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Minor degenerations of the full matrix algebra over a field

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Abstract. Given a positive integer $n \geq 2$, an arbitrary field K and an n-block $q = [q^{(1)}| \cdots |q^{(n)}]$ of $n \times n$ square matrices $q^{(1)}, \dots, q^{(n)}$ with coefficients in K satisfying certain conditions, we define a multiplication $\cdot_q : \boldsymbol{M}_n(K) \otimes_K \boldsymbol{M}_n(K) \longrightarrow \boldsymbol{M}_n(K)$ on the K-module $\boldsymbol{M}_n(K)$ of all square $n \times n$ matrices with coefficients in K in such a way that $\cdot_q$ defines a K-algebra structure on $\boldsymbol{M}_n(K)$. We denote it by $\boldsymbol{M}_n^q(K)$, and we call it a minor q-degeneration of the full matrix K-algebra $\boldsymbol{M}_n(K)$. The class of minor degenerations of the algebra $\boldsymbol{M}_n(K)$ and their modules are investigated in the paper by means of the properties of q and by applying quivers with relations. The Gabriel quiver of $\boldsymbol{M}_n^q(K)$ is described and conditions for q to be $\boldsymbol{M}_n^q(K)$ a Frobenius algebra are given. In case K is an infinite field, for each $n \geq 4$ a one-parameter K-algebraic family $\{C_\mu\}_{\mu \in K^*}$ of basic pairwise non-isomorphic Frobenius K-algebras of the form $C_\mu = \boldsymbol{M}_n^{q_\mu}(K)$ is constructed. We also show that if $A_q = \boldsymbol{M}_n^q(K)$ is a Frobenius algebra such that $J(A_q)^3 = 0$, then A_q is representation-finite if and only if $n = 3$, and A_q is tame representation-infinite if and only if $n = 4$.

1. Introduction.

Let R be a discrete valuation ring with a unique maximal ideal πR. It is standard to reduce homological properties of R-orders Λ to those of factor algebras $\Lambda/\pi\Lambda$. For example, Gorenstein R-orders can be reduced to quasi-Frobenius $R/\pi R$-algebras. However, the study of such factor algebras is very limited, while its importance is well-recognized by many authors, see e.g. [**16**] and [**24**] for the relationship of homological dimensions, [**6**] and [**25**] for Gorenstein tiled R-orders and their factor algebras, and [**14**] and [**17**] for further information. In [**7**], Fujita introduced full matrix algebras with structure systems as a framework for such factor algebras $\Lambda/\pi\Lambda$ of tiled R-orders Λ.

Let $n \geq 2$ be an integer and K a field. A structure system is an n-tuple of $n \times n$ matrices over K with certain properties. A full matrix algebra with a structure system is an n^2-dimensional K-vector space with an associative multiplication defined by a structure system. In [**7**] and [**8**], we mainly studied full matrix algebras with $(0, 1)$-structure systems, that is, their components are 0 or 1, just as structure systems of factor algebras $\Lambda/\pi\Lambda$ of tiled D-orders Λ, and we are interested in Frobenius full matrix algebras and showed that the class of Frobenius full matrix algebras is a strictry larger class than that of the factor algebras of Gorenstein tiled orders. Then one may ask, as a next step, whether there are full matrix algebras which are not isomorphic to ones with

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$(0,1)$-structure systems at all. This is one of the motivations for our study. In this paper, we provide such examples in Sections 4 and 5.

The other motivation for our study is the fact that we are able to treat the class of full matrix algebras with structure systems by an elementary algebraic geometry technique and study them in a deformation theory context [**13**]. It turns out that, for suitable choice of the structure matrix q, the algebra $\boldsymbol{M}_n^q(K)$ is a degeneration of the full matrix algebra $\boldsymbol{M}_n(K)$, see [**12**] and Section 2. So, in this paper, we consider the class of full matrix algebras with structure systems as a subclass of minor degenerations of the full matrix algebra $\boldsymbol{M}_n(K)$, see Section 2 for definition. We would like to note here that we are also following an old idea of the skew matrix ring construction by Kupisch in [**19**] and [**20**], see also Oshiro and Rim [**22**].

There is also another motivation coming from the fact proved in [**28**] that, given a prime $p \geq 2$ and an algebraically closed field K of characteristic zero, any Hopf K-algebra of dimension p^2 is semisimple or is isomorphic to the Taft Hopf algebra. In connection with this result and the facts that Hopf algebras are Frobenius algebras and the Taft Hopf algebra is a Nakayama algebra, the existence of a Hopf algebra structure on a Frobenius algebra of the form $\boldsymbol{M}_n^q(K)$ (of dimension $n^2!$), seems to be a natural problem to solve, see (2.8). We do not solve it here, but we shall study it in a subsequent paper. Here we only describe Nakayama algebras (Section 3) and Frobenius algebras (Section 5) of the form $\boldsymbol{M}_n^q(K)$ for a class of matrices q.

Section 2 contains basic definitions, examples and properties of minor q-degenerations $\boldsymbol{M}_n^q(K)$ of the full matrix K-algebra $\boldsymbol{M}_n(K)$. In particular, we give a criterion for the existence of a K-algebra isomorphism $\boldsymbol{M}_n^q(K) \cong \boldsymbol{M}_n^{q'}(K)$ in terms of an action

$$* : \boldsymbol{G}_n(K) \times \boldsymbol{ST}_n(K) \longrightarrow \boldsymbol{ST}_n(K)$$

of an algebraic group $\boldsymbol{G}_n(K) = \mathscr{T}_n \ltimes S_n$ (containing the symmetric group S_n and the torus $\mathscr{T}_n$) on the algebraic K-variety $\boldsymbol{ST}_n(K) \subseteq \boldsymbol{M}_{n\times n^2}(K)$ of the minor constant matrices $q = [q^{(1)}|\cdots|q^{(n)}]$, see (2.16), (2.17) and (2.18). The algebras $\boldsymbol{M}_n^q(K)$ and their modules are investigated by means of the properties of q and by applying quivers with relations. In case the algebra is basic, the Gabriel quiver of $\boldsymbol{M}_n^q(K)$ is described.

A complete classification, up to isomorphism, of basic algebras $\boldsymbol{M}_n^q(K)$ in case $n = 2$ and $n = 3$ is given in Section 4. The matrices $q = [q^{(1)}|\cdots|q^{(n)}]$ in $\boldsymbol{ST}_n(K)$ such that $\boldsymbol{M}_n^q(K)$ is a Nakayama algebra are described in Section 3, where also $(0,1)$-limits of algebras $\boldsymbol{M}_n^q(K)$ are studied.

Conditions for the matrices $q = [q^{(1)}|\cdots|q^{(n)}]$ in $\boldsymbol{ST}_n(K)$ to be $A_q = \boldsymbol{M}_n^q(K)$ a Frobenius algebra are given in Section 5, by extending some of the Fujita's results in [**7**, Section 4]. All matrices q such that A_q is a Frobenius algebra and the cube $J(A_q)^3$ of the Jacobson radical $J(A_q)$ of $A_q = \boldsymbol{M}_n^q(K)$ is zero are described in Theorem 5.5. In case K is an infinite field, for each $n \geq 4$, we construct a one-parameter K-algebraic family $\{C_\mu\}_{\mu\in K^*}$ of basic pairwise non-isomorphic Frobenius K-algebras of the form $C_\mu = \boldsymbol{M}_n^{q_\mu}(K)$.

