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L^p-independence of spectral bounds of Schrödinger-type operators with non-local potentials

By Yoshihiro Tawara

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Abstract. We establish a necessary and sufficient condition for spectral bounds of a non-local Feynman-Kac semigroup being L^p-independent. This result is an extension of that in [**24**] to more general symmetric Markov processes; in [**24**], we only treated a symmetric stable process on $\boldsymbol{R}^d$. For example, we consider a symmetric stable process on the hyperbolic space, the jump process generated by the fractional power of the Laplace-Beltrami operator, and prove that by adding a non-local potential, the associated Feynman-Kac semigroup satisfies the L^p-independence.

1. Introduction.

In this paper, we consider the L^p-independence of spectral bounds of Schrödinger-type operators with non-local potential. The main objective is to extend our results in [**23**] and [**24**] to more general Schrödinger-type operators.

Let X be a locally compact separable metric space and m a positive Radon measure on X with full support. Let $\boldsymbol{M} = (X_t, \boldsymbol{P}_x)$ be a conservative m-symmetric Hunt process on X and denote by $(N(x, dy), H_t)$ the Lévy system of $\boldsymbol{M}$ ([**10**, Definition A.3.7]). Let F be a symmetric function on $X \times X$ in a certain class $\mathscr{J}_\infty$ (see Definition 2.2) and define a discontinuous additive functional $A_t(F)$ by

$$A_t(F) = \sum_{0<s\le t} F(X_{s-}, X_s).$$

We denote by $\mathscr{L}$ the L^2-generator of $\boldsymbol{M}$ and define a Schrödinger-type operator formally by

$$\mathscr{H}^F f = \mathscr{L} f + \mu_H \boldsymbol{F} f, \quad \mu_H \boldsymbol{F} f = \int_X \left(e^{F(x,y)} - 1\right) f(y) N(x, dy) \mu_H(dx),$$

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where μ_H is the measure in the Revuz correspondence to the positive continuous additive functional H_t. We denote by $\{p_t^F\}_{t>0}$ the semigroup generated by $\mathscr{H}^F$, $p_t^F = \exp(t\mathscr{H}^F)$. Then the semigroup $\{p_t^F\}_{t>0}$ is expressed by the non-local Feynman-Kac semigroup,

$$p_t^F f(x) = \boldsymbol{E}_x[\exp(A_t(F))f(X_t)].$$

We define the L^p-spectral bound of $\{p_t^F\}_{t>0}$ by

$$\lambda_p(F) = -\lim_{t\to\infty}\frac{1}{t}\log\left\|p_t^F\right\|_{p,p} \quad 1 \le p \le \infty,$$

where $\|p_t^F\|_{p,p}$ is the operator norm from $L^p(X;m)$ to $L^p(X;m)$. The main theorem in this paper is as follows: Suppose that the function F belongs to the class $\mathscr{J}_\infty$. Then $\lambda_p(F)$ is independent of p if and only if $\lambda_2(F) \le 0$. In [**23**], Takeda proved this statement for Schrödinger-type operators with local potential, and in [**24**] we did it for Schrödinger-type operators whose principal part is the fractional Laplacian, $-\frac{1}{2}(-\Delta)^{\alpha/2}$.

For a classical Schrödinger-type operator $\frac{1}{2}\Delta + V$ on $\boldsymbol{R}^d$, B. Simon [**18**] proved the L^p-independence and K.-Th. Sturm in [**20**], [**21**] extended it to Schrödinger-type operator on Riemannian manifolds. For the proof of the L^p-independence, they used the heat kernel estimates of Schrödinger-type operators. Our method in this paper is completely different from those in [**18**], [**20**] and [**21**]. The approach in this paper is similar to that in [**23**] and [**24**]. We shall use arguments in Donsker-Varadhan's large deviation theory. However, our method is more general than that in [**24**]; we used in [**24**] the heat kernel estimate for the α-stable process on $\boldsymbol{R}^d$, due to Bass and Levin [**5**]. However, it is not applicable for general Hunt processes. Instead of the heat kernel estimate for the α-stable process, we use facts that the Feynman-Kac semigroup $\{p_t^F\}_{t>0}$ possesses the *doubly Feller property*, $p_t^F(\mathscr{B}_b(X)) \subset C_b(X)$ and $p_t^F(C_\infty(X)) \subset C_\infty(X)$. Here $C_\infty(X)$ is the space of continuous functions on X such that vanishing at infinity. Moreover, we derive the invariance of $C_u(X)$, $p_t^F(C_u(X)) \subset C_u(X)$, where $C_u(X)$ is the space of uniformly continuous bounded functions on X such that $\lim_{x\to\infty} f(x)$ exists. In our argument, the invariance of $C_u(X)$ plays a crucial role. In fact, we extend the Markov process on the one-point compactification X_∞ by making the adjoined point ∞ a trap, and use the upper bound of the large deviation for the extended Markov process. Then the so-called *Donsker-Varadhan's I-function*, say $\bar{I}_F$, of the extended Markov process is a function on the space of probability measures on X_∞ not X. We make a connection between the modified I-function and the original one. To show that $\bar{I}_F(\delta_\infty) = 0$, that is, there exists no contribution of adjoined point ∞, we need the invariance of $C_u(X)$. To prove the properties of the Feynman-Kac semigroup stated above, we apply a result of Chung [**8**] which was devoted to the stability of the doubly Feller property under transform by multiplicative functionals. We summarize in Proposition 3.1 properties equivalent to the invariance of $C_\infty(X)$, which is an extension of a result of Azencott [**4**].

We use qualitative properties of the Feynman-Kac semigroup for the proof of the L^p-independence. As a result, we can treat more general Schrödinger semigroups. In Section 5, we shall give an example of non-local Feynman-Kac semigroup satisfying the L^p-independence as follows: Employing results in McGillivray [**15**] and Ôkura [**16**], we prove that our assumptions (I)–(IV) are preserved by a certain subordination. We thus see that our main theorem is applicable for the α-stable process on the hyperbolic space, that is, the subordinated process of the Brownian motion generated by $-\frac{1}{2}(-\Delta)^{\alpha/2}$. Here Δ is the Laplace-Beltrami operator on the hyperbolic space. It is well-known that the spectral bounds of Laplace-Beltrami operator on the hyperbolic space is equal to $(d-1)^2/8$ (e.g. Davies [**9**]). By the spectral theorem, the L^2-spectral bound of the α-stable process is equal to $(d-1)^\alpha/2^{1+\alpha}$. We construct a function $F \in \mathscr{J}_\infty$ such that $\lambda_2(F) \leq 0$ by Lemmas 4.7 and 4.8. We thus conclude that the spectral bounds of $-\frac{1}{2}(-\Delta)^{\alpha/2} + \mu_V \boldsymbol{F}$ is L^p-independent, where μ_V is the Riemannian volume.

