{0\le r\le m} \xi_{k+2r}^2 .
\end{aligned}
$$

This entails the result. □

Now we come back to the proof for the proposition. By Lemma 3, we have

$$
\Psi^*(\Pi_k) = \frac{t}{4}(\xi_{k+1} - \xi_k) + \frac{1}{4}\big((\xi_{\text{total}} - t)(\xi_{k+1} - \xi_k) + (-1)^k \tau(\xi)\big).
$$

Hence we get

$$\begin{aligned}
&\Psi^*\left(u_{2k} - \frac{2}{t}(\Pi_{2k} - \alpha_{\mathrm{even}})\right) \\
&\quad = \xi_{2k} - \frac{1}{2t}\big((\xi_{2k+1} - \xi_{2k})(\xi_{\mathrm{total}} - t) + (\tau(\xi) - 4\alpha_{\mathrm{even}})\big), \\
&\Psi^*\left(u_{2k} + \frac{2}{t}(\Pi_{2k} - \alpha_{\mathrm{even}})\right) \\
&\quad = \xi_{2k+1} + \frac{1}{2t}\big((\xi_{2k+1} - \xi_{2k})(\xi_{\mathrm{total}} - t) + (\tau(\xi) - 4\alpha_{\mathrm{even}})\big).
\end{aligned}$$

Hence we have obtained $\Phi\,|_{W_{t,u}} \circ \Psi\,|_{W_{t,\xi}} = \mathrm{Id}_{W_{t,\xi}}$. This completes the proof. $\square$

Noticing $\alpha_{\mathrm{even}} + \alpha_{\mathrm{odd}} = 0$, we have

$$\begin{aligned}
\Psi^*(f_k) &= \Psi^*(u_k)\big(\Psi^*(\Pi_k) - (-1)^k\alpha_{\mathrm{even}}\big) + \alpha_k\frac{t}{2} \\
&= \frac{\xi_{k+1} + \xi_k}{8}\big(\xi_{\mathrm{total}}(\xi_{k+1} - \xi_k) + (-1)^k(\tau(\xi) - 4\alpha_{\mathrm{even}})\big) + \alpha_k\frac{t}{2} \\
&= \frac{t}{2}\left(\frac{1}{4}(\xi_{k+1}^2 - \xi_k^2) + \alpha_k\right) \\
&\quad + \frac{\xi_{k+1}^2 - \xi_k^2}{8}(\xi_{\mathrm{total}} - t) + (-1)^k\frac{\xi_{k+1} + \xi_k}{8}(\tau(\xi) - 4\alpha_{\mathrm{even}}).
\end{aligned}$$

Since Ψ^* gives a sheaf isomorphism between $\Psi^{-1}\mathscr{O}_{W_{t,u}}$ and $\mathscr{O}_{W_{t,\xi}}$, we get

$$\begin{aligned}
\Psi^{-1}\mathscr{N}\Big|_{W_{t,\xi}} &= \Psi^{-1}\left(\frac{\mathscr{O}_{W_{t,u}}}{\mathscr{O}_{W_{t,u}}(f_0, f_1, \dots, f_{2m+1})}\right)\Bigg|_{W_{t,\xi}} \\
&\overset{\Psi^*}{\simeq} \frac{\mathscr{O}_{W_{t,\xi}}}{\mathscr{O}_{W_{t,\xi}}(\Psi^*(f_0), \Psi^*(f_1), \dots, \Psi^*(f_{2m+1}))}\Bigg|_{W_{t,\xi}} \\
&= \frac{\mathscr{O}_{W_{t,\xi}}}{\mathscr{O}_{W_{t,\xi}}(g_0, g_1, \dots, g_{2m+1})}\Bigg|_{W_{t,\xi}} \\
&= \frac{\mathscr{O}_{U_{t,\xi}}}{\mathscr{O}_{U_{t,\xi}}(g_0, g_1, \dots, g_{2m+1}, \xi_{\mathrm{total}} - t)}\Bigg|_{W_{t,\xi}}.
\end{aligned}$$

Hence we have obtained the following theorem:

Theorem 2. *The map Ψ gives an isomorphism between the analytic sets $V \cap U_{t,u}$ and $V_{t,\xi} \cap U_{t,\xi}$. Moreover the sheaves $\Psi^{-1}\mathscr{N} \mid_{W_{t,\xi}}$ and $\mathscr{N}_{t,\xi} \mid_{W_{t,\xi}}$ on $W_{t,\xi}$ are isomorphic as $\pi^{-1}\mathscr{O}_{\boldsymbol{C}_t}$ modules. Here $\pi : X_{t,\xi} \to \boldsymbol{C}_t$ is the canonical projection.*

4. A characteristic polynomial on the leading variety.

Let us recall that the characteristic polynomial associated with $(NY)_l$ is given by

$$\Lambda(\lambda, u) := \det\left(\lambda I_{l+1} + \frac{\partial(f_0, \dots, f_l)}{\partial(u_0, \dots, u_l)}\right).$$

As we have seen in the introduction, we need to know the concrete form of $\Lambda(\lambda, u) \mid_{\check{V}}$ to study the Stokes geometry of $(NY)_l$. See also [**T**] and [**AHU**] for the related topics. Its explicit form was first obtained by [**T**] for even l in general and for odd l with an additional assumption $\alpha_{\mathrm{odd}} = \alpha_{\mathrm{even}} = 0$. The result of the previous section enables us to calculate it easily for both cases in general, that is, for any $l \geq 2$ with $\alpha_{\mathrm{total}} = 0$. We first state the results.

Theorem 3. *The characteristic polynomial $\Lambda(\lambda, u)$ can be written in the following form.*

1. *If l is an even number $2m$, then we have*

$$\Psi^*(\Lambda)(\lambda, \xi) = \frac{1}{2}(\Lambda^+(\lambda, \xi) + \Lambda^-(\lambda, \xi)),$$

where $\Lambda^{\pm}(\lambda, \xi)$ is given by

$$\Lambda^+(\lambda, \xi) = (\lambda + \xi_0)(\lambda + \xi_1) \cdots (\lambda + \xi_{2m}),$$
$$\Lambda^-(\lambda, \xi) = (\lambda - \xi_0)(\lambda - \xi_1) \cdots (\lambda - \xi_{2m}).$$

2. *If l is an odd number $2m+1$, then, on $W_{t,\xi}$, we have*

$$\Psi^*(\Lambda)(\lambda, \xi) = \frac{(\lambda^2 - \alpha_{\mathrm{even}}^2)}{t^2\lambda}(\Lambda^+(\lambda, \xi) - \Lambda^-(\lambda, \xi)),$$

where $\Lambda^{\pm}(\lambda, \xi)$ is given by

$$\Lambda^+(\lambda,\xi)=\left(\lambda+\frac{t\xi_0}{2}\right)\left(\lambda+\frac{t\xi_1}{2}\right)\cdots\left(\lambda+\frac{t\xi_{2m}}{2}\right)\left(\lambda+\frac{t\xi_{2m+1}}{2}\right),$$
$$\Lambda^-(\lambda,\xi)=\left(\lambda-\frac{t\xi_0}{2}\right)\left(\lambda-\frac{t\xi_1}{2}\right)\cdots\left(\lambda-\frac{t\xi_{2m}}{2}\right)\left(\lambda-\frac{t\xi_{2m+1}}{2}\right).$$

Proof. Let us prove the first claim. For a function φ, we denote by $\nabla_u\varphi$ (resp. $\nabla_\xi\varphi$) the column vector

$$^t(\partial_{u_0}\varphi,\partial_{u_1}\varphi,\ldots,\partial_{u_l}\varphi)\quad\big(\text{resp. }{}^t(\partial_{\xi_0}\varphi,\partial_{\xi_1}\varphi,\ldots,\partial_{\xi_l}\varphi)\big)$$

of size $l+1$. We have

$$\begin{aligned}\Psi^*(\Lambda)&=\det\big(\lambda I+(\Psi^*(\nabla_u f_0),\Psi^*(\nabla_u f_1),\ldots,\Psi^*(\nabla_u f_l))\big)\\&=\det\big(\lambda I+(\Phi'\nabla_\xi g_0,\Phi'\nabla_\xi g_1,\ldots,\Phi'\nabla_\xi g_l)\big).\end{aligned}$$

As $\Psi'\circ\Phi'=\mathrm{Id}$ holds, we have

$$\Psi^*(\Lambda)=\frac{1}{\det(\Psi')}\det\big(\lambda\Psi'+(\nabla_\xi g_0,\nabla_\xi g_1,\ldots,\nabla_\xi g_l)\big).$$

We can easily calculate Ψ' and $\det(\Psi')$, and hence, we obtain

$$\Psi^*(\Lambda)=\frac{1}{2}\det\begin{pmatrix}\lambda-\xi_0 & & & & \lambda+\xi_0\\ \lambda+\xi_1 & \lambda-\xi_1 & & & \\ & \lambda+\xi_2 & & & \vdots\\ & & \cdots & & \\ & & & \lambda-\xi_{l-1} & \\ & & & \lambda+\xi_l & \lambda-\xi_l\end{pmatrix}.$$

This completes the proof for the first claim.