Finally, we show that if $A_q = \boldsymbol{M}_n^q(K)$ is a Frobenius algebra such that $J(A_q)^3 = 0$, then the representation type of A_q is completely determined as follows:

(i) A_q is representation-finite if and only if $n = 3$,
(ii) A_q is tame representation-infinite [**26**, Section 14.4] if and only if $n = 4$, and
(iii) A_q is representation-wild [**26**, Section 14.2] if and only if $n \geq 5$,

where we assume in (ii) and in (iii) that the field K is algebraically closed.

Throughout this paper K is a field and R is a ring with an identity element. We denote by $J(R)$ the Jacobson radical of R, and by $\mathrm{mod}(R)$ the category of finitely generated right R-modules. Given $n \geq 1$, we denote by $\boldsymbol{M}_n(R)$ the full matrix R-algebra consisiting of all square $n \times n$ matrices with coefficients in R and by e_{ij} the matrix unit in $\boldsymbol{M}_n(R)$ with 1 on the (i,j) entry, and zero elsewhere. We denote by $e_1, \ldots, e_n$ the standard matrix idempotents $e_{11}, \ldots, e_{nn}$ of $\boldsymbol{M}_n(R)$.

2. Minor constant structure matrices and minor degenerations.

Throughout, we fix an integer $n \geq 2$. We suppose that K is an arbitrary field and R is a ring with an identity element. We recall that, given a finite dimensional K-algebra A and a complete set $e_1, \ldots, e_n$ of pairwise orthogonal primitive idempotents of A, we define the Cartan matrix of A to be the matrix $C_A = [c_{ij}] \in \boldsymbol{M}_n(\boldsymbol{Z})$, where $c_{ij} = \dim_K e_i A e_j$. The algebra A is said to be basic if $e_j A \not\cong e_i A$ for $i \neq j$, and A is said to be connected if A is not a direct product of two K-algebras (see [**1**] and [**2**]).

Following Fujita [**7**], we introduce the following definition.

Definition 2.1. Assume that $n \geq 2$. A minor constant structure matrix of size $n \times n^2$, with coefficients in a ring R, is the n-block matrix

$$q = \left[q^{(1)} | q^{(2)} | \cdots | q^{(n)}\right] \tag{2.2}$$

where $q^{(1)} = [q^{(1)}_{ij}], \ldots, q^{(n)} = [q^{(n)}_{ij}] \in \boldsymbol{M}_n(R)$ are $n \times n$ square matrices with coefficients in the center $Z(R)$ of R satisfying the following two conditions

(C1) $q^{(r)}_{rj} = 1$ and $q^{(r)}_{jr} = 1$, for all $j, r \in \{1, \ldots, n\}$.

(C2) $q^{(r)}_{ij} q^{(j)}_{is} = q^{(r)}_{is} q^{(j)}_{rs}$, for all $i, j, r, s \in \{1, \ldots, n\}$.

We call q basic if, in addition, the following condition is satisfied

(C3) $q^{(r)}_{jj} = 0$, for $r = 1, \ldots, n$ and all $j \in \{1, \ldots, n\}$ such that $j \neq r$.

The minor constant structure matrix q is called $(0,1)$-matrix, if each entry $q^{(r)}_{ij}$ is either 0 or 1. Throughout this paper, a minor constant structure matrix will be often called a structure matrix of $\boldsymbol{M}_n(R)$, in short. We denote by

$$\boldsymbol{ST}_n(R) \subseteq \boldsymbol{M}_{n \times n^2}(R) \tag{2.3}$$

the set of all minor constant structure matrices q of size $n \times n^2$, with coefficients in R.

Lemma 2.4.

(a) *Let $n \geq 2$ and let $q = [q^{(1)} | q^{(2)} | \cdots | q^{(n)}]$ be a matrix of the form* (2.2) *satisfying the condition* (C1). *Then the equality $q^{(r)}_{ij} q^{(j)}_{is} = q^{(r)}_{is} q^{(j)}_{rs}$ in* (C2) *holds, if $i = r$, or $r = j$, or $j = s$, and $i, j, r, s \in \{1, \ldots, n\}$.*

(b) *Assume that* $q = [q^{(1)}|q^{(2)}|\cdots|q^{(n)}]$ *is a structure matrix* (2.2) *in* $\boldsymbol{ST}_n(R)$.
 (b1) $q_{jj}^{(r)} = q_{rr}^{(j)}$, *for all* $j, r \in \{1, \dots, n\}$.
 (b2) $q_{jj}^{(r)} = q_{js}^{(r)} q_{rs}^{(j)} = q_{sj}^{(r)} q_{sr}^{(j)}$, *for any triple of elements* $j, r, s \in \{1, \dots, n\}$.
 (b3) *Assume that* R *is a domain. If* $q_{jj}^{(r)} \neq 0$ *and* $q_{ss}^{(r)} \neq 0$, *then* $q_{ss}^{(j)} = q_{jj}^{(s)} \neq 0$.

(c) *If* $n \geq 3$ *and the matrix* $q = [q^{(1)}|q^{(2)}|\cdots|q^{(n)}]$ *is basic then, for any* $i, j, r \in \{1, \dots, n\}$, $q_{ij}^{(r)} q_{ir}^{(j)} = 0$ *if* $j \neq r$, *and* $q_{rj}^{(i)} q_{ij}^{(r)} = 0$ *if* $i \neq r$.

PROOF.

(a) Let $i = r$. Then (C1) yields $q_{rj}^{(r)} = 1$, $q_{rs}^{(r)} = 1$ and we get $q_{ij}^{(r)} q_{is}^{(j)} = q_{rj}^{(r)} = q_{rs}^{(r)} q_{rs}^{(j)} = q_{is}^{(r)} q_{rs}^{(j)}$. If $r = j$ or $j = s$, the equality $q_{ij}^{(r)} q_{is}^{(j)} = q_{is}^{(r)} q_{rs}^{(j)}$ follows in a similar way.

(b) (b1) Apply (C2) with $i = j$, $s = r$ and then use (C1).
 (b2) By (C2), we have $q_{jj}^{(r)} q_{js}^{(j)} = q_{js}^{(r)} q_{rs}^{(j)}$. Since $q_{js}^{(j)} = 1$, the first equality holds. The second one follows in a similar way.
 (b3) By (C2), we have $q_{sr}^{(j)} q_{ss}^{(r)} = q_{ss}^{(j)} q_{js}^{(r)}$. Since $q_{jj}^{(r)} \neq 0$ then, according to (b2), $q_{sr}^{(j)}$ is non-zero and the equation yields $q_{ss}^{(j)} \neq 0$.

(c) By applying (C2) with $s = r$ we get $q_{ij}^{(r)} q_{ir}^{(j)} = q_{ir}^{(r)} q_{rr}^{(j)} = 0$, because $j \neq r$ implies $q_{rr}^{(j)} = 0$, by (C3). The equality $q_{rj}^{(i)} q_{ij}^{(r)} = 0$ follows in a similar way. □

Now we introduce the minor q-degeneration $\boldsymbol{M}_n^q(R)$ of the algebra $\boldsymbol{M}_n(R)$.