We close the introduction with some words on notation. For a topological space X, we use $\mathscr{B}(X)$ to denote the set of all Borel set (or functions) on X. If $\mathscr{C} \subset \mathscr{B}(X)$, then $\mathscr{C}_b$ (resp. $\mathscr{C}_+$) denotes the set of bounded (resp. non-negative) functions in $\mathscr{C}$. For a subset $A \subset X$, we denote by 1_A the indicator function of A and by A^c the complement of A. We use $c, C, \ldots$, etc as positive constants which may be different at different occurrences.

2. Notations.

Let X be a locally compact separable metric space and X_∞ the one-point compactification of X with adjoined point ∞. Let m be a positive Radon measure on X with full support. Let $\boldsymbol{M} = (\Omega, \mathscr{M}, \mathscr{M}_t, \theta_t, \boldsymbol{P}_x, X_t, \zeta)$ be an m-symmetric Hunt process on (X, m). Here $\{\mathscr{M}_t\}_{t \geq 0}$ is the minimal (augmented) admissible filtration, θ_t, $t \geq 0$ is the shift operator satisfying $X_s(\theta_t) = X_{s+t}$ identically for $s, t \geq 0$, and ζ is the lifetime of $\boldsymbol{M}$. We denote that $(N, H) = (N(x, dy), H_t)$ is the Lévy system of $\boldsymbol{M}$, that is, N is a kernel on $X_\infty \times \mathscr{B}(X_\infty)$ and H is a positive continuous additive functional of $\boldsymbol{M}$ such that for any nonnegative measurable function F on $X_\infty \times X_\infty$ vanishing on the diagonal set and any $x \in X_\infty$,

$$\boldsymbol{E}_x\left[\sum_{0<s\leq t} F(X_{s-}, X_s)\right] = \boldsymbol{E}_x\left[\int_0^t \int_{X_\infty} N(X_s, dy) F(X_s, y) dH_s\right].$$

From Assumption (II) below, we may replace X_∞ by X in the definition of the Lévy system. Throughout this paper, we assume that the Hunt process $\boldsymbol{M}$ is transient. Moreover, we assume that the semigroup of $\boldsymbol{M}$, $p_t f(x) = \boldsymbol{E}_x[f(X_t)]$, possesses the following properties:

(I) (Irreducibility) If a Borel set A is p_t-invariant, that is, for any $f \in L^2(X;m) \cap \mathscr{B}_b(X)$ and $t > 0$, $p_t(1_A f)(x) = 1_A(x) p_t f(x)$ m-a.e. x, then A satisfies either $m(A) = 0$ or $m(X \setminus A) = 0$.
(II) (Conservativeness) $p_t 1 = 1$.
(III) (Strong Feller Property) $p_t(\mathscr{B}_b(X)) \subset C_b(X)$.
(IV) (Invariance of $C_\infty(X)$) $p_t(C_\infty(X)) \subset C_\infty(X)$.

Let us denote by $(\mathscr{E}, \mathscr{F})$ the Dirichlet form on $L^2(X;m)$ generated by $\boldsymbol{M}$; by the right continuity of sample paths of $\boldsymbol{M}$, $\{p_t\}_{t>0}$ can be extended to an $L^2(X;m)$-strongly continuous semigroup, say $\{T_t\}$ ([**10**, Lemma 1.4.3]). Then $(\mathscr{E}, \mathscr{F})$ is defined by

$$\begin{cases} \mathscr{F} = \left\{ u \in L^2(X;m) : \lim_{t\to 0} \dfrac{1}{t}(u - T_t u, u)_m < \infty \right\}, \\ \mathscr{E}(u,v) = \lim_{t\to 0} \dfrac{1}{t}(u - T_t u, v)_m, \quad u, v \in \mathscr{F}. \end{cases}$$

It follows from Assumption (IV) that $(\mathscr{E}, \mathscr{F})$ is regular and thus each function u in $\mathscr{F}$ admits a quasi-continuous version $\tilde{u}$ (cf. [**10**, Theorem 2.1.3]). In the sequel we always assume that every function $u \in \mathscr{F}$ is represented by its quasi-continuous version.

We call a Borel measure μ on X *smooth* if it satisfies the following conditions:

1. μ charges no set of zero capacity,
2. there exists an increasing sequence $\{F_n\}$ of closed sets such that $\mu(F_n) < \infty$ for all n and $\lim_{n\to\infty} \mathrm{Cap}(K \setminus F_n) = 0$ for any compact set K.

For given smooth measure μ, we denote by $A_t(\mu)$ the positive continuous additive functional in the Revuz correspondence (cf. [**10**, Theorem 5.1.4]): For any $f \in \mathscr{B}_+(X)$ and γ-excessive function h,

$$\lim_{t\to 0} \frac{1}{t} \boldsymbol{E}_{h\cdot m}\left[\int_0^t f(X_s) dA_s(\mu) \right] = \int_X f(x) h(x) \mu(dx).$$

Under Assumption (II), we obtain the next expression of the Dirichlet form $\mathscr{E}$ due to Beurling and Deny:

$$\mathscr{E}(u,u) = \mathscr{E}^{(c)}(u,u) + \frac{1}{2}\int_{X\times X}(u(x)-u(y))^2 N(x,dy)\mu_H(dx).$$

Here, $\mathscr{E}^{(c)}$ is the continuous part of $(\mathscr{E},\mathscr{F})$ and μ_H is the Revuz measure of positive additive functional H (see [**10**, Section 3.2]).

REMARK 2.1. We see from Assumption (III) and symmetry of $\{p_t\}_{t>0}$ that the semigroup $\{p_t\}_{t>0}$ admits an integral kernel $\{p(t,x,y)\}$ with respect to the measure m.

Let $\{G_\beta(x,y)\}_{\beta>0}$ the resolvent kernel defined by

$$G_\beta(x,y) = \int_0^\infty e^{-\beta t}p(t,x,y)dt, \quad \beta > 0.$$

We simply write $G(x,y)$ for the Green function $G_0(x,y)$. The existence of the Green function follows from the transience of $\boldsymbol{M}$.

DEFINITION 2.1 (Kato measure and Green tight measure). Suppose that μ is a signed smooth measure associated with a positive continuous additive functional $A_t(\mu)$.

1. A smooth measure μ is said to be *Kato measure* (in notation, $\mu \in \mathscr{K}$) if

$$\lim_{t\to 0}\sup_{x\in X}\boldsymbol{E}_x[A_t(|\mu|)] = 0.$$

2. A measure $\mu \in \mathscr{K}$ is said to be *Green tight measure* (in notation, $\mu \in \mathscr{K}_\infty$) if for any $\epsilon > 0$ there exist a compact subset K and a positive constant $\delta > 0$ such that

$$\sup_{x\in X}\int_{K^c} G(x,y)|\mu|(dy) \le \epsilon$$

and for any Borel set $B \subset K$ with $|\mu|(B) < \delta$,

$$\sup_{x\in X}\int_B G(x,y)|\mu|(dy) \le \epsilon.$$

REMARK 2.2. A Green tight measure μ is Green-bounded:

$$\sup_{x\in X}\int_X G(x,y)|\mu|(dy)<\infty$$

(see Chen [**6**, Remark 2.1]).