Now let us prove the second claim. The strategy of the proof is similar to that for the first claim. However, in this case, the map Ψ gives an isomorphism only on $W_{t,\xi}$. Hence, to overcome this difficulty, we need to prepare a lemma. Let $0<d<n$ and $(x_1,\ldots,x_n)$ be a system of coordinates of $\boldsymbol{C}^n$. Let $i_W:W\hookrightarrow\boldsymbol{C}^n$ be a closed complex submanifold of complex dimension d whose defining ideal is generated by $(n-d)$-holomorphic functions $\varphi_1,\varphi_2,\ldots,\varphi_{n-d}$. We assume

$$D_{n-d}(x) := \begin{pmatrix} \partial_{x_{d+1}}\varphi_1 & \dots & \partial_{x_n}\varphi_1 \\ \partial_{x_{d+1}}\varphi_2 & \dots & \partial_{x_n}\varphi_2 \\ \vdots & \dots & \vdots \\ \partial_{x_{d+1}}\varphi_{n-d} & \dots & \partial_{x_n}\varphi_{n-d} \end{pmatrix} \neq 0.$$

Note that, by the assumption, we can take the variables $(x_1, \dots, x_d)$ as a system of local coordinates of W. For a function f on W, we denote by $\nabla^d f$ the column vector ${}^t(\partial_{x_1} f, \partial_{x_2} f, \dots, \partial_{x_d} f)$ of size d. Let C be a square matrix of size d with components in functions of the variables x, and we define the square matrix $\tilde{C}$ of size n by $\left(\begin{smallmatrix} C & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Then the following lemma is easily proved:

Lemma 4. *Let $\rho_1, \dots, \rho_d$ be d-functions on $\boldsymbol{C}^n$. We have*

$$\begin{aligned} &\det\big(\tilde{C} + (\nabla^n\rho_1, \dots \nabla^n\rho_d, \nabla^n\varphi_1, \dots, \nabla^n\varphi_{n-d})\big) \\ &\quad = \det(D_{n-d}) \det\big(C + (\nabla^d i_W^*(\rho_1), \dots, \nabla^d i_W^*(\rho_d))\big). \end{aligned}$$

For a function f on $X_{t,u}$, we define the vector $\nabla_u f$ of size $2m+2$ and $\check{\nabla}_u f$ of size $2m$ by

$$\begin{aligned} \nabla_u f &:= {}^t\big(\partial_{u_0} f, \partial_{u_1} f, \dots, \partial_{u_{2m+1}} f\big), \\ \check{\nabla}_u f &:= {}^t\big(\partial_{u_0} f, \partial_{u_1} f, \dots, \partial_{u_{2m-1}} f\big). \end{aligned}$$

Let e_i $(0 \le i \le 2m+1)$ be the column vector of size $2m+2$ such that its i-th component is 1 and the other components are all zero, and we set

$$e_{\text{total}} = \sum_{k=0}^{2m+1} e_k, \quad e_{\text{even}} = \sum_{k=0}^{m} e_{2k}, \quad e_{\text{odd}} = \sum_{k=0}^{m} e_{2k+1}.$$

Using these notations, we rewrite the characteristic polynomial $\Lambda(\lambda, u)$ in the form

$$\det\big(\lambda e_0 + \nabla_u f_0, \lambda e_1 + \nabla_u f_1, \dots, \lambda e_{2m+1} + \nabla_u f_{2m+1}\big).$$

By noticing (16), this reduces to

$$\begin{aligned} \det\Bigg(\{\lambda e_k + \nabla_u f_k\}_{k=0}^{2m-1},\ &\lambda e_{\text{even}} + \nabla_u\Big(-\alpha_{\text{even}}\Big(u_{\text{even}} - \frac{t}{2}\Big)\Big), \\ &\qquad \lambda e_{\text{odd}} + \nabla_u\Big(-\alpha_{\text{odd}}\Big(u_{\text{odd}} - \frac{t}{2}\Big)\Big)\Bigg) \end{aligned}$$

$$= \left(\lambda^2 - \alpha_{\mathrm{even}}^2\right) \det \left(\{\lambda e_k + \nabla_u f_k\}_{k=0}^{2m-1},\ \nabla_u \left(u_{\mathrm{even}} - \frac{t}{2} \right),\ \nabla_u \left(u_{\mathrm{odd}} - \frac{t}{2} \right) \right).$$

Then, applying the lemma to the closed embedding $i_{W_{t,u}} : W_{t,u} \hookrightarrow U_{t,u}$, we find that $\Lambda(\lambda, u)$ is equal to

$$\left(\lambda^2 - \alpha_{\mathrm{even}}^2\right) \det \left(\left\{ \lambda e_k + \check{\nabla}_u i_{W_{t,u}}^*(f_k) \right\}_{k=0}^{2m-1} \right). \tag{34}$$

As $\Psi\mid_{W_{t,\xi}} \circ \Phi \mid_{W_{t,u}} = \mathrm{Id}_{W_{t,u}}$ and $\Psi \mid_{W_{t,\xi}}^* (i_{W_{t,u}}^*(f_k)) = (t/2) i_{W_{t,\xi}}^*(g_k)$ hold for the closed embedding $i_{W_{t,\xi}} : W_{t,\xi} \hookrightarrow U_{t,\xi}$, we have

$$\check{\nabla}_u i_{W_{t,u}}^*(f_k) = \frac{t}{2} \check{\nabla}_u \big(\Phi \mid_{W_{t,u}}^* (i_{W_{t,\xi}}^*(g_k)) \big) = \frac{t}{2} (\Phi \mid_{W_{t,u}})' \check{\nabla}_\xi i_{W_{t,\xi}}^*(g_k).$$

Therefore we find that the second factor of (34) equals

$$\det \left(\lambda I_{2m} + (\Phi \mid_{W_{t,u}})' \left(\left\{ \frac{t}{2} \check{\nabla}_\xi i_{W_{t,\xi}}^*(g_k) \right\}_{k=0}^{2m-1} \right) \right).$$

Now let us calculate the $2m \times 2m$ matrix $(\Psi \mid_{W_{t,\xi}})'$. Clearly we have, on $W_{t,\xi}$,

$$d\xi_0 + d\xi_1 + \cdots + d\xi_{2m+1} = 0,$$
$$\sum_{k=0}^{m} \xi_{2k} d\xi_{2k} - \sum_{k=0}^{m} \xi_{2k+1} d\xi_{2k+1} = 0.$$

As at least one of $\xi_{2k} + \xi_{2k+1}$ $(k = 0, 1 \ldots, 2m)$ never vanishes on $W_{t,\xi}$, we may assume $\xi_{2m} + \xi_{2m+1} \neq 0$. Then we have

$$\Psi^*(du_k) = \frac{d\xi_k + d\xi_{k+1}}{2} \quad (k = 0, 1, \ldots, 2m-2),$$
$$\Psi^*(du_{2m-1}) = \frac{d\xi_{2m-1} + d\xi_{2m}}{2}$$
$$= \frac{1}{2} \left(d\xi_{2m-1} - \sum_{k=0}^{2m-1} \frac{(\xi_{2m+1} + (-1)^k \xi_k)}{\xi_{2m} + \xi_{2m+1}} d\xi_k \right).$$