DEFINITION 2.5. Let $n \geq 2$ be an integer and let $q = [q^{(1)}|\cdots|q^{(n)}]$ be a minor constant structure matrix (2.2) in $\boldsymbol{ST}_n(R)$ with coefficients in the center of a ring R. A q-degeneration $\boldsymbol{M}_n^q(R)$ of the full matrix ring $\boldsymbol{M}_n(R)$ is defined to be the R-module $\boldsymbol{M}_n(R)$ equipped with the q-multiplication

$$\cdot_q : \boldsymbol{M}_n(R) \otimes_R \boldsymbol{M}_n(R) \longrightarrow \boldsymbol{M}_n(R)$$

that associates to any pair of matrices $\lambda' = [\lambda'_{ij}], \lambda'' = [\lambda''_{ij}] \in \boldsymbol{M}_n(R)$ the matrix

$$\lambda' \cdot_q \lambda'' = [\lambda_{ij}], \text{ where } \lambda_{ij} = \sum_{s=1}^{n} \lambda'_{is} q_{ij}^{(s)} \lambda''_{sj}, \tag{2.6}$$

for $i, j \in \{1, \dots, n\}$. Throughout, we simply write $\lambda'\lambda''$ instead of $\lambda' \cdot_q \lambda''$.

A straightforward computation shows that $\boldsymbol{M}_n^q(R)$ is a ring and the identity matrix $E = \mathrm{diag}(1, \dots, 1)$ of $\boldsymbol{M}_n(R)$ is the identity of $\boldsymbol{M}_n^q(R)$.

By a minor degeneration of the full matrix ring $\boldsymbol{M}_n(R)$ we mean a q-degeneration ring $\boldsymbol{M}_n^q(R)$, where $n \geq 2$ and q is a structure matrix (2.2) in $\boldsymbol{ST}_n(R)$.

Elementary properties of the K-algebra $\boldsymbol{M}_n^q(K)$ are collected in Theorem 2.9 below. In particular, it follows that $\boldsymbol{M}_n^q(K)$ is a non-semisimple basic K-algebra, if q is basic, $n \geq 2$, and K is a field.

We remark that if $q = [q^{(1)}|\cdots|q^{(n)}]$ is the matrix (2.2) with $q_{ij}^{(s)} = 1$ for all $i, j, s \in \{1, \dots, n\}$, then the conditions (C1) and (C2) are satisfied, but the condition (C3) is not. In this case, we have $\boldsymbol{M}_n^q(R) = \boldsymbol{M}_n(R)$, because the formula (2.6) defines the usual matrix multiplication on $\boldsymbol{M}_n(R)$.

It turns out that, under a suitable choice of q, the algebra $\boldsymbol{M}_n^q(K)$ is a degeneration of $\boldsymbol{M}_n(K)$ in the sense of [**13**], if K is a field, see Examples 2.8 and 2.14. We recall from [**13**] and [**11**] that given two K-algebras A_1 and A_0 (with an underlying K-space K^m) defined by the constant structure matrices μ_1 and μ_0, respectively, μ_1 and μ_0 are viewed as elements of the algebraic variety $\mathscr{A}lg(K^m)$ of associative unitary K-algebra structures on the vector space K^m. The general linear group $\mathrm{Gl}(K^m)$ acts on $\mathscr{A}lg(K^m)$ by the transport of structures, see also [**18**, p. 225]. An algebra A_1 is said to be a deformation of the algebra A_0 (or A_0 is a degeneration of the algebra A_1), if μ_0 lies in the closure of the $\mathrm{Gl}(K^m)$-orbit of μ_1 in $\mathscr{A}lg(K^m)$, see [**11**], [**12**] and [**18**]. We note that the set $\boldsymbol{ST}_n(K) \subseteq \boldsymbol{M}_{n\times n^2}(K)$ of minor constant structure matrices (2.2) of size $n \times n^2$ is an algebraic K-variety. Moreover, there is a variety embedding

$$\boldsymbol{ST}_n(K) \subseteq \mathscr{A}lg(K^{n^2}) = \mathscr{A}lg(\boldsymbol{M}_n(K)) \tag{2.7}$$

defined by attaching to any minor constant structure matrix q the matrix of constants of the multiplication $\cdot_q : \boldsymbol{M}_n^q(K) \otimes \boldsymbol{M}_n^q(K) \longrightarrow \boldsymbol{M}_n^q(K)$ in the matrix unit basis, see (2.10) below. It is clear that $\boldsymbol{ST}_n(K)$ is a locally closed subset of $\mathscr{A}lg(K^{n^2})$.

In this paper we study the basic K-algebras $\boldsymbol{M}_n^q(K)$ and their modules by means of quivers with relations. We recall that, given a quiver $Q = (Q_0, Q_1)$, by an oriented paths in Q starting from the vertex $i = i_0$ and ending at the vertex $j = i_m$ we mean a formal composition

$$\beta_1\beta_2\cdots\beta_m \equiv \left(i_0 \xrightarrow{\beta_1} i_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_m} i_m\right)$$

of arrows $\beta_1, \dots, \beta_m$. We denote by KQ the path K-algebra, that is, the K-algebra generated by all oriented paths in Q, see [**1**, Chapter II], [**2**], [**26**, Chapter 14], and [**30**].

Now we illustrate the notion of a minor degeneration algebra by the following example.

EXAMPLE 2.8. Assume that $n = 2$ and R is a ring with identity. It follows from Lemma 2.4(b) and the conditions (C1) and (C2) in Definition 2.1 that $q = [q^{(1)}|q^{(2)}]$ is a structure matrix (2.2) in $\boldsymbol{ST}_2(R)$ if and only if q has the form $q(\mu) = \left[\begin{smallmatrix} 1 & 1 \\ 1 & \mu \end{smallmatrix}\middle|\begin{smallmatrix} \mu & 1 \\ 1 & 1 \end{smallmatrix}\right]$, where $\mu = q_{22}^{(1)} = q_{11}^{(2)}$ is a scalar in R. The matrix $q = q(0) = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\middle|\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ is a unique basic structure matrix in $\boldsymbol{ST}_2(R)$.

Assume that K is a field, $q(\mu)$ is the structure matrix presented above with $\mu \in K$, and let $A(\mu) = \boldsymbol{M}_2^{q(\mu)}(K)$. We claim that:

- The K-algebra $A(\mu)$ is semisimple and $A(\mu) \cong A(1) = \boldsymbol{M}_2(K)$ if and only if $\mu \neq 0$.
- For each $\mu \in K$, $A(\mu)$ is a degeneration of the full matrix algebra $A(1) = \boldsymbol{M}_2(K)$.
- $A(0)$ is a non-semisimple self-injective Nakayama K-algebra of finite representation type.
- The algebra $A(0)$ admits a Hopf algebra structure (by [**28**]). If $\operatorname{char} K \neq 2$, then the Hopf algebra $A(0)$ is isomorphic to the Sweedler Hopf algebra, see [**21**, p. 8].

The first statement and the second one are easily verified. To see the third one we note that, by the multiplication rule (2.6), the Jacobson radical $J(A)$ of the K-algebra $A = A(0)$ has the form $J(A) = \left(\begin{smallmatrix} 0 & K \\ K & 0 \end{smallmatrix}\right) = Ke_{12} \oplus Ke_{21}$. Note also that $J(A)^2 = 0$ and $\operatorname{soc} A_A = J(A)$. Hence we easily conclude that there is a K-algebra isomorphism

$$A = A(0) \cong KQ/I,$$

where Q is the quiver

$$Q: \quad 1 \underset{\beta_{21}}{\overset{\beta_{12}}{\rightleftarrows}} 2$$

and $I = (\beta_{12}\beta_{21}, \beta_{21}\beta_{12})$ is the two-sided ideal of the path K-algebra KQ of Q generated by two zero relations $\beta_{12}\beta_{21}$ and $\beta_{21}\beta_{12}$ (see [**1**], [**26**, Chapter 14]). The K-algebra isomorphism $A(0) \cong KQ/I$ is given by the formulae $e_1 \mapsto \varepsilon_1$, $e_2 \mapsto \varepsilon_2$, $e_{12} \mapsto \beta_{12}$ and $e_{21} \mapsto \beta_{21}$, where ε_1 and ε_2 are the primitive idempotent of the path algebra KQ defined by the stationary paths at the vertices 1 and 2. Hence easily follows that A is a non-semisimple self-injective Nakayama K-algebra of finite representation type.