Definition 2.2 (Class $\mathscr{J}_\infty$). Let F be a bounded measurable function on $X\times X$ vanishing on the diagonal $\triangle=\{(x,x):x\in X\}$. We say that F belongs to the class $\mathscr{J}_\infty$, if

$$\mu_F(dx)=\left(\int_X F(x,y)N(x,dy)\right)\mu_H(dx)\in\mathscr{K}_\infty. \tag{2.1}$$

Here, μ_H is the Revuz measure corresponding to H.

In the remainder of this paper, we assume that F is symmetric, $F(x,y)=F(y,x)$. For $F\in\mathscr{J}_\infty$, we define a symmetric Dirichlet form $(\mathscr{E}_F,\mathscr{F})$ by

$$\mathscr{E}_F(u,u)=\mathscr{E}^{(c)}(u,u)+\frac{1}{2}\int_{X\times X}(u(x)-u(y))^2e^{F(x,y)}N(x,dy)\mu_H(dx).$$

Furthermore, we set $F_1=e^F-1\in\mathscr{J}_\infty$, and define another bilinear form $\mathscr{E}^F$ by

$$\begin{aligned}\mathscr{E}^F(u,u)&=\mathscr{E}_F(u,u)-\int_X u^2d\mu_{F_1}\\&=\mathscr{E}(u,u)-\int_{X\times X}u(x)u(y)F_1(x,y)N(x,dy)\mu_H(dx),\quad u\in\mathscr{F}.\end{aligned}$$

We see that $(\mathscr{E}^F,\mathscr{F})$ is a lower semi-bounded closed symmetric form by Albeverio and Ma [**2**, Theorem 4.1], [**3**, Proposition 3.3]. Denote by $\mathscr{L}^F$ the self-adjoint operator associated with $(\mathscr{E}_F,\mathscr{F})$ and $\mathscr{H}^F$ the self-adjoint operator associated with $(\mathscr{E}^F,\mathscr{F})$. Then $\mathscr{L}^F$ and $\mathscr{H}^F$ are formally written by

$$\mathscr{L}^Ff=\mathscr{L}f+\left(\int_X(f(y)-f(x))F_1(x,y)N(x,dy)\right)\mu_H(dx)$$

and

$$\mathscr{H}^Ff=\mathscr{L}f+\mu_H\boldsymbol{F}f=\mathscr{L}^Ff+\mu_HV^Ff,$$

where

$$\mu_H \boldsymbol{F} f = \left(\int_X f(y) F_1(x,y) N(x,dy) \right) \mu_H(dx),$$
$$\mu_H V^F f = \left(\int_X F_1(x,y) N(x,dy) \right) f(x) \mu_H(dx)$$

(Chen and Song [**7**, Remark 1]). Indeed, we have by the formal calculation,

$$\begin{aligned}
\mathscr{E}_F(f,g) &= \mathscr{E}^{(c)}(f,g) + \frac{1}{2}\int_{X\times X} e^{F(x,y)}(f(y)-f(x))(g(y)-g(x))N(x,dy)\mu_H(dx) \\
&= (-\mathscr{L}^{(c)} f, g)_m + \frac{1}{2}\int_{X\times X} (f(y)-f(x))(g(y)-g(x))N(x,dy)\mu_H(dx) \\
&\quad + \frac{1}{2}\int_X (e^{F(x,y)}-1)(f(y)-f(x))(g(y)-g(x))N(x,dy)\mu_H(dx) \\
&= (-\mathscr{L} f, g)_m + \frac{1}{2}\int_{X\times X} (e^{F(x,y)}-1)(f(y)g(y)+f(x)g(x))N(x,dy)\mu_H(dx) \\
&\quad - \frac{1}{2}\int_{X\times X} (e^{F(x,y)}-1)(f(y)g(x)+f(x)g(y))N(x,dy)\mu_H(dx)
\end{aligned}$$

where $\mathscr{L}^{(c)}$ is the self-adjoint operator associated with $(\mathscr{E}^{(c)}, \mathscr{F})$. Furthermore, by the symmetry of the Lévy system,

$$\begin{aligned}
&= (-\mathscr{L} f, g)_m + \int_{X\times X} F_1(x,y) f(x) g(x) N(x,dy)\mu_H(dx) \\
&\quad - \int_{X\times X} F_1(x,y) f(y) g(x) N(x,dy)\mu_H(dx) \\
&= (-\mathscr{L} f, g)_m - \int_{X\times X} F_1(x,y)(f(y)-f(x)) g(x) N(x,dy)\mu_H(dx) \\
&= (-\mathscr{L}^F f, g)_m.
\end{aligned}$$

Analogously, $\mathscr{E}^F(f,g) = (-\mathscr{H}^F f, g)_m$.

Let $\{p_t^F\}_{t>0}$ be the L^2-semigroup generated by $\mathscr{H}^F$: $p_t^F = \exp(t\mathscr{H}^F)$. Then, using the discontinuous additive functional $A_t(F) = \sum_{0<s\le t} F(X_{s-}, X_s)$, the semigroup $\{p_t^F\}_{t>0}$ is expressed by

$$p_t^F f(x) = \boldsymbol{E}_x[\exp(A_t(F)) f(X_t)]. \tag{2.2}$$

In fact, for $F \in \mathscr{J}_\infty$, let $M_t = A_t(F_1) - A_t^p(F_1)$, where

$$A_t^p(F_1) = \int_0^t \left(\int_X F_1(X_s, y) N(X_s, dy) \right) dH_s. \tag{2.3}$$

By the definition of the Lévy system, we see that M_t is a local martingale. Then the Doléans-Dade exponential M_t^F of M_t is defined by

$$M_t^F = \exp(M_t) \prod_{0<s\le t} (1 + \Delta M_s) \exp(-\Delta M_s),\ \Delta M_s = M_s - M_{s-}$$

(cf. He, Wang and Yan [**13**, Theorem 9.39]). Noting that $\Delta M_{s-} = F_1(X_{s-}, X_s)$, we have

$$\begin{aligned} M_t^F &= \exp\big(A_t(F) - A_t^p(F) + A_t(F) - A_t(F_1)\big) \\ &= \exp\big(A_t(F) - A_t^p(F_1)\big). \end{aligned} \tag{2.4}$$

The semigroup

$$T_t^F f(x) = \boldsymbol{E}_x\left[M_t^F f(X_t)\right]$$

is identical to the one generated by $(\mathscr{E}_F, \mathscr{F})$ (cf. [**7**, Theorem 4.8]). Let $(X_t, \boldsymbol{P}_x^M)$ be the transformed process of $\boldsymbol{M}$ by M_t^F: $\boldsymbol{P}_x^M(d\omega) = M_t^F \cdot \boldsymbol{P}_x(d\omega)$. We then see from (2.4) that the transformed semigroup by the non-local Feynman-Kac functional $\exp(A_t(F))$ is identical to the transformed semigroup of $\boldsymbol{P}_x^M$ by the Feynman-Kac functional $\exp(A_t^p(F_1))$:

$$p_t^F f(x) = \boldsymbol{E}_x^M\left[\exp(A_t^p(F_1)) f(X_t)\right]. \tag{2.5}$$

3. Non-local Feynman-Kac semigroups.

In this section, we shall show some properties of the non-local Feynman-Kac semigroup $\{p_t^F\}_{t>0}$ transformed by $F \in \mathscr{J}_\infty$. Let K be a Borel set and σ_K the first hitting time of K, $\sigma_K = \inf\{t > 0 : X_t \in K\}$. The next proposition is an extension of Proposition 3.1 in Azencott [**4**]. We think that the proposition is of independent interest. Hence we state the proposition in a complete way, while we only use a part of Proposition 3.1.