Hence we obtain

$$C(\lambda,\xi) := \lambda(\Psi\,|_{W_{t,\xi}})' = \frac{\lambda}{2}\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 & \kappa_0 \\ 1 & 1 & 0 & & 0 & 0 & 0 & \kappa_1 \\ 0 & 1 & 1 & & 0 & 0 & 0 & \kappa_2 \\ & & & \dots & & & & \dots \\ & & & \dots & & & & \dots \\ & & & & 1 & 1 & 0 & \kappa_{2m-3} \\ & & & & 0 & 1 & 1 & \kappa_{2m-2} \\ & & & & 0 & 0 & 1 & \kappa_{2m-1}+1 \end{pmatrix}.$$

Here we set $\kappa_k = -(\xi_{2m+1} + (-1)^k \xi_k)/(\xi_{2m} + \xi_{2m+1})$. It is easy to see

$$\det C(1,\xi) = \frac{\xi_{\mathrm{total}}}{2^{2m}(\xi_{2m}+\xi_{2m+1})}.$$

Therefore, noticing $(\Psi\,|_{W_{t,\xi}})'(\Phi\,|_{W_{t,u}})' = I_{2m}$, we see that the second factor of (34) is equal to

$$\frac{\det\left(C(\lambda,\xi) + \left(\left\{\frac{t}{2}\check{\nabla}_\xi i^*_{W_{t,\xi}}(g_k)\right\}_{k=0}^{2m-1}\right)\right)}{\det C(1,\xi)}. \tag{35}$$

Then, by applying the lemma to the closed embedding $i_{W_{t,\xi}} : W_{t,\xi} \hookrightarrow U_{t,\xi}$ again, the numerator of (35) can be reduced to

$$-\frac{\det\left(\tilde{C}(\lambda,\xi) + \left(\left\{\frac{t}{2}\nabla_\xi g_k\right\}_{k=0}^{2m-1}, \nabla_\xi \xi_{\mathrm{total}}, \nabla_\xi \frac{\tau(\xi)}{2}\right)\right)}{\xi_{2m}+\xi_{2m+1}}. \tag{36}$$

Since the relation

$$\frac{1}{\xi_{2m}+\xi_{2m+1}}\left(\xi_{2m+1}\nabla_\xi \xi_{\mathrm{total}} + \nabla_\xi \frac{\tau(\xi)}{2}\right) + \begin{pmatrix} \kappa_0 \\ \kappa_1 \\ \vdots \\ \kappa_{2m-1} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

holds, by eliminating the variable λ in $(2m-1)$-th column by using $(2m)$-th and $(2m+1)$-th columns, the numerator of (36) is written in the form

$$\frac{1}{4^{2m}}\det\begin{pmatrix} 2\lambda - t\xi_0 & 0 & \dots & 0 & 1 & \xi_0 \\ 2\lambda + t\xi_1 & 2\lambda - t\xi_1 & & 0 & 1 & -\xi_1 \\ 0 & 2\lambda + t\xi_2 & \dots & 0 & 1 & \xi_2 \\ & & \dots & \vdots & \vdots & \vdots \\ & & & 0 & 1 & \xi_{2m-2} \\ & & & 2\lambda - t\xi_{2m-1} & 1 & -\xi_{2m-1} \\ & & & 2\lambda + t\xi_{2m} & 1 & \xi_{2m} \\ & & & 0 & 1 & -\xi_{2m+1} \end{pmatrix}. \tag{37}$$

Finally, by simple reductions of the matrix, (37) is equal to

$$\frac{1}{4^{2m+1}t\lambda}\det\begin{pmatrix} 2\lambda - t\xi_0 & 0 & & 0 & 2\lambda + t\xi_0 \\ 2\lambda + t\xi_1 & 2\lambda - t\xi_1 & \dots & 0 & 0 \\ 0 & 2\lambda + t\xi_2 & & 0 & 0 \\ & & \dots & \vdots & \vdots \\ & & & 0 & 0 \\ & & & 2\lambda - t\xi_{2m} & 0 \\ & & & 2\lambda + t\xi_{2m+1} & 2\lambda - t\xi_{2m+1} \end{pmatrix}.$$

This completes the proof for the theorem. □

5. Formulas for the number of solutions and turning points of the first kind.

In this section, we establish formulas which give the number of solutions and that of turning points of the first kind. We first investigate properties of the variety defined by the hyperbolic system with a normalization condition.

Let $(t, \xi_0, \dots, \xi_l; \eta)$ (resp. $(\xi_0, \dots, \xi_l; \eta)$) be a system of homogeneous coordinates of the projective space $\boldsymbol{P}^{l+2}_{t,\xi}$ (resp. $\boldsymbol{P}^{l+1}_{\xi}$), and we identify the space $\boldsymbol{C}_t \times \boldsymbol{C}^{l+1}_{\xi}$ (resp. $\boldsymbol{C}^{l+1}_{\xi}$) with the set $\{\eta \neq 0\}$. We define the homogeneous polynomials

$$g_k := \frac{1}{4}\left(\xi_{k+1}^2 - \xi_k^2\right) + \alpha_k\eta^2, \quad k = 0, 1, \dots, l, \tag{38}$$

and $\xi_{\mathrm{total}} := \xi_0 + \xi_1 + \dots + \xi_l$ on $\boldsymbol{P}^{l+1}_{\xi}$. We also define the ideals

$$\mathscr{I}_{t,\xi} := \mathscr{O}_{\boldsymbol{P}^{l+2}_{t,\xi}}(g_0, g_1, \dots, g_l, \xi_{\mathrm{total}} - t), \quad \mathscr{J}_{\xi} := \mathscr{O}_{\boldsymbol{P}^{l+1}_{\xi}}(g_0, g_1, \dots, g_l), \tag{39}$$

and the modules

$$\mathscr{N}_{t,\xi} := \frac{\mathscr{O}_{\boldsymbol{P}^{l+2}_{t,\xi}}}{\mathscr{I}_{t,\xi}}, \qquad \mathscr{M}_{\xi} := \frac{\mathscr{O}_{\boldsymbol{P}^{l+1}_{\xi}}}{\mathscr{J}_{\xi}}. \tag{40}$$

Set $V_{t,\xi} := \operatorname{supp}(\mathscr{N}_{t,\xi})$ and $Z_\xi := \operatorname{supp}(\mathscr{M}_\xi)$.

Let $H_{t,\xi}$ be the hypersurface $\{\xi_{\mathrm{total}} - t = 0\} \subset \boldsymbol{P}^{l+2}_{t,\xi}$ and $\pi : H_{t,\xi} \to \boldsymbol{P}^{l+1}_{\xi}$ the isomorphism defined by

$$\pi(t, \xi; \eta) = (\xi; \eta), \quad (t, \xi; \eta) \in H_{t,\xi}.$$

Note that $V_{t,\xi} \subset H_{t,\xi}$ holds. Clearly π gives the isomorphism between the analytic sets $V_{t,\xi}$ and Z_ξ, and it also induces the isomorphism of $\pi^{-1}\mathscr{O}_{\boldsymbol{P}^{l+1}_{\xi}}$ modules

$$\mathscr{N}_{t,\xi}\,|_{H_{t,\xi}} \simeq \frac{\mathscr{O}_{H_{t,\xi}}}{\mathscr{O}_{H_{t,\xi}}(g_0, g_1, \dots, g_l)} \simeq \pi^{-1}\mathscr{M}_\xi.$$

Hence all the properties for $\mathscr{N}_{t,\xi}$ can be reduced to those of $\mathscr{M}_\xi$ and, in what follows, we consider the problems on $\boldsymbol{P}^{l+1}_{\xi}$.