We extend [**7**, 1.2(1)–1.3] as follows.

THEOREM 2.9. *Assume that K is a field, $n \geq 2$ is an integer, $q = [q^{(1)}|\cdots|q^{(n)}]$ is a minor constant structure matrix* (2.2) *in $\boldsymbol{ST}_n(K)$, and let $A_q = \boldsymbol{M}_n^q(K)$.*

(a) *A_q is an associative K-algebra such that*

$$e_{is} \cdot_q e_{tj} = \begin{cases} q_{ij}^{(s)} e_{ij}, & \text{for } s = t, \\ 0, & \text{for } s \neq t, \end{cases} \tag{2.10}$$

and $e_i \cdot_q e_{ij} = e_{ij} = e_{ij} \cdot_q e_j$, for all $i, j, s, t \in \{1, \ldots, n\}$, where e_{ij} is the (i,j)-matrix unit.

(b) *The standard matrix idempotents $e_1 = e_{11}, \ldots, e_n = e_{nn}$ of the algebra $\boldsymbol{M}_n(K)$ are pairwise orthogonal primitive idempotents of the algebra A_q. Moreover, there is a right ideal decomposition $A_q = e_1 A_q \oplus \cdots \oplus e_n A_q$, there are K-algebra isomorphisms $\operatorname{End}_{A_q}(e_i A_q) \cong e_i A_q e_i \cong K$, for $i = 1, \ldots, n$, and an isomorphism $\operatorname{Hom}_{A_q}(e_j A_q, e_i A_q) \cong e_i A_q e_j \cong K e_{ij}$ of K-vector spaces, for $i \neq j$. Moreover, there is an isomorphism $e_i A_q \cong e_j A_q$ of right ideals if and only if $q_{jj}^{(i)} = q_{ii}^{(j)} \neq 0$.*

(c) *The algebra A_q is basic if and only if the matrix q is basic.*

(d) *If A_q is basic then*

(i) *A_q is connected, the ideal J of A_q consisting of all matrices $\lambda = [\lambda_{ij}]$ with $\lambda_{11} = \cdots = \lambda_{nn} = 0$ is the Jacobson radical $J(A_q)$ of A_q, and $J(A_q)^n = 0$,*

(ii) *the group $\operatorname{Gl}(A_q)$ of units of A_q consists of all matrices $\lambda = [\lambda_{ij}] \in \boldsymbol{M}_n(K)$ with $\lambda_{11} \cdot \lambda_{22} \cdot \cdots \cdot \lambda_{nn} \neq 0$,*

(iii) *every non-zero two-sided ideal of A_q is generated by a finite subset of the set $\{e_{ij}; i, j = 1, \ldots, n\}$ of the matrix units e_{ij} of A_q, and*

(iv) *the global dimension of the algebra A_q is infinite.*

PROOF.

(a) The definition of the multiplication $\cdot_q$ (2.6) in $A_q = \boldsymbol{M}_n^q(K)$ yields the formula (2.10). Hence, in view of (C1), we get the equalities $e_i \cdot_q e_{ij} = e_{ij} = e_{ij} \cdot_q e_j$. It follows that the matrix of structure constants of $\boldsymbol{M}_n^q(K)$ in the matrix units basis $\{e_{ij}\}_{i,j}$ is obtained from $q = [q^{(1)}| \cdots |q^{(n)}]$ by completing it with zeros at the remaining entries, see [**23**]. Moreover, the multiplication rule (2.6) yields

$$(e_{is} \cdot_q e_{sj}) \cdot_q e_{jt} = q_{ij}^{(s)} e_{ij} \cdot_q e_{jt} = q_{ij}^{(s)} q_{it}^{(t)} e_{it} \quad \text{and}$$
$$e_{is} \cdot_q (e_{sj} \cdot_q e_{jt}) = e_{is} \cdot_q \left(q_{st}^{(j)} e_{st}\right) = q_{it}^{(s)} q_{st}^{(j)} e_{it}.$$

Hence we easily conclude that the multiplication $\cdot_q$ in $A_q = \boldsymbol{M}_n^q(K)$ defined by (2.6) is associative if and only if the condition (C2) is satisfied, see [**23**, Section 1.5]. It follows that $A_q = \boldsymbol{M}_n^q(K)$ is an associative K-algebra, the identity matrix $E = \mathrm{diag}(1, \ldots, 1)$ of $\boldsymbol{M}_n(K)$ is the identity of A_q and the equalities (2.10) hold.

(b) Given a matrix $\lambda = [\lambda_{ij}] \in A_q = \boldsymbol{M}_n^q(K)$ and $p \leq n$, we have $\lambda = \sum_{i,j} \lambda_{ij} e_{ij}$ and, according to (2.10), we get $e_p \cdot_q \lambda \cdot_q e_p = e_p \cdot_q \left(\sum_{i,j} \lambda_{ij} e_{ij}\right) \cdot_q e_p = \lambda_{pp} q_{pp}^{(p)} q_{pp}^{(p)} e_p = \lambda_{pp} e_p$, because $q_{pp}^{(p)} = 1$. It follows that the map $e_p \cdot_q \lambda \cdot_q e_p \mapsto \lambda_{pp}$ defines a K-algebra isomorphism $e_p A_q e_p \cong K$. The K-algebra isomorphism $\mathrm{End}_{A_q}(e_p A_q) \cong e_p A_q e_p$ is given by $f \mapsto f(e_p)$. The vector space isomorphisms $\mathrm{Hom}_{A_q}(e_j A_q, e_i A_q) \cong e_i A_q e_j \cong K e_{ij}$ follow in a similar way.

To prove the remaining part of (b), assume that $q_{jj}^{(i)} \neq 0$, where $i \neq j$. By Lemma 2.4(a), $q_{jj}^{(i)} = q_{ii}^{(j)} \neq 0$. Consider the A_q-module homomorphisms $e_i A_q \underset{e_{ij}\cdot}{\overset{e_{ji}\cdot}{\rightleftarrows}} e_j A_q$ defined as the left hand side multiplication by e_{ji} and by e_{ij}, respectively. Since $e_{ji} \cdot_q e_{ij} \cdot_q e_j = e_j q_{ii}^{(j)}$ and $e_{ij} \cdot_q e_{ji} \cdot_q e_i = e_i q_{jj}^{(i)}$ then the right ideals $e_i A_q$ and $e_j A_q$ of A_q are isomorphic. Conversely, assume that there exists an isomorphism $h : e_i A_q \longrightarrow e_j A_q$, and let $h(e_i) = e_j \cdot_q a$, where $a = \sum_{s,r} \lambda_{sr} e_{sr}$ and $\lambda_{sr} \in K$. Then

$$0 \neq h(e_{ij}) = h(e_i \cdot_q e_{ij}) = h(e_i) \cdot_q e_{ij} = e_j \cdot_q a \cdot_q e_{ij} = \lambda_{ji} e_{ji} \cdot_q e_{ij} = e_j \lambda_{ji} q_{jj}^{(i)}.$$

In view of Lemma 2.4(a), this yields $q_{jj}^{(i)} = q_{ii}^{(j)} \neq 0$.