PROPOSITION 3.1. *Let $\boldsymbol{M}$ be a Hunt process that satisfies the properties* (II) *and* (III). *Then the following properties are equivalent to each other*:

(A): *$\boldsymbol{M}$ possesses the property* (IV), *that is, for each $t > 0$ and $f \in C_\infty(X)$,*

$$\lim_{x \to \infty} p_t f(x) = 0.$$

(B): *For each $\beta > 0$ and $f \in C_\infty(X)$,*

$$\lim_{x \to \infty} G_\beta f(x) = 0.$$

(C): *For each $t > 0$ and compact set K,*

$$\lim_{x \to \infty} P_x(\sigma_K \leq t) = 0.$$

(D): *For each $\beta > 0$ and compact set K,*

$$\lim_{x \to \infty} \boldsymbol{E}_x[e^{-\beta \sigma_K}] = 0.$$

PROOF.

(A) $\Rightarrow$ (B): Let f be a strictly positive function in $C_\infty(X)$. By properties (II) and (IV), $G_\beta f$ is a strictly positive continuous function in $C_\infty(X)$.

(B) $\Rightarrow$ (C): Put $c = \inf_{x \in K} G_\beta f(x) > 0$. Since for $\beta > 0$,

$$\boldsymbol{P}_x[\sigma_K \leq t] \leq e^{\beta t} \boldsymbol{E}_x\left[e^{-\beta \sigma_k}\right] \leq \frac{e^{\beta t}}{c} \boldsymbol{E}_x\left[e^{-\beta \sigma_K} G_\beta f(X_{\sigma_K})\right]$$

and

$$\begin{aligned}\boldsymbol{E}_x\left[e^{-\beta \sigma_K} G_\beta f(X_{\sigma_K})\right] &= \boldsymbol{E}_x\left[e^{-\beta \sigma_K} \boldsymbol{E}_{X_{\sigma_K}}\left[\int_0^\infty e^{-\beta t} f(X_t) dt\right]\right] \\ &\leq \boldsymbol{E}_x\left[\int_{\sigma_K}^\infty e^{-\beta t} f(X_t) dt\right] \leq G_\beta f(x),\end{aligned}$$

we have the implication.

(C) $\Rightarrow$ (A): Let f be a nonnegative function in $C_\infty(X)$. By the property (III), we only have to show that $\lim_{x \to \infty} p_t f(x) = 0$. For any $\epsilon > 0$, there exists compact set K such that $f(x) < \epsilon$ for all $x \notin K$. Then $f(X_t) \leq \|f\|_\infty 1_{\{\sigma_K \leq t\}} + \epsilon 1_{\{\sigma_K > t\}} \leq \|f\|_\infty 1_{\{\sigma_K \leq t\}} + \epsilon$. Thus,

$$p_t f(x) = \boldsymbol{E}_x[f(X_t)] \le \|f\|_\infty \boldsymbol{P}_x(\sigma_K \le t) + \epsilon.$$

(C) $\Rightarrow$ (D): By the property (C), for arbitrary $\beta > 0$, compact set K and $\epsilon > 0$, there exist $t > 0$ and $U \subset K^c$ such that $e^{-\beta t} < \epsilon$ and $\boldsymbol{P}_x(\sigma_K \le t) < \epsilon$ for all $x \in U$. Hence we have,

$$\begin{aligned}\boldsymbol{E}_x\left[e^{-\beta\sigma_K}\right] &= \boldsymbol{E}_x\left[e^{-\beta\sigma_K}; \sigma_K \le t\right] + \boldsymbol{E}_x\left[e^{-\beta\sigma_K}; \sigma_K > t\right] \\ &\le \boldsymbol{P}_x(\sigma_K \le t) + e^{-\beta t}\boldsymbol{P}_x(\sigma_K > t) \le 2\epsilon.\end{aligned}$$

We get desired claim.

(D) $\Rightarrow$ (C): It follows from the following inequality:

$$\boldsymbol{E}_x\left[e^{-\beta\sigma_K}\right] \ge \boldsymbol{E}_x\left[e^{-\beta\sigma_K} 1_{\{\sigma_K \le t\}}\right] \ge e^{-\beta t}\boldsymbol{P}_x(\sigma_K \le t). \qquad \square$$

Theorem 3.2. *Let $F \in \mathscr{J}_\infty$.*

(i) *There exist constants c and $\kappa(F)$ such that*

$$\left\|p_t^F\right\|_{p,p} \le c e^{\kappa(F)t}, \quad 1 \le \forall p \le \infty,\ t > 0.$$

Here $\|\cdot\|_{p,q}$ means the operator norm from $L^p(X;m)$ to $L^q(X;m)$,

(ii) *$\{p_t^F\}_{t>0}$ is a strongly continuous symmetric semigroup on $L^2(X;m)$ and the closed form corresponding to p_t^F is identical to $(\mathscr{E}^F, \mathscr{F})$,*

(iii) *$p_t^F(\mathscr{B}_b(X)) \subset C_b(X)$,*

(iv) *$p_t^F(C_\infty(X)) \subset C_\infty(X)$,*

(v) *$p_t^F(C_u(X)) \subset C_u(X)$ and $\lim_{x\to\infty} p_t^F f(x) = \lim_{x\to\infty} f(x)$, where $C_u(X)$ is the space of uniformly continuous bounded functions on X such that $\lim_{x\to\infty} f(x)$ exists.*