We set, for $\hat{t} \in \boldsymbol{C} \cup \{\infty\}$,

$$H_{\hat{t}} := \bigl\{(\xi; \eta) \in \boldsymbol{P}^{l+1}_{\xi}; \xi_{\mathrm{total}} = \hat{t}\eta\bigr\}. \tag{41}$$

In particular, we have $H_0 = \{\xi_{\mathrm{total}} = 0\}$ and $H_\infty = \{\eta = 0\}$.

5.1. Smoothness and connectedness of Z_ξ.

We first give fundamental properties of the variety Z_ξ that are needed in later subsections. Let us recall the definitions of the parameter space A^l, E^l_{cup}, etc. given in Subsection 2.3. Note that, for $\alpha \in A^l$, we have

$$g_0 + g_1 + \cdots + g_l = 0. \tag{42}$$

Theorem 4. *Let α be an element of A^l.*

1. *The complex dimension of every irreducible component of Z_ξ is one. The set $Z_\xi \cap H_\infty$ consists of 2^l-points and Z_ξ is smooth near every point in $Z_\xi \cap H_\infty$.*
2. *The multiplicity of the $\mathscr{O}_{\boldsymbol{P}^{l+1}_{\xi}}$ module $\mathscr{M}_\xi$ is one along every irreducible components of Z_ξ.*
3. *If $\alpha \in A^l \setminus E^l_{\mathrm{cup}}$, then Z_ξ is smooth and connected.*

PROOF. The first and the second claims are easy to see. We will show the third claim. Assume $\alpha \notin E^l_{\mathrm{cup}}$. Let $p = (\xi_0, \dots, \xi_l; 1)$ be a point in $Z_\xi \setminus H_\infty$. We first note that, since

$$\xi_j^2 - \xi_i^2 = -4\alpha(i; j-i-1) \neq 0 \quad \text{for } i < j$$

holds, different coordinates ξ_i and ξ_j $(i \neq j)$ never vanish simultaneously. Let us consider, on $Z_\xi \setminus H_\infty$, the $l \times (l+1)$ matrix

$$2\left(\frac{\partial g_i}{\partial \xi_j}\right)_{0 \le i \le l-1, 0 \le j \le l} = \begin{pmatrix} -\xi_0 & \xi_1 & & & & \\ & -\xi_1 & \xi_2 & & & \\ & & & \cdots & & \\ & & & & \cdots & \\ & & & & -\xi_{l-1} & \xi_l \end{pmatrix}.$$

The determinant of the above matrix without k-th column $(0 \le k \le l)$ is given by

$$(-1)^k \xi_0 \xi_1 \cdots \check{\xi}_k \cdots \xi_{l-1} \xi_l. \tag{43}$$

Hence, for every point $p \in Z_\xi \setminus H_\infty$, there exists an index k such that (43) does not vanish, and from which the smoothness of Z_ξ follows.

To show the connectedness of Z_ξ, it suffices to prove that of $Z_\xi \setminus H_\infty$. Note that $Z_\xi \setminus H_\infty$ is parametrized by $x \in \boldsymbol{C}$ as

$$\xi_0 = x,\ \xi_1 = \pm\sqrt{x^2 - \beta_1},\ \dots,\ \xi_l = \pm\sqrt{x^2 - \beta_l}, \quad \eta = 1, \tag{44}$$

where $\beta_k = 4\alpha(0; k-1)$ $(k = 1, \dots, l)$. Then the condition implies that all branching points $\{\pm\sqrt{\beta_1}, \dots, \pm\sqrt{\beta_l}\}$ with respect to x are mutually distinct, from which the connectedness of $Z_\xi \setminus H_\infty$ easily follows. $\square$

We can easily obtain the counterparts of Theorems 1 and 2 in the original space. Notation that appears in the following theorem was introduced in Subsection 2.2.

THEOREM 5. *Let α be an element of A^l.*

1. *Every irreducible component of $\check{V}$ is of complex dimension one, and the multiplicity of $\mathscr{N}$ along each irreducible component of $\check{V}$ is one. In particular, the polynomials $h_0, \dots, h_l$ that were defined by* (17) *form a regular sequence over*

$\mathscr{O}_p$ for any $p \in \check{V}$.

2. *If $\alpha \in A^l \setminus E^l_{\mathrm{cup}}$, then $\check{V}$ is a smooth connected complex curve.*

5.2. A formula for the number of solutions.

In this subsection, we show that the set of solutions satisfying the algebraic equations associated with $(NY)_l$ consists of a finite number of points for a fixed $t = \hat{t} \in \check{\boldsymbol{C}}_t$. We also give a formula for the number of these points. To be more precise, as we need to take the multiplicity of these points into account, the number is given by

$$\mathrm{NSol}(\check{V}; \hat{t}\,) := \sum_{p \in \check{V} \cap Y_{\hat{t}}} \mathrm{mul}\left(\check{V}, Y_{\hat{t}}; p\right), \tag{45}$$

where $\check{V}$ was defined by (13) and $Y_{\hat{t}} = \check{\pi}_t^{-1}(\hat{t}\,)$, and $\mathrm{mul}(\check{V}, Y_{\hat{t}}; p)$ is the intersection multiplicity number of $\check{V}$ and $Y_{\hat{t}}$ at p. Note that, in our case, the intersection multiplicity number can be calculated by the formula

$$\mathrm{mul}\left(\check{V}, Y_{\hat{t}}; p\right) = \dim_{\boldsymbol{C}} \left(\mathscr{N} \underset{\mathscr{O}_{\boldsymbol{C}_t}}{\otimes} \frac{\mathscr{O}_{\boldsymbol{C}_t}}{\mathscr{O}_{\boldsymbol{C}_t}(t - \hat{t}\,)} \right)_p .$$

Here the module $\mathscr{N}$ was defined in Subsection 2.2. First we evaluate the number of points in $(Z_\xi \cap H_{\hat{t}}) \cap \boldsymbol{C}^{l+1}_\xi$.

Proposition 2. *Let $\hat{t}$ be a point in $\boldsymbol{C}_t$.*

1. *Assume either that l is even or that l is odd and $\hat{t} \neq 0$. If $\alpha \in A^l$, then $Z_\xi \cap H_{\hat{t}}$ consists of a finite number of points, and we have*

$$\sum_{p \in (Z_\xi \cap H_{\hat{t}}) \setminus H_\infty} \mathrm{mul}(Z_\xi, H_{\hat{t}}; p) = \begin{cases} 2^l & \text{if } l \text{ is even,} \\ 2^l - {}_lC_{[l/2]} & \text{if } l \text{ is odd.} \end{cases}$$

2. *Assume that l is odd and $\hat{t} = 0$. If α belongs to $A^l \setminus E^l_{\mathrm{cap}}$, then the number of points in $Z_\xi \cap H_0$ is also finite. The number satisfies the estimate*

$$\sum_{p \in (Z_\xi \cap H_0) \setminus H_\infty} \mathrm{mul}(Z_\xi, H_0;\, p) \leq 2^l - 2_l C_{[l/2]}.$$

In particular, for a generic parameter α, the equality of the above estimate holds.

PROOF. Set

$$\phi_{\hat{t}}(\xi) = \xi_{\text{total}} - \hat{t}\eta.$$

We first show that Z_ξ and $H_{\hat{t}} = \{\phi_{\hat{t}}(\xi) = 0\}$ intersect properly. Let $\{Z_{\xi,i}\}$ be a irreducible decomposition of Z_ξ. Since $\dim(Z_{\xi,i}) = 1$ and $\dim(H_\infty) = l$, the set $Z_{\xi,i} \cap H_\infty$ is not empty. Moreover Z_ξ is smooth near $Z_\xi \cap H_\infty$. Therefore it suffices to show the claim that $\phi_{\hat{t}}(\xi) \,|_{Z_\xi}$ is not identically zero near every point in $Z_\xi \cap H_\infty$.