(c) Assume that A_q is basic and suppose, to the contrary, that q is not basic, that is, $q_{jj}^{(r)} \neq 0$, for some r and $j \neq r$. Then $n \geq 2$ and by Lemma 2.4(b), $q_{jj}^{(i)} = q_{ii}^{(j)} \neq 0$. It follows from (b) that the right ideals $e_i A_q$ and $e_j A_q$ of A_q are isomorphic; contrary to the assumption that A_q is basic.

Conversely, assume that q is basic. By (b), there is a right ideal decomposition $A_q = e_1 A_q \oplus \cdots \oplus e_n A_q$ and the vector space $\mathrm{Hom}_{A_q}(e_j A_q, e_i A_q)$ is non-zero, for all $i, j \in \{1, \ldots, n\}$. It follows that A_q is connected. Moreover, a simple calculation shows that J is a two-sided ideal of A_q such that $J^n = 0$ and $A_q / J \cong K \times \cdots \times K$. Hence we conclude that $J = J(A_q)$ and the algebra A_q is basic.

(d) Assume that q is basic. The statement (i) is proved above. To prove (ii), assume that $\lambda = [\lambda_{ij}] \in \boldsymbol{M}_n^q(K)$. First we show that

$$\lambda \text{ is invertible in } A_q \text{ if and only if } \lambda_{11} \cdot \lambda_{22} \cdot \dots \cdot \lambda_{nn} \neq 0.$$

To prove the sufficiency, assume that $\lambda_{11} \cdot \lambda_{22} \cdot \dots \cdot \lambda_{nn} \neq 0$ and consider the diagonal matrix $d_\lambda := \mathrm{diag}(\lambda_{11}, \lambda_{22}, \dots, \lambda_{nn}) \in \boldsymbol{M}_n^q(K)$ with the coefficients $\lambda_{11}, \lambda_{22}, \dots, \lambda_{nn}$ on the main diagonal. Now we view the matrix $d_\lambda^{-1} \cdot_q \lambda = \mathrm{diag}(\lambda_{11}^{-1}, \lambda_{22}^{-1}, \dots, \lambda_{nn}^{-1}) \cdot_q \lambda$ in the form $d_\lambda^{-1} \cdot_q \lambda = E - \check{\lambda}$, where $\check{\lambda} \in J(A_q)$, see (i). It follows that $\check{\lambda}^n = 0$ and therefore

$$d_\lambda^{-1} \lambda \cdot_q \left(E + \check{\lambda} + \check{\lambda}^2 + \dots + \check{\lambda}^{n-1}\right) = \left(E - \check{\lambda}\right) \cdot_q \left(E + \check{\lambda} + \check{\lambda}^2 + \dots + \check{\lambda}^{n-1}\right) = E.$$

This shows that λ is invertible in A_q and the matrix

$$\lambda^{-1} = d_\lambda \cdot_q \left(E + \check{\lambda} + \check{\lambda}^2 + \dots + \check{\lambda}^{n-1}\right)$$

is the inverse of λ in A_q. Conversely, assume that λ is invertible in A_q and assume, to the contrary, that $\lambda_{11} \cdot \lambda_{22} \cdot \dots \cdot \lambda_{nn} = 0$; say $\lambda_{11} = 0$. It follows from (i) that λ has the form $\lambda = \lambda_{22} e_2 + \dots + \lambda_{nn} e_n + \check{\lambda}$, where $\check{\lambda} \in J(A_q)$ and $\check{\lambda}^n = 0$. If μ is an inverse of λ in A_q then

$$\begin{aligned} E = \lambda \cdot_q \mu &= \left(\lambda_{22} e_2 + \dots + \lambda_{nn} e_n + \check{\lambda}\right) \cdot_q \mu \\ &= \lambda_{22} e_2 \cdot_q \mu + \dots + \lambda_{nn} e_n \cdot_q \mu + \check{\lambda} \cdot_q \mu = c_{22} e_2 + \dots + c_{nn} e_n + \lambda', \end{aligned}$$

where $c_{22}, \dots, c_{nn} \in K$ and $\lambda' \in J(A_q)$. It follows that the coefficient at the $(1,1)$ entry of the matrix $c_{22} e_2 + \dots + c_{nn} e_n + \lambda'$ is zero, and we get a contradiction. This finishes the proof of (ii).

(iii) Assume that $\mathfrak{A}$ is a non-zero two-sided ideal of A_q. If $\lambda = [\lambda_{ij}]$ is a non-zero matrix in $\mathfrak{A}$, with $\lambda_{ij} \in K$, then $\lambda = \sum_{i,j} \lambda_{ij} e_{ij}$. It follows that, given i and j such that $\lambda_{ij} \neq 0$, the element $e_i \cdot_q \lambda \cdot_q e_j = \lambda_{ij} e_{ij}$ belongs to $\mathfrak{A}$ and, consequently, the matrix unit e_{ij} belongs to $\mathfrak{A}$, because $\lambda_{ij} \neq 0$. Hence (iii) follows.

(iv) Since, by (b), $e_i A_q e_j \cong K e_{ij}$, for all $i, j \in \{1, \dots, n\}$, then C_{A_q} has the form

$$C_{A_q} = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}.$$

On the other hand, it is well-known that the determinant of the Cartan matrix of any K-algebra R is 1 or -1, if R is basic of finite global dimension, see [**1**, Chapter I]. Then (iv) follows and the proof of the theorem is complete. □

COROLLARY 2.11. *If K is a field and $q = [q^{(1)} | \cdots | q^{(n)}] \in \boldsymbol{ST}_n(K)$ is a structure matrix. There is a K-algebra isomorphism $\boldsymbol{M}_n^q(K) \cong \boldsymbol{M}_n(K)$ if and only if $q_{22}^{(1)} \neq 0$, $q_{33}^{(1)} \neq 0, \dots, q_{nn}^{(1)} \neq 0$.*

PROOF. Let $A_q = \boldsymbol{M}_n^q(K)$. We recall from Lemma 2.4(b) that $q_{jj}^{(r)} = q_{rr}^{(j)}$, for all $j, r \in \{1, \dots, n\}$. Hence, in view of Theorem 2.9(b), there are isomorphisms $e_1A_q \cong \dots \cong e_nA_q$ of right ideals of A_q if and only if $q_{22}^{(1)} \neq 0$, $q_{33}^{(1)} \neq 0, \dots, q_{nn}^{(1)} \neq 0$. Since $\operatorname{End} e_1A_q \cong K$, the corollary follows. □

DEFINITION 2.12.

(a) Given a matrix $\lambda = [\lambda_{pr}] \in \boldsymbol{M}_n(R)$ and a permutation $\sigma \in S_n$ of the set $\{1, \dots, n\}$, we denote by $\sigma * \lambda = [\lambda_{pr}^{\sigma}]$ the matrix in $\boldsymbol{M}_n(R)$ with $\lambda_{pr}^{\sigma} = \lambda_{\sigma^{-1}(p)\,\sigma^{-1}(r)}$.