Proof. The statements (i) and (ii) follow from results in Albeverio, Blanchard and Ma [**1**]. Next, we show the invariance of $C_\infty(X)$ and the strong Feller property of $\{p_t^F\}_{t>0}$ using Theorem 3 in Chung [**8**]. By the definition of the Lévy system, we have

$$\boldsymbol{E}_x[A_t(F_1)] = \boldsymbol{E}_x\left[\int_0^t \left(\int_X F_1(X_s, y)N(X_s, dy)\right)dH_s\right] = \boldsymbol{E}_x[A_t(\mu_{F_1})],$$

$$\lim_{t\to 0}\sup_{x\in X} \boldsymbol{E}_x[A_t(|F_1|)] = \lim_{t\to 0}\sup_{x\in X} \boldsymbol{E}_x[A_t(|\mu_{F_1}|)] = 0$$

for all $F_1 \in \mathscr{J}_\infty$. We have

$$\begin{aligned}
\boldsymbol{E}_x\Big[\exp(A_t(F))\Big] &= \boldsymbol{E}_x\Big[\exp\Big(\sum_{0<s\le t} F(X_{s-}, X_s)\Big)\Big] \\
&= \boldsymbol{E}_x\Big[\prod_{0<s\le t}(1+F_1(X_{s-}, X_s))\Big]. \qquad (3.1)
\end{aligned}$$

Furthermore, the Stieltjes exponential of $A_t(F_1)$ is equal to $\prod_{0<s\le t}(1 + F_1(X_{s-}, X_s))$ (see e.g. Sharpe [**17**, Section 71] and Ying [**25**]). Lemma 2.1 in [**25**] says that the right hand side of (3.1) is less than or equal to $(1 - \sup_{x\in X} \boldsymbol{E}_x[A_t(F_1)])^{-1}$. Thus, $\exp(A_t(F))$ satisfies conditions (a)–(c) in [**8**], that is, Theorem 3 in [**8**] is applicable for $\exp(A_t(F))$. Hence we show properties (iii) and (iv).

(v) Since $f(x) - f(\infty) \in C_\infty(X)$ and $p_t^F f(x) = p_t^F(f(x) - f(\infty)) + f(\infty)p_t^F 1(x)$, it is enough to prove that

$$\lim_{x\to\infty} p_t^F 1(x) = \lim_{x\to\infty} \boldsymbol{E}_x[\exp(A_t(F))] = 1.$$

For a non-negative function $F \in \mathscr{J}_\infty$ and a compact set $K \subset X$, define $F_K(x,y) = 1_K(x)F(x,y)$. We then have

$$\begin{aligned}
\boldsymbol{E}_x[\exp(A_t(F_K))] &= \boldsymbol{E}_x\big[\exp(A_t(F_K)); \sigma'_K > t\big] + \boldsymbol{E}_x\big[\exp(A_t(F_K)); \sigma'_K \le t\big] \\
&= \boldsymbol{P}_x\big(\sigma'_K > t\big) + \boldsymbol{E}_x\big[\exp(A_t(F_K)); \sigma'_K \le t\big].
\end{aligned}$$

Here, $\sigma'_K = \inf\{t > 0 : X_{t-} \in K\}$. By Theorem A.2.3 in [**10**] and Proposition 3.1, $\lim_{x\to\infty} \boldsymbol{P}_x(\sigma'_K > t) \ge \lim_{x\to\infty} \boldsymbol{P}_x(\sigma_K > t) = 1$. Since

$$\boldsymbol{E}_x\big[\exp(A_t(F_K)); \sigma'_K \le t\big] \le \boldsymbol{E}_x[\exp(A_t(2F))]^{1/2}\boldsymbol{P}_x\big(\sigma'_K \le t\big)^{1/2},$$

we see

$$\lim_{x\to\infty} \boldsymbol{E}_x[\exp(A_t(F_K))] = 1.$$

Moreover, using Lemma 2.1 in [**25**] again, we have

$$\begin{aligned}\sup_{x\in X}\boldsymbol{E}_x[\exp(A_t(F_{K^c}))] &= \sup_x \boldsymbol{E}_x\big[1+F_{1,K^c}(X_{s-},X_s)\big]\\ &\le \frac{1}{1-\sup_{x\in X}\boldsymbol{E}_x[A_t(F_{1,K^c})]}.\end{aligned}$$

By the definition of $\mathscr{J}_\infty$, for any $\epsilon>0$ there exists a compact set K such that

$$\begin{aligned}\sup_{x\in X}\boldsymbol{E}_x[A_t(F_{1,K^c})] &\le \sup_{x\in X}\int_{K^c} G(x,y)(\mu_{F_1})(dy)\\ &\le \epsilon.\end{aligned}$$

We then see that

$$\begin{aligned}\limsup_{x\to\infty}\boldsymbol{E}_x[\exp(A_t(F))] &= \limsup_{x\to\infty}\boldsymbol{E}_x\big[\exp(A_t(F_K))\exp(A_t(F_{K^c}))\big]\\ &\le \limsup_{x\to\infty}\big[\boldsymbol{E}_x[\exp(A_t(2F_K))]^{1/2}\boldsymbol{E}_x[\exp(A_t(2F_{K^c}))]^{1/2}\big]\\ &\le 1.\end{aligned}$$

In addition,

$$\begin{aligned}\liminf_{x\to\infty}\boldsymbol{E}_x[\exp(A_t(F))] &\ge \liminf_{x\to\infty}\boldsymbol{E}_x[\exp(-A_t(F^-)]\\ &\ge \Big\{\limsup_{x\to\infty}\boldsymbol{E}_x[\exp(A_t(F^-)]\Big\}^{-1}\ge 1.\end{aligned}$$

Hence we see that for any $F\in\mathscr{J}_\infty$, $\lim_{x\to\infty}\boldsymbol{E}_x[\exp(A_t(F))]=1$. □

4. L^p-independence of spectral bounds.

In this section, we give the sketch of the proof of the main theorem (see [**23**] and [**24**] for more details) and proofs of two lemmas (Lemmas 4.7 and 4.8) which play the important role of producing of the L^p-independence.

Let $\mathscr{P}(X)$ be the set of probability measures on X equipped with the weak topology. Define a function $I_{\mathscr{E}^F}$ on $\mathscr{P}(X)$ by

$$I_{\mathscr{E}^F}(\nu)=\begin{cases}\mathscr{E}^F(\sqrt{f},\sqrt{f}) & \text{if } \nu=f\cdot dm,\\ \infty & \text{otherwise.}\end{cases}$$

Let $\{R^F_\alpha\}_{\alpha>\kappa(F)}$ be the resolvent of the Schrödinger-type operator $\mathscr{H}^F$, that is, for $f \in \mathscr{B}_b(X)$,

$$\begin{aligned} R_\alpha^F f(x) &= \int_0^\infty e^{-\alpha t} p_t^F f(x) dt \\ &= \boldsymbol{E}_x\left[\int_0^\infty \exp(-\alpha t + A_t(F)) f(X_t) dt\right]. \end{aligned}$$

Here, $\kappa(F)$ is the constant in Theorem 3.2 (i). Set

$$\mathscr{D}_{++}(\mathscr{H}^F) = \left\{\phi = R_\alpha^F g : \alpha > \kappa(F), g \in C_u(X) \text{ with } g \geq \epsilon \text{ for some } \epsilon > 0\right\}.$$

For $\phi = R_\alpha^F g \in \mathscr{D}_{++}(\mathscr{H}^F)$, let

$$\mathscr{H}^F \phi = \alpha\phi - g,$$

and define a function I_F on $\mathscr{P}(X)$ by

$$I_F(\nu) = -\inf_{\phi \in \mathscr{D}_{++}(\mathscr{H}^F)} \int_X \frac{\mathscr{H}^F \phi}{\phi} d\nu.$$