Let $p = (1, \sigma_1, \sigma_2, \ldots, \sigma_l; 0)$ $(\sigma_i = \pm 1)$ be a point in $Z_\xi \cap H_\infty$. Then, in a neighborhood of p, the analytic set Z_ξ can be parametrized by the sufficiently small parameter $s \in \boldsymbol{C}$ as

$$\begin{aligned} &\eta = s, \ \xi_0 = 1, \\ &\xi_1 = \sigma_1\sqrt{1-\beta_1 s^2}, \, \xi_2 = \sigma_2\sqrt{1-\beta_2 s^2}, \, \ldots, \, \xi_l = \sigma_l\sqrt{1-\beta_l s^2}, \end{aligned} \tag{46}$$

where we set $\beta_k = 4\alpha(0; k-1)$ $(k = 1, \ldots, l)$ and we take a branch of the square root so that it takes positive real values on the positive real axis. By putting (46) into $\phi_{\hat{t}}(\xi)$, we have

$$\begin{aligned} \phi_{\hat{t}}(s) = (1 + \sigma_1 + \cdots + \sigma_l) - \hat{t}s - \frac{1}{2}(\sigma_1\beta_1 + \cdots + \sigma_l\beta_l)s^2 \\ - \frac{1}{4}\big(\sigma_1\beta_1^2 + \cdots + \sigma_l\beta_l^2\big)s^4 - \frac{3}{8}\big(\sigma_1\beta_1^3 + \cdots + \sigma_l\beta_l^3\big)s^6 - \cdots . \end{aligned} \tag{47}$$

If l is even, then we have

$$1 + \sigma_1 + \cdots + \sigma_l \equiv 1 + 1 + \cdots + 1 \equiv l + 1 \equiv 1 \neq 0 \mod 2,$$

from which the claim follows.

Assume that l is an odd number $2m+1$. If $\hat{t} \neq 0$, then the function $\phi(s)$ is not identically zero by (47). Let us consider the case $\hat{t} = 0$. Since α does not belong to E^l_{cap}, we may assume $\alpha \notin E^l_0$, that is, we suppose $\beta_1\beta_2\cdots\beta_l \neq 0$. If $1 + \sigma_1 + \sigma_2 + \cdots + \sigma_l \neq 0$, then the claim clearly holds. Hence we consider the claim near a point that satisfies $1 + \sigma_1 + \sigma_2 + \cdots + \sigma_l = 0$. Then the claim is an immediate consequence of the following lemma. Set

$$\begin{aligned} I_+ &= \{k \in \{1, 2, \ldots, l\}; \sigma_k = 1\}, \\ I_- &= \{k \in \{1, 2, \ldots, l\}; \sigma_k = -1\}. \end{aligned}$$

Note that $\#I_+ = m$ and $\#I_- = m+1$ hold.

Lemma 5. *Assume that* $\sigma_1\beta_1^k + \sigma_2\beta_2^k + \cdots + \sigma_l\beta_l^k = 0$ *for any* $1 \le k \le m+1$. *Then there exists* $i_0 \in I_-$ *such that* $\beta_{i_0} = 0$. *Moreover there exists a bijective map* $\kappa : I_+ \to I_- \setminus \{i_0\}$ *that satisfies* $\beta_i = \beta_{\kappa(i)}$ *for any* $i \in I_+$.

Proof. Let s_k $(k = 1, 2, \dots, r)$ denote the k-th fundamental symmetric polynomial of r-independent variables $y_1, y_2, \dots, y_r$, and set $S_k = y_1^k + y_2^k + \cdots + y_r^k$ $(k = 1, 2, \dots)$. It follows from the well-known Newton formulas that we have

$$\begin{aligned} &S_k + s_1 S_{k-1} + s_2 S_{k-2} \cdots + s_{k-1} S_1 + k s_k = 0, && k < r, \\ &S_k + s_1 S_{k-1} + s_2 S_{k-2} \cdots + s_{k-r} S_{k-r} = 0, && k \ge r. \end{aligned}$$

By applying these formulas to the set of variables $\{\beta_k\}_{k \in I_+}$ and that of $\{\beta_k\}_{k \in I_-}$, we can easily obtain the result. $\square$

Since Z_ξ and $H_{\hat{t}}$ properly intersect, by the theorem of Bézout (see T. Suwa [**S**]), we have

$$\sum_{p \in Z_\xi \cap H_{\hat{t}}} \mathrm{mul}(Z_\xi, H_{\hat{t}}; p) = 2^l.$$

If l is even, then the set $(Z_\xi \cap H_{\hat{t}}) \cap H_\infty$ is empty. If l is odd, then the set $(Z_\xi \cap H_{\hat{t}}) \cap H_\infty$ consists of points that satisfy $1 + \sigma_1 + \sigma_2 + \cdots + \sigma_l = 0$. The number of these points is ${}_l C_{[l/2]}$. Moreover, it follows from (47) that the intersection multiplicity number of Z_ξ and $H_{\hat{t}}$ is one if $\hat{t} \neq 0$ and greater than one if $\hat{t} = 0$. Summing up, we obtain

$$\sum_{p \in (Z_\xi \cap H_{\hat{t}}) \cap H_\infty} \mathrm{mul}(Z_\xi, H_{\hat{t}}; p) = \begin{cases} 0 & \text{if } l \text{ is even}, \\ {}_l C_{[l/2]} & \text{if } l \text{ is odd and } \hat{t} \neq 0, \end{cases}$$

$$\sum_{p \in (Z_\xi \cap H_{\hat{t}}) \cap H_\infty} \mathrm{mul}(Z_\xi, H_{\hat{t}}; p) \ge 2\,{}_l C_{[l/2]} \quad \text{if } l \text{ is odd and } \hat{t} = 0.$$

Set

$$E^{(1),l} = \bigcup_{\substack{1+\sigma_1+\cdots+\sigma_l=0, \\ \sigma_i = \pm 1}} \big\{(\alpha_0, \alpha_1, \dots, \alpha_l) \in A^l; \sigma_1\beta_1 + \sigma_2\beta_2 + \cdots + \sigma_l\beta_l = 0\big\}. \quad (48)$$

Note that $E^{(1),l}$ is a proper analytic subset in A^l. We can easily see that, if

$\alpha \notin E^{(1),l}$, the equality holds in the above estimate. This completes the proof. $\square$

As $\check{V}$ and $Y_{\hat{t}}$ intersect properly, $(l+2)$-polynomials $h_0, \ldots, h_l$ and $t - \hat{t}$ form a regular sequence over $\mathscr{O}_p$ for any $p \in \check{V} \cap Y_{\hat{t}}$. Hence the intersection multiplicity number of $\check{V}$ and $Y_{\hat{t}}$ at p is calculated by

$$\mathrm{mul}(\check{V}, Y_{\hat{t}}; p) = \dim_{\boldsymbol{C}} \left(\mathscr{N} \underset{\mathscr{O}_{\boldsymbol{C}_t}}{\otimes} \frac{\mathscr{O}_{\boldsymbol{C}_t}}{\mathscr{O}_{\boldsymbol{C}_t}(t - \hat{t}\,)} \right)_p .$$

Let $q = (\hat{t}; \hat{\xi})$ be a point in $V_{t,\xi} \cap (\check{\boldsymbol{C}}_t \times \boldsymbol{C}_\xi^{l+1})$. It follows form Theorems 1 and 2 that we have

$$\begin{aligned}
&\dim_{\boldsymbol{C}} \left(\mathscr{N} \underset{\mathscr{O}_{\boldsymbol{C}_t}}{\otimes} \frac{\mathscr{O}_{\boldsymbol{C}_t}}{\mathscr{O}_{\boldsymbol{C}_t}(t - \hat{t}\,)} \right)_{\Psi(q)} \\
&= \dim_{\boldsymbol{C}} \left(\mathscr{N}_{t,\xi} \underset{\mathscr{O}_{\boldsymbol{C}_t}}{\otimes} \frac{\mathscr{O}_{\boldsymbol{C}_t}}{\mathscr{O}_{\boldsymbol{C}_t}(t - \hat{t}\,)} \right)_q \\
&= \dim_{\boldsymbol{C}} \left(\frac{\mathscr{O}_{\boldsymbol{P}_{t,\xi}^{l+2}}}{\mathscr{O}_{\boldsymbol{P}_{t,\xi}^{l+2}}(g_0, g_1, \ldots, g_l, \xi_{\mathrm{total}} - t, \xi_{\mathrm{total}} - \hat{t}\eta)} \right)_q \\
&= \dim_{\boldsymbol{C}} \left(\frac{\mathscr{O}_{\boldsymbol{P}_{\xi}^{l+1}}}{\mathscr{O}_{\boldsymbol{P}_{\xi}^{l+1}}(g_0, g_1, \ldots, g_l, \phi_{\hat{t}})} \right)_{\hat{\xi}} .
\end{aligned}$$