(b) Given a structure matrix $q = [q^{(1)}|\cdots|q^{(n)}] \in \boldsymbol{ST}_n(K)$ and $\sigma \in S_n$, we set

$$\sigma * q = \big[\sigma * q^{(\sigma^{-1}(1))}|\cdots|\sigma * q^{(\sigma^{-1}(n))}\big].$$

We also define the transpose of q to be the n-block matrix $q^{tr} = \widetilde{q} = [\widetilde{q}^{(1)}|\cdots|\widetilde{q}^{(n)}]$, where $\widetilde{q}^{(j)} = [q^{(j)}]^{tr}$ is the transpose of $q^{(j)}$, for $j = 1, \dots, n$.

It is clear that the map $(\sigma, q) \mapsto \sigma * q$ defines an action

$$* : S_n \times \boldsymbol{ST}_n(K) \longrightarrow \boldsymbol{ST}_n(K) \tag{2.13}$$

of the symmetric group S_n on the K-variety $\boldsymbol{ST}_n(K)$ of all minor constant structure matrices q (2.2) of size $n \times n^2$. The subsets consisting of all basic matrices and of all basic $(0,1)$-matrices are S_n-invariant.

EXAMPLE 2.14. A simple calculation shows that, in case $n = 3$, every matrix $q = [q^{(1)}|q^{(2)}|q^{(3)}]$ in $\boldsymbol{ST}_3(K)$ has one of the following four forms, up to the S_3-action,

$$q_1 = \begin{bmatrix} 1\,1\,1 & \lambda\,1\,\frac{\lambda}{\mu} & \xi\;\frac{\xi}{\nu}\;1 \\ 1\,\lambda\,\mu & 1\,1\,1 & \frac{\xi}{\mu}\,\frac{\lambda\xi}{\mu\nu}\,1 \\ 1\,\nu\,\xi & \frac{\lambda}{\nu}\,1\,\frac{\lambda\xi}{\mu\nu} & 1\;1\;1 \end{bmatrix}, \quad q_2 = \begin{bmatrix} 1\,1\,1 & \lambda\,1\,\frac{\lambda}{\mu} & 0\,0\,1 \\ 1\,\lambda\,\mu & 1\,1\,1 & 0\,0\,1 \\ 1\,\nu\,0 & \frac{\lambda}{\nu}\,1\,0 & 1\,1\,1 \end{bmatrix},$$

$$q_3 = \begin{bmatrix} 1\,1\,1 & 0\,1\,0 & 0\,0\,1 \\ 1\,0\,\mu & 1\,1\,1 & 0\,0\,1 \\ 1\,\nu\,0 & 0\,1\,0 & 1\,1\,1 \end{bmatrix}, \qquad q_4 = \begin{bmatrix} 1\,1\,1 & 0\,1\,\lambda & 0\,0\,1 \\ 1\,0\,0 & 1\,1\,1 & \tau\,0\,1 \\ 1\,\nu\,0 & 0\,1\,0 & 1\,1\,1 \end{bmatrix},$$

where $\lambda, \mu, \nu, \xi, \tau \in K$; and we assume that $\mu\nu \neq 0$ in the matrices q_1 and q_2. Note that $q_2 = q_1|_{\xi=0}$ and $q_3 = q_2|_{\lambda=0}$. It follows from Corollary 2.11 that, if $\lambda\xi \neq 0$, then the algebra $A_1 = \boldsymbol{M}_3^{q_1}(K)$ is isomorphic to $\boldsymbol{M}_n(K)$, because $(q_1)_{22}^{(1)} = \lambda \neq 0$ and $(q_1)_{33}^{(1)} = \xi \neq 0$. Note also that the algebra $A_2 = \boldsymbol{M}_3^{q_2}(K)$ is Morita equivalent to the algebra $A(0) = \boldsymbol{M}_2^{q(0)}(K)$ of Example 2.8. Indeed, by Theorem 2.9(b), there is an isomorphism $e_1A_2 \cong e_2A_2$, because $(q_2)_{11}^{(2)} = (q_2)_{22}^{(1)} = \lambda \neq 0$. Moreover, the right ideals e_1A_2 and e_3A_2 are not isomorphic, because $(q_2)_{33}^{(1)} = 0$.

The following simple result is very useful.

LEMMA 2.15. *Let $n \geq 2$ and let $q = [q^{(1)}|\cdots|q^{(n)}]$ be a basic structure matrix* (2.2) *in $\boldsymbol{ST}_n(K)$, with coefficients in K. Let $\boldsymbol{M}_n^q(K)$ be the q-degeneration of $\boldsymbol{M}_n(K)$.*

(a) *The transpose $q^{tr} = \widetilde{q} = [\widetilde{q}^{(1)}|\cdots|\widetilde{q}^{(n)}]$ of q is a basic structure matrix in $\boldsymbol{ST}_n(K)$ and the K-linear map $\boldsymbol{M}_n^q(R) \longrightarrow \boldsymbol{M}_n^{q^{tr}}(K)$, defined by $\lambda \mapsto \lambda^{tr}$, is a K-algebra anti-isomorphism, that is, it defines a K-algebra isomorphism $(\boldsymbol{M}_n^q(K))^{op} \cong \boldsymbol{M}_n^{q^{tr}}(K)$.*

(b) *If $\sigma \in S_n$ is a permutation of the set $\{1, \ldots, n\}$ then $\sigma * q$ is a basic structure matrix in $\boldsymbol{ST}_n(K)$ and the map $\lambda \mapsto \sigma * \lambda$ defines an isomorphism $\boldsymbol{M}_n^q(R) \cong \boldsymbol{M}_n^{\sigma * q}(R)$ of R-algebras such that $e_{ij} \mapsto e_{\sigma^{-1}(i)\,\sigma^{-1}(j)}$, for all i and j.*

Proof. The proof is straightforward, and is left to the reader. □

Now we extend the action $* : S_n \times \boldsymbol{ST}_n(K) \longrightarrow \boldsymbol{ST}_n(K)$ of the symmetric group S_n to an action of the following semidirect product algebraic group

$$\boldsymbol{G}_n(K) = \mathscr{T}_n \ltimes S_n \tag{2.16}$$

containing S_n, where $\mathscr{T}_n \ltimes S_n = \mathscr{T}_n \times S_n$ is the Cartesian product,

$$\mathscr{T}_n = \big\{T = [t_{ij}] \in \boldsymbol{M}_n(K); t_{11} = \cdots = t_{nn} = 1 \text{ and } t_{ij} \neq 0, \text{ for all } i, j\big\}$$

is viewed as a group with the coordinate-wise multiplication $[t_{ij}] \cdot [t'_{ij}] = [t_{ij}t'_{ij}]$ and the multiplication in $\boldsymbol{G}_n(K)$ is defined by the formula $(T, \sigma) \cdot (T', \sigma') = (T \cdot (\sigma * T'),\ \sigma\sigma')$, for $T, T' \in \mathscr{T}_n$ and $\sigma, \sigma' \in S_n$.

It is clear that the group $\mathscr{T}_n$ is isomorphic to the $(n^2 - n)$-dimensional K-torus $T_{n^2-n}(K) = K^* \times K^* \times \cdots \times K^*$ (the product of $n^2 - n$ copies of the multiplicative group $K^* = K \setminus \{0\}$ of K).