It is known in Takeda [**22**, Proposition 4.3] that

$$I_{\mathscr{E}^F}(\nu) = I_F(\nu), \quad \forall \nu \in \mathscr{P}(X).$$

We define a transition probability $\bar{p}_t(x, dy)$ on $(X_\infty, \mathscr{B}(X_\infty))$; for $E \in \mathscr{B}(X_\infty)$,

$$\bar{p}_t(x, E) = \begin{cases} p_t(x, E \setminus \{\infty\}), & x \in X, \\ \delta_\infty(E), & x = \infty. \end{cases}$$

Let $\bar{\boldsymbol{M}}$ be a Markov process on X_∞ with transition probability $\bar{p}_t(x, dy)$. $\bar{\boldsymbol{M}}$ is an extension of $\boldsymbol{M}$ with ∞ being a trap. Furthermore, for $F \in \mathscr{J}_\infty$, we define $\{\bar{p}_t^F\}_{t>0}$ and $\{\bar{R}_\alpha^F\}_{\alpha > \kappa(F)}$ by

$$\begin{aligned} \bar{p}_t^F f(x) &= \bar{\boldsymbol{E}}_x[\exp(A_t(F)) f(X_t)], \\ \bar{R}_\alpha^F f(x) &= \int_0^\infty e^{-\alpha t} \bar{p}_t^F f(x) dt, \quad f \in \mathscr{B}_b(X_\infty). \end{aligned}$$

Then $\bar{R}_\alpha^F f(x) = R_\alpha^F f(x)$ for $x \in X$ and $\bar{R}_\alpha^F f(\infty) = f(\infty)/\alpha$. Set

$$\mathscr{D}_{++}(\bar{\mathscr{H}}^F) = \{\phi = \bar{R}_\alpha^F g : \alpha > \kappa(F), g \in C(X_\infty) \text{ with } g(x) > 0\}.$$

We see that for $\phi = \bar{R}_\alpha^F g \in \mathscr{D}_{++}(\bar{\mathscr{H}}^F)$, $\lim_{x\to\infty} \phi(x) = g(\infty)/\alpha$ by Theorem 3.2 (v). Let us define a function on $\mathscr{P}(X_\infty)$ the set of probability measures on X_∞, by

$$\bar{I}_F(\nu) = -\inf_{\phi \in \mathscr{D}_{++}(\bar{\mathscr{H}}^F)} \int_{X_\infty} \frac{\bar{\mathscr{H}}^F \phi}{\phi} d\nu, \quad \nu \in \mathscr{P}(X_\infty)$$

where $\bar{\mathscr{H}}^F \phi = \alpha \bar{R}_\alpha^F g - g$ for $\phi = \bar{R}_\alpha^F g \in \mathscr{D}_{++}(\bar{\mathscr{H}}^F)$. We then have

$$\bar{I}_F(\delta_\infty) = 0, \tag{4.1}$$

because $\bar{\mathscr{H}}^F \phi(\infty) = 0$ for any $\phi \in \mathscr{D}_{++}(\bar{\mathscr{H}}^F)$.

Let L_t be the *occupation distribution*, that is,

$$L_t(A) = \frac{1}{t}\int_0^t 1_A(X_s) ds, \quad t > 0,\ A \in \mathscr{B}(X). \tag{4.2}$$

Then $L_t \in \mathscr{P}(X)$.

Proposition 4.1 (Kim [**14**, Theorem 4.1 and Remark 4.1]). *Let $F \in \mathscr{J}_\infty$. Then for a closed set $K \subset \mathscr{P}(X_\infty)$ and an open set $G \subset \mathscr{P}(X)$*

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{x \in X} \boldsymbol{E}_x[\exp(A_t(F)); L_t \in K] \le -\inf_{\nu \in K} \bar{I}_F(\nu),$$

$$-\inf_{\nu \in G} I_F(\nu) \le \liminf_{t\to\infty} \frac{1}{t} \log \boldsymbol{E}_x[\exp(A_t(F)); L_t \in G].$$

Note that $\mathscr{P}(X_\infty)\setminus\{\delta_\infty\}$ and $(0,1]\times\mathscr{P}(X)$ are in one-to-one correspondence through the map:

$$\nu \in \mathscr{P}(X_\infty) \setminus \{\delta_\infty\} \mapsto \big(\nu(X), \hat{\nu}(\bullet) = \nu(\bullet)/\nu(X)\big) \in (0,1] \times \mathscr{P}(X). \tag{4.3}$$

Then, the next lemma can be proved by the same manner as that in [**23**, Lemma 3.1] and [**24**, Lemma 3.3]:

Lemma 4.2. *For $\nu \in \mathscr{P}(X_\infty) \setminus \{\delta_\infty\}$,*

$$\bar{I}_F(\nu) = I_F(\nu) = \nu(X) \cdot I_{\mathscr{E}^F}(\hat{\nu}).$$

We have the next inequality through the one-to-one map (4.3).

$$\inf_{\nu \in \mathscr{P}(X_\infty)\setminus\{\delta_\infty\}} \bar{I}_F(\nu) = \inf_{0<\theta\leq 1, \nu \in \mathscr{P}(X)} (\theta I_{\mathscr{E}^F}(\nu)) \leq 0.$$

Moreover, $\bar{I}_F(\delta_\infty) = 0$ from (4.1). Thus, the next corollary holds as follows.

Corollary 4.3.

$$\inf_{\nu \in \mathscr{P}(X_\infty)} \bar{I}_F(\nu) = \inf_{0\leq\theta\leq 1, \nu \in \mathscr{P}(X)} (\theta I_{\mathscr{E}^F}(\nu)) = \inf_{0\leq\theta\leq 1} \Big(\theta \inf_{\nu \in \mathscr{P}(X)} I_{\mathscr{E}^F}(\nu)\Big). \tag{4.4}$$

Let us denote by $\|p_t^F\|_{p,p}$ the operator norm of p_t^F from $L^p(X)$ to $L^p(X)$ and define

$$\lambda_p(F) = -\lim_{t\to\infty} \frac{1}{t} \log \|p_t^F\|_{p,p}, \quad 1 \leq p \leq \infty.$$

Noting that $\sup_{x\in X} \boldsymbol{E}_x[\exp(A_t(F))]$ equals $\|p_t^F\|_{\infty,\infty}$, we see that

$$\lim_{t\to\infty} \frac{1}{t} \log \sup_{x\in X} \boldsymbol{E}_x[\exp(A_t(F))] = -\lambda_\infty(F).$$

Hence we have

$$\lambda_\infty(F) \geq \inf_{0\leq\theta\leq 1} \Big(\theta \inf_{\nu \in \mathscr{P}(X)} I_{\mathscr{E}^F}(\nu)\Big) \tag{4.5}$$

by Proposition 4.1 and the equation (4.4).