This implies

$$\mathrm{NSol}(\check{V}, \hat{t}\,) := \sum_{p \in \check{V} \cap Y_{\hat{t}}} \mathrm{mul}\left(\check{V}, Y_{\hat{t}}; p\right) = \sum_{q \in (Z_\xi \cap H_{\hat{t}}) \setminus H_\infty} \mathrm{mult}\left(Z_\xi, H_{\hat{t}}; q\right).$$

Hence we obtain

Theorem 6. *Let $\alpha \in A^l$ and $\hat{t} \in \check{\boldsymbol{C}}$.*

1. *The map $\check{\pi}_t \mid_{\check{V}}$ is proper and finite. Here $\check{\pi}_t$ is defined in* (14) *and $\check{\pi}_t \mid_{\check{V}}$ denotes its restriction to $\check{V}$.*
2. *We have the formula*

$$\mathrm{NSol}(\check{V}, \hat{t}\,) = \begin{cases} 2^l & \text{if } l \text{ is even,} \\ 2^l - {}_lC_{[l/2]} & \text{if } l \text{ is odd.} \end{cases}$$

5.3. A formula for the number of turning points of the first kind. Let J_u be the Jacobian $\det(\partial(h_0,\dots,h_l)/\partial(u_0,\dots,u_l))$ and let D_u (resp. $[D_u]$) be the zero set (resp. the analytic cycle) of J_u. Then a turning point of the first kind is nothing but a point in $D_u \cap \check{V}$. We show, in this subsection, that the set $D_u \cap \check{V}$ consists of finite points, and we also give a formula for the number $\mathrm{NTp}(\check{V})$ of turning points of the first kind, that is, we evaluate

$$\mathrm{NTp}(\check{V}) := \sum_{p\in\check{V}\cap D_u} \mathrm{mul}(\check{V},[D_u];p). \tag{49}$$

We first establish the corresponding results in $\boldsymbol{P}_\xi^{l+1}$. Let D_ξ (resp. $[D_\xi]$) be the zero set (resp. the analytic cycle) of $J_\xi := \det(\partial(g_0,\dots,g_{l-1},\xi_{\mathrm{total}})/\partial(\xi_0,\dots,\xi_l))$. Note that, since J_ξ is a homogeneous polynomial of the variables ξ, D_ξ and $[D_\xi]$ are well-defined in $\boldsymbol{P}_\xi^{l+1}$.

Proposition 3. *Let α be a point in A^l, and let us assume $\alpha \notin E^l_{\mathrm{cap}}$ if l is odd. Then $Z_\xi \cap D_\xi$ is a finite set, and we have*

$$\sum_{p\in(Z_\xi\cap D_\xi)\setminus H_\infty} \mathrm{mul}(Z_\xi,[D_\xi];p) = l2^l, \qquad \text{if } l \text{ is even},$$
$$\sum_{p\in(Z_\xi\cap D_\xi)\setminus H_\infty} \mathrm{mul}(Z_\xi,[D_\xi];p) \le l2^l - 2_lC_{[l/2]}, \quad \text{if } l \text{ is odd}.$$

In particular, the equality holds for a generic parameter α when l is odd.

Remark 3. If l is odd and α belongs to $A^l_{\mathrm{e}} \setminus E^l_{\mathrm{cap}}$, then we have

$$\sum_{p\in(Z_\xi\cap D_\xi)\setminus H_\infty} \mathrm{mul}(Z_\xi,[D_\xi],p) \le l2^l - 2_lC_{[l/2]} - 2. \tag{50}$$

In particular, the equality holds for a generic point in A^l_{e}.

Proof. By the direct calculation, we have

$$2^l J_\xi = \sum_{k=0}^{l} \xi_0\xi_1\cdots\check{\xi}_k\cdots\xi_l.$$

Let $p=(1,\sigma_1,\sigma_2,\dots,\sigma_l;0)$ $(\sigma_i=\pm1)$ be a point in $p\in Z_\xi\cap H_\infty$. In a neighborhood of p, the determinant 2^lJ_ξ can be written in the form

$$\xi_0\xi_1\cdots\xi_l\left(\frac{1}{\xi_0}+\frac{1}{\xi_1}+\cdots+\frac{1}{\xi_l}\right)=\xi_0\xi_1\cdots\xi_l\theta(\xi).$$

Then, by putting (46) into $\theta(\xi)$, we get

$$\begin{aligned}\theta(s)&=(1+\sigma_1+\cdots+\sigma_l)+\frac{1}{2}(\sigma_1\beta_1+\cdots+\sigma_l\beta_l)s^2\\&\quad+\frac{3}{4}\big(\sigma_1\beta_1^2+\cdots+\sigma_l\beta_l^2\big)s^4+\frac{15}{8}\big(\sigma_1\beta_1^3+\cdots+\sigma_l\beta_l^3\big)s^6+\cdots.\end{aligned}\tag{51}$$

Therefore the proof proceeds in the same way as that for Proposition 2. □

Next we show the claim given in Remark 3. As

$$\alpha_{\text{even}}=\beta_1-\beta_2+\beta_3-\beta_4+\beta_5-\cdots-\beta_{l-1}+\beta_l\tag{52}$$

holds, the set A^l_{e} is contained in $E^{(1),l}$. It follows from (51) and (52) that $J_\xi\mid_{Z_\xi}$ has zero of at least degree 4 at the point

$$(1,-1,1,-1,\ldots,1,-1;0)\in Z_\xi\cap H_\infty.$$

Therefore we have

$$\sum_{p\in(Z_\xi\cap D_\xi)\cap H_\infty}\mathrm{mul}(Z_\xi,[D_\xi];p)\geq 2{}_lC_{[l/2]}+2.$$

It is clear that the equality holds for a generic parameter. This implies (50).

We note the following fact. Let p be a point in $X:=\boldsymbol{C}_t\times\boldsymbol{C}_u^{l+1}$ and let $\{\varphi_0,\varphi_1,\ldots,\varphi_l\}$ (resp. $\{\psi_0,\psi_1,\ldots,\psi_l\}$) be a family of generators of the ideal $\mathscr{I}$ at p. If each family of generators forms a regular sequence over $\mathscr{O}_{X,p}$, then we can easily see

$$\mathscr{O}_X(\varphi_0,\varphi_1,\ldots,\varphi_l,J_{u,\varphi})_p=\mathscr{O}_X(\psi_0,\psi_1,\ldots,\psi_l,J_{u,\psi})_p.$$

Here $J_{u,\varphi}$ is defined by $\det(\partial(\varphi_0,\ldots,\varphi_l)/\partial(u_0,\ldots,u_l))$ and $J_{u,\psi}$ is also similarly defined.