We define the algebraic group action

$$* : \boldsymbol{G}_n(K) \times \boldsymbol{ST}_n(K) \longrightarrow \boldsymbol{ST}_n(K) \tag{2.17}$$

by the formula $(T, \sigma) * q = [\widehat{q}^{(1)}|\cdots|\widehat{q}^{(n)}]$, where $T = [t_{ij}] \in \mathscr{T}_n$, $\sigma \in S_n$, and $\widehat{q}^{(r)} = [\widehat{q}_{ij}^{(r)}] \in \boldsymbol{M}_n(K)$ is defined by the formula

$$\widehat{q}_{ij}^{(r)} = q_{\sigma^{-1}(i)\,\sigma^{-1}(j)}^{(\sigma^{-1}(r))} \cdot t_{ir}^{-1} t_{ij} t_{rj}^{-1},$$

for $i, j, r \in \{1, \ldots, n\}$.

The following result shows that the $\boldsymbol{G}_n(K)$-orbits of $\boldsymbol{ST}_n(K)$ classify the isomorhism classes of the basic algebras $\boldsymbol{M}_n^q(K)$ of dimension n^2.

Theorem 2.18. *Assume that K is a field and that $n \geq 2$ is an integer.*

(a) *The map (2.17) is an action of the algebraic group $\boldsymbol{G}_n(K)$ (2.16) on the algebraic K-variety $\boldsymbol{ST}_n(K)$ of structure matrices $q = [q^{(1)}|\cdots|q^{(n)}]$ (2.2). The subvariety of $\boldsymbol{ST}_n(K)$ consisting of the basic structure matrices is $\boldsymbol{G}_n(K)$-invariant.*

(b) *Given two basic structure matrices $q = [q^{(1)}|\cdots|q^{(n)}]$ and $q' = [q'^{(1)}|\cdots|q'^{(n)}]$ in*

$\boldsymbol{ST}_n(K)$, *the following statements are equivalent.*

(b1) *The* K*-algebras* $\boldsymbol{M}_n^q(K)$ *and* $\boldsymbol{M}_n^{q'}(K)$ *are isomorphic.*

(b2) *The matrices* q *and* q' *belong to the same* $\boldsymbol{G}_n(K)$*-orbit.*

(b3) *There exist a permutation* $\sigma : \{1,\dots,n\} \longrightarrow \{1,\dots,n\}$ *and a square matrix* $T = [t_{ij}] \in \boldsymbol{M}_n(K)$ *such that*
- $t_{11} = \cdots = t_{nn} = 1$,
- $t_{ij} \neq 0$, *for all* $i,j \in \{1,\dots,n\}$, *and*
- $t_{ir} \cdot {q'}_{ij}^{(r)} \cdot t_{rj} = q_{\sigma(i)\,\sigma(j)}^{(\sigma(r))} \cdot t_{ij}$, *for all* $i,r,j \in \{1,\dots,n\}$.

PROOF.

(a) The proof is straightforward and we leave it to the reader.

(b) A simple calculation shows that q' belongs to the $\boldsymbol{G}_n(K)$-orbit of q if and only if there exist a permutation $\sigma \in S_n$ and a matrix $T = [t_{ij}] \in \boldsymbol{M}_n(K)$ such that the conditions stated in (b3) are satisfied. Consequently, the statements (b2) and (b3) are equivalent.

(b3)⇒(b1) Suppose that $T = [t_{ij}] \in \boldsymbol{M}_n(K)$ and $\sigma \in S_n$, are such that the conditions stated in (b3) are satisfied. Then the map $e_{\sigma(i)\,\sigma(j)} \mapsto t_{ij}e_{ij}$ defines a K-algebra isomorphism $\boldsymbol{M}_n^q(K) \cong \boldsymbol{M}_n^{q'}(K)$.

(b1)⇒(b3) Assume that there is an K-algebra isomorphism $h : \boldsymbol{M}_n^q(K) \longrightarrow \boldsymbol{M}_n^{q'}(K)$. The elements $h(e_1),\dots,h(e_n)$ are primitive orthogonal idempotents of $\boldsymbol{M}_n^{q'}(R)$ such that $1 = h(e_1) + \cdots + h(e_n)$. By [**5**, Theorem 3.4.1], there exist a permutation $\sigma : \{1,\dots,n\} \longrightarrow \{1,\dots,n\}$ and an invertible element $B \in \boldsymbol{M}_n^{q'}(K)$ such that $e_j = B \cdot h(e_{\sigma(j)}) \cdot B^{-1}$, for $j = 1,\dots,n$. Hence we conclude that there exists a K-algebra isomorphism $h' : \boldsymbol{M}_n^q(K) \longrightarrow \boldsymbol{M}_n^{q'}(K)$ such that $e_1 = h'(e_{\sigma(1)}),\dots,e_n = h'(e_{\sigma(n)})$. Since $h'(e_{\sigma(i)\,\sigma(j)}) = h'(e_{\sigma(i)} \cdot e_{\sigma(i)\,\sigma(j)} \cdot e_{\sigma(j)}) = e_i \cdot h'(e_{\sigma(i)\,\sigma(j)}) \cdot e_j$, then there exists a non-zero element $t_{ij} \in K^*$ such that $h'(e_{\sigma(i)\,\sigma(j)}) = t_{ij}e_{ij}$, for $i,j \in \{1,\dots,n\}$. It is clear that $t_{11} = \cdots = t_{nn} = 1$. Moreover, the equality $h'(e_{\sigma(i)\,\sigma(r)} \cdot e_{\sigma(r)\,\sigma(j)}) = h'(e_{\sigma(i)\,\sigma(r)}) \cdot h'(e_{\sigma(r)\,\sigma(j)})$ yields $q_{\sigma(i)\,\sigma(j)}^{(\sigma(r))} t_{ij} = t_{ir} {q'}_{ij}^{(r)} t_{rj}$, for all $i,r,j \in \{1,\dots,n\}$. Consequently, the matrix $T = [t_{ij}] \in \boldsymbol{M}_n(K)$ satisfies the conditions stated in (b3) and (T,σ) is an element of the group $\boldsymbol{G}_n(K)$. This completes the proof. □

As a consequence of Theorem 2.18 we get the following isomorphism criterion.

COROLLARY 2.19. *Let* K *be a field,* $n \geq 2$, *and let* $q = [q^{(1)}|\cdots|q^{(n)}]$, $q' = [{q'}^{(1)}|\cdots|{q'}^{(n)}]$ *be basic structure* $(0,1)$*-matrices* (2.2) *in* $\boldsymbol{ST}_n(K)$. *The* K*-algebras* $\boldsymbol{M}_n^q(K)$ *and* $\boldsymbol{M}_n^{q'}(K)$ *are isomorphic if and only if* q *and* q' *are in the same* S_n*-orbit, that is, there exists a permutation* $\sigma : \{1,\dots,n\} \longrightarrow \{1,\dots,n\}$ *such that* $q_{\sigma(i)\,\sigma(j)}^{(\sigma(r))} = {q'}_{ij}^{(r)}$, *for all* $i,r,j \in \{1,\dots,n\}$.