By the spectral theorem, $\lambda_2(F)$ is identical to the bottom of the spectrum of $-\mathscr{H}^F$ and by the variational formula for the bottom of spectrum

$$\lambda_2(F) = \inf_{\nu \in \mathscr{P}(X)} I_{\mathscr{E}^F}(\nu). \tag{4.6}$$

Combining (4.5) and (4.6), we then have the following inequality: For any $F \in \mathscr{J}_\infty$,

$$\lambda_\infty(F) \geq \inf_{0\leq\theta\leq 1}\left(\theta \inf_{\nu\in\mathscr{P}(X)} I_{\mathscr{E}^F}(\nu)\right) = \inf_{0\leq\theta\leq 1}(\theta\lambda_2(F)). \tag{4.7}$$

If $\lambda_2(F) \leq 0$, then $\inf_{0\leq\theta\leq 1}(\theta\lambda_2(F)) = \lambda_2(F)$. Hence we have:

Corollary 4.4. *If $\lambda_2(F) \leq 0$, then*

$$\lambda_\infty(F) \geq \lambda_2(F).$$

The inequality, $\lambda_2(F) \geq \lambda_\infty(F)$, holds generally because

$$\|p_t^F\|_{2,2} \leq \|p_t^F\|_{\infty,\infty}$$

by the symmetry and the positivity of p_t^F. Since

$$\|p_t^F\|_{2,2} \leq \|p_t^F\|_{p,p} \leq \|p_t^F\|_{\infty,\infty}$$

by the Riesz-Thorin interpolation theorem, we can conclude that if $\lambda_2(F) \leq 0$, then the L^p-independence holds. Now we state main theorem.

Theorem 4.5. *Let $F \in \mathscr{J}_\infty$. Then $\lambda_2(F) = \lambda_p(F)$ for all $1 \leq p \leq \infty$ if and only if $\lambda_2(F) \leq 0$.*

Proof. On account of Corollary 4.4, we have only to prove the "only if" part. Suppose that $\lambda_2(F) > 0$. Then

$$\lambda_\infty(F) \geq \inf_{0\leq\theta\leq 1}\theta \inf_{\nu\in\mathscr{P}(X)} I_{\mathscr{E}^F}(\nu) = \inf_{0\leq\theta\leq 1}\theta(\lambda_2(F)) = 0$$

by (4.7). By Theorem 3.2 (v), $\lim_{x\to\infty} p_t^F 1(x) = 1$, which implies that $\|p_t^F\|_{\infty,\infty} \geq 1$ and $\lambda_\infty(F) \leq 0$. Therefore if $\lambda_2(F) > 0$, then $\lambda_\infty(F) = 0$. □

Corollary 4.6. *Suppose that $\lambda_2(0) = 0$. If $F \in \mathscr{J}_\infty$, then $\lambda_2(F) = \lambda_p(F)$ for all $1 \leq p \leq \infty$.*

Proof. By Theorem 4.5, we only have to prove that $\lambda_2(F) \leq 0$ for any $F \in \mathscr{J}_\infty$. That is, for any positive $\mu \in \mathscr{K}_\infty$,

$$\lambda_2(\mu) = \inf\left\{\mathscr{E}_F(u,u) + \int_X u^2 d\mu : u \in \mathscr{F}, \|u\|_2 = 1\right\} = 0.$$

We see from [**19**, Theorem 3.1], for all $u \in \mathscr{F}$ such that $\|u\|_2 = 1$,

$$\int_X u^2 d\mu \le C\|G\mu\|_\infty \mathscr{E}(u,u).$$

Since the boundedness of F, there exists a constant C' such that $\mathscr{E}_F(u,u) \le C'\mathscr{E}(u,u)$ for all $u \in \mathscr{F}$. We then have

$$\begin{aligned}\lambda_2(\mu) &\le \left(\mathscr{E}_F(u,u) + \int_X u^2 d\mu\right) \\ &\le \big(C' + C\|G\mu\|_\infty\big)\mathscr{E}(u,u).\end{aligned}$$

We get desired claim. □

Next two lemmas play the important role of producing of the L^p-independence.

LEMMA 4.7. *If*

$$\inf\left\{\mathscr{E}(u,u) : u \in \mathscr{F}, \int_{X\times X} u(x)u(y)F_1(x,y)N(x,dy)\mu_H(dx) = 1\right\} < 1, \tag{4.8}$$

then

$$\inf\left\{\mathscr{E}^F(u,u) : u \in \mathscr{F}, \|u\|_2 = 1\right\} < 0.$$

PROOF. Let ϕ be a function such that (4.8) holds. Let $\psi = \phi/\|\phi\|$. Then we have

$$\begin{aligned}\mathscr{E}^F(\psi,\psi) &= \mathscr{E}(\psi,\psi) - \int_{X\times X} \psi(x)\psi(y)F_1(x,y)N(x,dy)\mu_H(dx) \\ &= \frac{1}{\|\phi\|_2^2}\left(\mathscr{E}(\phi,\phi) - \int_{X\times X} \phi(x)\phi(y)F_1(x,y)N(x,dy)\mu_H(dx)\right) < 0. \quad \square\end{aligned}$$

LEMMA 4.8. *Let $F \in \mathscr{J}_\infty$, $F \ge 0$ and $F \not\equiv 0$ and define $F_1^\theta = e^{\theta F} - 1$. Then there exists $u \in \mathscr{F}$ such that*

$$\mathscr{E}(u,u) < 1 \textit{ and } \int_{X\times X} u(x)u(y)F_1^\theta(x,y)N(x,dy)\mu_H(dx) = 1 \tag{4.9}$$

holds for sufficiently large θ.

Proof. Let $v \in \mathscr{F}$ such that $v \geq 0$, $\int v(x)v(y)F_1(x,y)N(x,dy)\mu_H(dx) = 1$ and

$$k(\theta) = \frac{\int v(x)v(y)F_1(x,y)N(x,dy)\mu_H(dx)}{\int u(x)u(y)F_1^\theta(x,y)N(x,dy)\mu_H(dx)} = \frac{1}{\int v(x)v(y)F_1^\theta(x,y)N(x,dy)\mu_H(dx)}.$$

Obviously, $k(\theta) \to 0$ as $\theta \to \infty$. Thus $u = \sqrt{k(\theta)}v$ satisfies (4.9) for sufficiently large θ. □

5. Examples.

5.1. α-stable processes on Euclidean space.