Taking the above fact into account, for a point $q=(\hat{t};\hat{\xi})\in V_{t,\xi}\cap(\check{\boldsymbol{C}}_t\times\boldsymbol{C}_\xi^{l+1})$, we have

$$\begin{aligned}\mathrm{mul}(\check{V},[D_u];\Psi(q)) &= \dim_{\boldsymbol{C}}\left(\frac{\mathscr{O}_X}{\mathscr{I}+\mathscr{O}_X J_u}\right)_{\Psi(q)}\\ &= \dim_{\boldsymbol{C}}\left(\frac{\mathscr{O}_{\boldsymbol{P}_\xi^{l+1}}}{\mathscr{J}_\xi+\mathscr{O}_{\boldsymbol{P}_\xi^{l+1}} J_\xi}\right)_{\hat{\xi}} = \mathrm{mul}\left(Z_\xi,[D_\xi];\hat{\xi}\right).\end{aligned}$$

Hence we have obtained

$$\mathrm{NTp}(\check{V}) := \sum_{p\in\check{V}\cap D_u}\mathrm{mul}(\check{V},[D_u];p) = \sum_{q\in G}\mathrm{mul}(Z_\xi,[D_\xi];q).$$

Here we set

$$G := \begin{cases}(Z_\xi\cap D_\xi)\cap \boldsymbol{C}_\xi^{l+1} & \text{for even } l,\\ ((Z_\xi\cap D_\xi)\setminus H_0)\cap \boldsymbol{C}_\xi^{l+1} & \text{for odd } l.\end{cases}$$

Thanks to Proposition 3, if l is even, we can immediately obtain a formula for the number of turning points of the first kind. However, if l is odd, we need to consider a point in $Z_\xi\cap D_\xi\cap H_0$. In what follows, we will show the claim that the set $(Z_\xi\cap D_\xi\cap H_0)\cap\boldsymbol{C}_\xi^{l+1}$ is empty for a generic parameter. Note that the set $Z_\xi\cap D_\xi\cap H_0$ is always non-empty.

To prove the claim, we need several preparations. Let $\pi_\alpha : \boldsymbol{P}_\xi^{l+1}\times A^l\to A^l$ be the canonical projection with respect to the parameter space and $T_{\xi,\alpha}\subset\boldsymbol{P}_\xi^{l+1}\times A^l$ an analytic subset defined by

$$\left\{(\xi;\eta;\alpha)\in\boldsymbol{P}_\xi^{l+1}\times A^l; g_0=g_1=\cdots=g_l=J_\xi=\xi_{\mathrm{total}}=0\right\}.$$

We set

$$\overline{T}_{\xi,\alpha} := \overline{T_{\xi,\alpha}\setminus(H_\infty\times A^l)}\subset\boldsymbol{P}_\xi^{l+1}\times A^l.$$

We also define a proper analytic subset $E^{(2),l}$ in A^l by

$$\bigcup_{\substack{1+\sigma_1+\cdots+\sigma_l=0,\\ \sigma_i=\pm1}}\left\{(\alpha_0,\alpha_1,\ldots,\alpha_l)\in A^l;\sigma_1\beta_1^2+\sigma_2\beta_2^2+\cdots+\sigma_l\beta_l^2=0\right\}.$$

Note that, by (52), the set $E^{(2),l}\cap A_{\mathrm{e}}^l$ is also a proper analytic subset in A_{e}^l.

LEMMA 6. *Assume $\alpha \in A^l \setminus E^{(2),l}$. Then the set $(\overline{T} \cap (H_\infty \times A^l)) \cap \pi_\alpha^{-1}(\alpha)$ is empty.*

PROOF. Suppose that $(\overline{T} \cap (H_\infty \times A^l)) \cap \pi_\alpha^{-1}(\alpha)$ is non-empty. Then we take a point $(1, \sigma_1, \sigma_2, \dots, \sigma_l; 0; \alpha)$ $(\sigma_j = \pm 1)$ in $(\overline{T} \cap (H_\infty \times A^l)) \cap \pi_\alpha^{-1}(\alpha)$. Note that $1 + \sigma_1 + \cdots + \sigma_l = 0$ is satisfied. By the definition, there exist $s_k \in \boldsymbol{C} \setminus \{0\}$ and $\alpha_k = (\alpha_{0,k}, \alpha_{1,k}, \dots, \alpha_{l,k}) \in A^l$ $(k = 1, 2, \dots)$ that satisfy

$$(\xi_{0,k}, \dots, \xi_{l,k}; s_k; \alpha_k) \to (1, \sigma_1, \sigma_2, \dots, \sigma_l; 0; \alpha) \quad k \to \infty.$$

Here $\xi_{0,k}, \dots, \xi_{l,k}$ are given by (46) with $s = s_k$. It follows from (47) with $\hat{t} = 0$ and (51) that we have

$$0 = \frac{1}{2} s_k^4 \big((\sigma_1 \beta_{1,k}^2 + \cdots + \sigma_l \beta_{l,k}^2) + 3(\sigma_1 \beta_{1,k}^3 + \cdots + \sigma_l \beta_{l,k}^3) s_k^2 + \cdots \big). \tag{53}$$

The condition $\alpha \in A^l \setminus E^{(2),l}$ implies there exists a positive $\delta > 0$ such that $|\sigma_1 \beta_{1,k}^2 + \cdots + \sigma_l \beta_{l,k}^2| > \delta$, which contradicts to (53) when k is sufficiently large. Hence $(\overline{T} \cap (H_\infty \times A^l)) \cap \pi_\alpha^{-1}(\alpha)$ is an empty set. $\square$

Let us consider the set $(\overline{T} \cap (\boldsymbol{C}_\xi^{l+1} \times A^l))$, for which we have the following result.

LEMMA 7. *There exists $\hat{\alpha} \in A^l$ such that $\hat{\alpha} \notin E^{(2),l}$ and*

$$\big(\overline{T} \cap (\boldsymbol{C}_\xi^{l+1} \times A^l)\big) \cap \pi_\alpha^{-1}(\hat{\alpha}) = \varnothing.$$

Moreover we can choose such an $\hat{\alpha}$ in A_{e}^l.

PROOF. Set, for $a \neq 0$,

$$\hat{\alpha} = (\alpha_0, \dots, \alpha_{l-4}, \alpha_{l-3}, \alpha_{l-2}, \alpha_{l-1}, \alpha_l) = \frac{1}{4}(0, \dots, 0, -1, -a, 1, a).$$

Note that, because of $\hat{\alpha}_{\mathrm{even}} = 0$, we have $\hat{\alpha} \in A_{\mathrm{e}}^l \subset A^l$. Moreover, as

$$(\beta_1, \dots, \beta_{l-3}, \beta_{l-2}, \beta_{l-1}, \beta_l) = (0, \dots, 0, -1, -(1+a), -a),$$

we have $\hat{\alpha} \notin E^{(2),l}$ if $\hat{\alpha}$ does not belong to a finite number of exceptional points.

Let us consider a family of polynomials $\{g_0, \dots, g_{l-3}, g_{l-1}, \xi_{\mathrm{total}}, J_\xi\}$. Note that the family does not contain the polynomial g_{l-2} and g_l. We also note that all the polynomials of the family do not depend on a, and hence, we regard these polynomials as ones defined on $\boldsymbol{C}_\xi^{l+1}$ by setting $\eta = 1$. Let $C \subset \boldsymbol{C}_\xi^{l+1}$ denote the set of the common zeros of polynomials belonging to the family, and set

$$\phi := \xi_{l-1}^2 - \xi_{l-2}^2 = 4g_{l-2} + a.$$

Then, to show the claim, it suffices to prove that the image set $\phi(C) \subset \boldsymbol{C}$ consists of finite points. Indeed, it is easy to see $(\overline{T} \cap (\boldsymbol{C}_\xi^{l+1} \times A^l)) \cap \pi_\alpha^{-1}(\hat{\alpha}) = \varnothing$ if we choose an $\hat{\alpha}$ determined by a that does not belong to $\phi(C)$.