Let $(X_t, \boldsymbol{P}_x)$ be a symmetric α-stable process on $\boldsymbol{R}^d$ $(0 < \alpha < 2,\ \alpha < d)$, the pure jump process generated by $\frac{1}{2}(-\Delta)^{\alpha/2}$. Let $(\mathscr{E}^{(\alpha)}, \mathscr{F}^{(\alpha)})$ be the symmetric Dirichlet form generated by $(X_t, \boldsymbol{P}_x)$. Then $\lambda_2(0) = 0$. We thus have by Corollary 4.6;

Theorem 5.1 ([**24**, Theorem 3.8]). *Let $F \in \mathscr{J}_\infty$. Then*

$$\lambda_p(F) = \lambda_2(F) \quad 1 \leq \forall p \leq \infty.$$

5.2. Subordination.

In this section, we consider "α-stable processes" on (X, m) generated by the subordination procedure ([**15**] and [**16**]). Let $(X_t, \boldsymbol{P}_x)$ be a Hunt process on (X, m) satisfying assumptions (I)–(IV). Let $\gamma_t^\alpha(s)$ $(s > 0, 0 < \alpha < 2)$ be the unique continuous function satisfying

$$e^{-ta^{\alpha/2}} = \int_0^\infty e^{-as}\gamma_t^{(\alpha)}(s)ds, \quad a, t > 0$$

(see Yosida [**26**, Chapter IX Section 11] for more details). Define

$$p_t^{(\alpha)} f(x) = \int_0^\infty \boldsymbol{E}_x[f(X_s)]\gamma_t^{(\alpha)}(s)ds, \quad t > 0. \tag{5.1}$$

Then $\{p_t^{(\alpha)}\}_{t>0}$ is a strongly continuous sub-Markovian semigroup on $L^2(X; m)$.

We have the corresponding Dirichlet form by

$$\begin{cases} \mathscr{E}^{(\alpha)}(u,u) = \displaystyle\int_0^\infty \lambda^{\alpha/2} d(E_\lambda u, u), \quad u \in \mathscr{F}^{(\alpha)}, \\ \mathscr{F}^{(\alpha)} = \left\{ u \in L^2(X;m) : \displaystyle\int_0^\infty \lambda^{\alpha/2} d(E_\lambda u, u) < \infty \right\}. \end{cases}$$

Furthermore, there exists a Hunt process $\boldsymbol{M}^{(\alpha)}$ properly associated to $(\mathscr{E}^{(\alpha)}, \mathscr{F}^{(\alpha)})$ ([**15**, Theorem 3.2]).

THEOREM 5.2 ([**16**, Theorem 3.2]). *If a Hunt process $\boldsymbol{M}$ is transient, then so is $\boldsymbol{M}^{(\alpha)}$.*

THEOREM 5.3. *If a Hunt process $\boldsymbol{M}$ satisfies* (I)–(IV)*, then so is $\boldsymbol{M}^{(\alpha)}$.*

PROOF. (I): Take any $p_t^{(\alpha)}$-invariant set A and $f \in L^2(X;m)$, $f > 0$.

$$\begin{aligned} 1_A(x)(p_t^{(\alpha)} f(x)) &= 1_A(x) \int_0^\infty \boldsymbol{E}_x[f(X_s)] \gamma_t^{(\alpha)}(s) ds \\ &= \int_0^\infty 1_A(x) \boldsymbol{E}_x[f(X_s)] \gamma_t^{(\alpha)}(s) ds \\ &= \int_0^\infty 1_A(x) p_s f(x) \gamma_t^{(\alpha)}(s) ds. \end{aligned}$$

Furthermore,

$$p_t^{(\alpha)}(1_A f(x)) = \int_0^\infty p_s(1_A f(x)) \gamma_t^{(\alpha)}(s) ds.$$

Since $\gamma_t^{(\alpha)}(s) > 0$, $p_s(1_A f(x)) = 1_A p_s f(x)$ a.e. s and the irreducibility of p_t, $m(A) = 0$ or $m(X \setminus A) = 0$.

(II): It is obvious by the conservativeness of $\{p_t\}_{t>0}$ and $\int_0^\infty \gamma_t^{(\alpha)}(s) ds = 1$.

(III) and (IV): From $\gamma_t^{(\alpha)}(s) ds$ being bounded measure and the dominated convergence theorem, (III) and (IV) hold. □

REMARK 5.1. In Theorem 5.3, each property (I)–(IV) holds independent on other properties.

5.3. "α-stable process" on the hyperbolic space.

Let $\boldsymbol{H}^d$ be a hyperbolic space of dimension d ($d \geq 2$) with volume element v. Let $\boldsymbol{M}$ be a Brownian motion on $\boldsymbol{H}^d$ with the Dirichlet form $(\mathscr{E}, \mathscr{F})$. Then the Brownian motion is transient (see e.g. Grigor'yan [**11**, pp. 148–149]) and the corresponding transition density has an exact expression (see e.g. Grigor'yan and Noguchi [**12**, Theorem 1.1]). The corresponding Dirichlet form is expressed by

$$\mathscr{E}(u,u) = \frac{1}{2}\int_{\boldsymbol{H}^d} (\nabla u, \nabla u)dv = \int_0^\infty \lambda d(E_\lambda u, u), \quad u \in \mathscr{F}, \tag{5.2}$$

where $\mathscr{F}$ is the closure of $C_0^\infty(\boldsymbol{H}^d)$ with respect to the norm; $\mathscr{E}_1(\cdot,\cdot)^{1/2} = (\mathscr{E}(\cdot,\cdot)+(\cdot,\cdot))^{1/2}$. We construct an example of producing the L^p-independence for non-local Feynman-Kac semigroups:

Example 5.1. Let $\boldsymbol{M}$ be the Brownian motion on $\boldsymbol{H}^d$ where the corresponding Dirichlet form is defined as in (5.2). Let $\boldsymbol{M}^{(\alpha)}$ be a Hunt process defined as in (5.1). It is well-known that

$$\inf\left\{\mathscr{E}^{(\alpha)}(u,u) : u \in \mathscr{F}^{(\alpha)}, \int u^2 dv = 1\right\} = \frac{1}{2}\left(\frac{(d-1)^2}{4}\right)^{\alpha/2}$$

from

$$\inf\left\{\mathscr{E}(u,u) : u \in \mathscr{F}, \int u^2 dv = 1\right\} = \frac{1}{2}\frac{(d-1)^2}{4}$$

(see [**9**, p. 177]). Thus, $\lambda_2(0) > 0$, i.e. the L^p-independence does *not* hold. Let F be in $\mathscr{J}_\infty$ such that $F \geq 0$ and $F \not\equiv 0$. Lemmas 4.7 and 4.8 yield that

$$\inf\left\{\mathscr{E}^{(\alpha),\theta F}(u,u) : u \in \mathscr{F}, \int_{\boldsymbol{H}^d} u^2 dv = 1\right\} < 0 \tag{5.3}$$

for sufficiently large θ. We can conclude that $\lambda_p(\theta F)$ is independent of p for large θ.

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Yoshihiro Tawara
Mathematical Institute
Tohoku University
Aoba, Sendai 980-8578, Japan
E-mail: sa5d11@math.tohoku.ac.jp

Current address:
Division of General Education
Nagaoka National College of Technology
888 Nishikatakai, Nagaoka
Niigata 940-8532, Japan
E-mail: tawara@nagaoka-ct.ac.jp