Let $B \subset \boldsymbol{C}_\xi^{l+1}$ be the common zero set of polynomials $g_0, \dots, g_{l-3}, g_{l-1}, \xi_{\mathrm{total}}$, that is, B is defined by the following system.

$$\begin{aligned} 4g_0 &= \xi_1^2 - \xi_0^2 = 0, \\ &\cdots \\ 4g_{l-4} &= \xi_{l-3}^2 - \xi_{l-4}^2 = 0, \\ 4g_{l-3} &= \xi_{l-2}^2 - \xi_{l-3}^2 - 1 = 0, \\ 4g_{l-1} &= \xi_l^2 - \xi_{l-1}^2 + 1 = 0, \\ \xi_{\mathrm{total}} &= \xi_0 + \xi_1 + \cdots + \xi_{l-1} + \xi_l = 0. \end{aligned}$$

Therefore B can be parametrized by $x \in \boldsymbol{C}$ as

$$\begin{aligned} \xi_0 &= x,\ \xi_1 = \sigma_1 x,\ \ \dots,\ \xi_{l-3} = \sigma_{l-3} x, \\ \xi_{l-2} &= \sigma_{l-2}\sqrt{1+x^2}, \\ \xi_{l-1} &= \frac{-1}{2}\left(\frac{1}{\tau_\sigma(x)} + \tau_\sigma(x)\right), \\ \xi_l &= \frac{1}{2}\left(\frac{1}{\tau_\sigma(x)} - \tau_\sigma(x)\right). \end{aligned} \tag{54}$$

Here $\sigma_k = \pm 1$ $(k = 1, 2, \dots, l-2)$ and

$$\tau_\sigma(x) := (1 + \sigma_1 + \cdots + \sigma_{l-3})x + \sigma_{l-2}\sqrt{1+x^2}.$$

Set

$$k := (1 + \sigma_1 + \cdots + \sigma_{l-3})\sigma_{l-3}.$$

Note that k is an odd integer and satisfies $-(l-2) \leq k \leq l-2$, and that $\{\tau_\sigma(x) = 0\}$ is a finite set. We also note that, on B, we have the equalities

$$\begin{aligned} \xi_{\mathrm{total}} &= k\xi_{l-3} + \xi_{l-2} + \xi_{l-1} + \xi_l = 0, \\ 2^l J_{\xi,g} &= \sigma_1\sigma_2\cdots\sigma_{l-4}x^{l-3}\big((k\xi_{l-2} + \xi_{l-3})\xi_l\xi_{l-1} + (\xi_l + \xi_{l-1})\xi_{l-2}\xi_{l-3}\big). \end{aligned} \tag{55}$$

Let $\pi_x : \boldsymbol{C}_\xi^{l+1} \to \boldsymbol{C}_x$ be the canonical projection with respect to the coordinate ξ_0. Since each irreducible component of B intersects $\pi_x^{-1}(0)$, it is enough to consider the problem near $\pi_x^{-1}(0)$.

CASE 1: On an irreducible component B_i of B that is described near $\pi_x^{-1}(0)$ by (54) with $k^2 = 1$. As $B \cap \pi_x^{-1}(0)$ is a finite set, we may assume $x \neq 0$. We will show that ϕ is constant on $B_i \cap \{J_{\xi,g} = 0\}$. It follows from (55) and $k^2 = 1$ that we have

$$(\xi_l + \xi_{l-1})(-k\xi_l\xi_{l-1} + \xi_{l-2}\xi_{l-3}) = 0.$$

Since $\xi_l + \xi_{l-1} = -\tau_\sigma(x)$ and $\tau_\sigma(x) \neq 0$ if $k^2 = 1$, we have

$$\xi_{l-2}\xi_{l-3} = k\xi_l\xi_{l-1} = -\frac{k}{4}\left(\frac{1}{\tau_\sigma^2} - \tau_\sigma^2\right).$$

Hence we obtain

$$\begin{aligned} \phi &= \xi_{l-1}^2 - \xi_{l-2}^2 \\ &= \frac{1}{4}\left(\frac{1}{\tau_\sigma^2} + \tau_\sigma^2 + 2\right) - \xi_{l-2}^2 \\ &= \frac{1}{4}\left(2\tau_\sigma^2 - 4k\xi_{l-2}\xi_{l-3} + 2\right) - \xi_{l-2}^2 \\ &= \frac{1}{2}\left(\xi_{l-2}^2 + \xi_{l-3}^2 + 1\right) - (\xi_{l-3}^2 + 1) \\ &= \frac{1}{2}\left(\xi_{l-2}^2 - \xi_{l-3}^2 - 1\right) = 0. \end{aligned}$$

CASE 2: On an irreducible component B_i of B that is described near $\pi_x^{-1}(0)$ by (54) with $k^2 \neq 1$. In this case, we will show $B_i \cap \{J_{\xi,g} = 0\}$ is a finite set. To show this, it suffices to prove that $J_{\xi,g}\,|_{B_i}$ is not identically zero near $\pi_x^{-1}(0)$. Since

$$\xi_{l-2} = \sigma_{l-2} + \frac{\sigma_{l-2}}{2} x^2 + \cdots ,$$
$$\tau_\sigma(x) = \sigma_{l-2} + k\sigma_{l-3} x + \frac{\sigma_{l-2}}{2} x^2 + \cdots$$

hold, we have

$$(k\xi_{l-2} + \xi_{l-3})\xi_l \xi_{l-1} = k^2 \sigma_{l-3} x + \left(k\sigma_{l-2} - \frac{k\sigma_{l-2}}{2}(k^2-1) \right) x^2 + \cdots ,$$
$$(\xi_{l-2} + k\xi_{l-3})\xi_{l-2}\xi_{l-3} = \sigma_{l-3} x + k\sigma_{l-2} x^2 + \cdots .$$

It follows from (55) that we get

$$\begin{aligned}
&(k\xi_{l-2} + \xi_{l-3})\xi_l \xi_{l-1} + (\xi_l + \xi_{l-1})\xi_{l-2}\xi_{l-3} \\
&\quad = (k\xi_{l-2} + \xi_{l-3})\xi_l \xi_{l-1} - (\xi_{l-2} + k\xi_{l-3})\xi_{l-2}\xi_{l-3} \\
&\quad = (k^2 - 1)\sigma_{l-3} x - \frac{\sigma_{l-2}}{2} k(k^2-1) x^2 + \cdots .
\end{aligned}$$

This implies that $J_\xi |_{B_i}$ has zero of degree $l-2$ if $k^2 \neq 1$, which shows the claim.

The proof has been completed. $\square$

Since π_α is a proper map, the set $E^l_{\mathrm{zero}} := \pi_\alpha(\overline{T})$ is an analytic set in A^l. If we take an $\hat{\alpha}$ that satisfies the above two lemma, then we have $\hat{\alpha} \notin E^l_{\mathrm{zero}}$. Hence E^l_{zero} is a proper analytic subset in A^l.

Finally, since the set $(Z_\xi \cap D_\xi \cap H_0) \cap \boldsymbol{C}^{l+1}_\xi$ is empty if $\hat{\alpha} \notin E^l_{\mathrm{zero}}$, we have obtained the following theorem.

Theorem 7. *Let $\alpha \in A^l$. Assume that $\alpha \notin E^l_{\mathrm{cap}}$ if l is odd. Then the number of turning points of the first kind is finite and it satisfies*

$$\begin{aligned}
&\mathrm{NTp}(\check{V}) = l2^l && \textit{if } l \textit{ is even,} \\
&\mathrm{NTp}(\check{V}) \leq l2^l - 2{}_lC_{[l/2]} && \textit{if } l \textit{ is odd.}
\end{aligned}$$

The equality holds for a generic parameter when l is odd.

We also have the following remark.

Remark 4. If l is odd and $\alpha \in A^l_{\mathrm{e}} \setminus E^l_{\mathrm{cap}}$, then we have the estimate

$$\mathrm{NTp}(\check{V}) \leq l2^l - 2{}_lC_{[l/2]} - 2.$$

In particular, the equality holds for a generic parameter in A^l_{e}.

We finally give an example of Theorem 7. Let us consider the system $(NY)_3$ $(l = 3)$ whose leading terms of parameters are given by

$$\alpha_0 = 0.24 + 0.5\sqrt{-1}, \quad \alpha_2 = -\alpha_0, \quad \alpha_1 = 0.5 - 0.25\sqrt{-1}, \quad \alpha_3 = -\alpha_1.$$

Note that $\alpha \in A^l_{\mathrm{e}} \setminus E^l_{\mathrm{cap}}$ is satisfied. By a numerical computation, we can find mutually distinct 16-turning points of the first kind in the domain $|t| < 10.0$. It follows Theorem 7 and its remark that the system has at most 16-turning points of the first kind. Hence we can conclude that all the turning points are exhausted by this numerical computation and they are located in the region $|t| < 10.0$.

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