PROOF. In this case the matrix $T = [t_{ij}] \in \mathscr{T}_n(K)$ required in Theorem 2.18(b) has $t_{ij} = 1$, for all i and j. □

Following P. Gabriel [**10**], we associate to any basic and connected finite dimensional K-algebra A, with a complete set of primitive orthogonal idempotents $\{e_1, e_2, \dots, e_n\}$, the Gabriel quiver $\mathscr{Q}(A) = (\mathscr{Q}(A)_0, \mathscr{Q}(A)_1)$ as follows. The set $\mathscr{Q}(A)_0 = \{1, 2, \dots, n\}$

is the set of points of $\mathscr{Q}(A)$, its elements are in bijective correspondence with the idempotents $e_1, e_2, \dots, e_n$. Given two points $i, j \in \mathscr{Q}(A)_0$, the arrows $\beta : i \to j$ in $\mathscr{Q}(A)_1$ are in bijective correspondence with the vectors in a basis of the K-vector space $e_i[J(A)/J(A)^2]e_j$, see [**1**, Chapter II]. For a completeness of the presentation we include here a proof of the following result presented in [**7**, 1.2(2)–(3)].

Corollary 2.20. *Let $n \geq 2$ and let $q = [q^{(1)}|\cdots|q^{(n)}] \in \boldsymbol{ST}_n(K)$ be a basic minor constant structure matrix* (2.2). *Let $A_q = \boldsymbol{M}_n^q(K)$ be the q-degeneration K-algebra of $\boldsymbol{M}_n(K)$ and let $\mathscr{Q}(A_q) = (\mathscr{Q}(A_q)_0, \mathscr{Q}(A_q)_1)$ be the Gabriel quiver of A.*

(a) $\mathscr{Q}(A_q)_0 = \{1, \dots, n\}$
(b) *Given $i, j \in \mathscr{Q}(A_q)_0$, there exists an arrow $i \to j$ in $\mathscr{Q}(A_q)_1$ if and only if $i \neq j$ and $q_{ij}^{(r)} = 0$, for all $r \notin \{i, j\}$. In this case, there is a unique arrow $\beta_{ij} : i \longrightarrow j$ that corresponds to the coset $\overline{e}_{ij} \in e_i[J(A_q)/J(A_q)^2]e_j$ of the matrix unit e_{ij}.*
(c) *The quiver $\mathscr{Q}(A_q)$ is connected and has no loops.*

Proof.

(a) It follows from Theorem 2.9 that the algebra $A_q = \boldsymbol{M}_n^q(K)$ is basic and $A_q/J(A_q) \cong K\overline{e}_1 \oplus \cdots \oplus K\overline{e}_n$. The points of the quiver $\mathscr{Q}(A_q)$ correspond to the primitive idempotents $e_1, \dots, e_n$ of A, and (a) follows.
(b) It follows from Theorem 2.9 that, given two primitive idempotents e_i and e_j, we have $\mathrm{Hom}_{A_q}(e_jA_q, e_iA_q) \cong K$, if $i = j$, and $\mathrm{Hom}_{A_q}(e_jA_q, e_iA_q) \cong e_iA_qe_j \cong e_{ij}K$, if $i \neq j$. Hence we get $e_iJ(A_q)e_i = 0$, that is, the quiver $\mathscr{Q}(A_q)$ has no loops. If $i \neq j$, we get $e_iJ(A_q)e_j \cong e_{ij}K$ and therefore $e_iJ(A_q)^2e_j = e_iJ(A)e_j$ if and only if there is an $s \in \{1, \dots, n\} \setminus \{i, j\}$ such that $e_{ij} = \mu e_{is}e_{sj}$, for some non-zero $\mu \in K$. Since $e_{is}e_{sj} = q_{ij}^{(s)}e_{ij}$, then $0 \neq \overline{e}_{ij} \in e_i[J(A_q)/J(A_q)^2]e_j$ if and only if $q_{ij}^{(s)} = 0$, for all $s \notin \{i, j\}$. Hence (b) follows.
(c) By Theorem 2.9(e), the algebra A_q is connected. Hence we conclude that the quiver $Q(A_q)$ is connected (see [**1**, Corollary II.3.4]). Since, by (C3), $q_{jj}^{(r)} = 0$, for $r = 1, \dots, n$ and all $j \in \{1, \dots, n\}$ such that $j \neq r$ then, according to (b), the quiver $Q(A_q)$ has no loops. This finishes the proof. □

Now assume that $A = \boldsymbol{M}_n^q(K)$ is a minor degeneration of the algebra $\boldsymbol{M}_n(K)$, where $q = [q^{(1)}|\cdots|q^{(n)}]$. Let I be a non-empty subset of $\{1, \dots, n\}$. Assume that $s = |I|$ is the cardinality of I and $I = \{i_1, \dots, i_s\}$. Define q_I to be the s-block matrix

$$q_I = \left[q_I^{(i_1)}|\cdots|q_I^{(i_s)}\right] \tag{2.21}$$

obtained from q by the restriction to I, that is, each matrix $q_I^{(i_s)} \in \boldsymbol{M}_s(K)$ is obtained from $q^{(i_s)} \in \boldsymbol{M}_n(K)$ by deleting the j-th row and the j-th column, for all $j \notin I$. It is clear that q_I is a structure matrix of size $s \times s^2$. We set

$$A_I = \boldsymbol{M}_s^{q_I}(K).$$

Let $e_I = \sum_{j \in I} e_j = e_{i_1} + \cdots + e_{i_s}$, where are the standard primitive idempotents of A. Then e_I is an idempotent of $A = \boldsymbol{M}_n^q(K)$ and there is a K-algebra isomorphism

$$e_I A e_I = e_I \boldsymbol{M}_n^q(K) e_I \cong \boldsymbol{M}_s^{q_I}(K) = A_I \tag{2.22}$$

given by associating to any matrix $e_I \lambda e_I \in e_I \boldsymbol{M}_n^q(K) e_I$ the restriction of $\lambda = [\lambda_{ij}] \in \boldsymbol{M}_n^q(K)$ to $I = \{i_1, \dots, i_s\}$.

Now we define three additive K-linear covariant functors

$$\operatorname{mod} \boldsymbol{M}_s^{q_I}(K) \underset{\operatorname{res}_I}{\overset{T_I,\, L_I}{\rightleftarrows}} \operatorname{mod} \boldsymbol{M}_n^q(K) \tag{2.23}$$

by the formulae $\operatorname{res}_I(-) = (-)e_I$, $T_I(-) = - \otimes_{e_I A e_I} e_I A$, $L_I(-) = \operatorname{Hom}_{e_I A e_I}(Ae_I, -)$, where $A = \boldsymbol{M}_n^q(K)$. If $f : X \longrightarrow X'$ is a homomorphism of A-modules, we define a homomorphism of $\boldsymbol{M}_s^{q_I}(K)$-modules $\operatorname{res}_I(f) : \operatorname{res}_I(X) \longrightarrow \operatorname{res}_I(X')$ by the formula $xe_I \mapsto f(x)e_I$, that is, $\operatorname{res}_I(f)$ is the restriction of f to the subspace Xe_I of X, see [**1**, Section I.6] and [**26**, Section 17.5].

The following result is very useful in applications.

Theorem 2.24. *Suppose that $A = \boldsymbol{M}_n^q(K)$ and $A_I = \boldsymbol{M}_I^{q_I}(K)$ are as above. Then there is a K-algebra isomorphism $A_I \cong e_I A e_I$ described above and the functors T_I, L_I* (2.23) *associated to I satisfy the following conditions.*

(a) *T_I and L_I are full and faithful K-linear functors such that $\operatorname{res}_I \circ T_I \cong \mathrm{id} \cong \operatorname{res}_I \circ L_I$, the functor L_I is right adjoint to res_I and T_I is left adjoint to res_I.*
(b) *The restriction functor res_I is exact, T_I is right exact and L_I is left exact.*
(c) *The functors T_I and L_I preserve indecomposability, T_I carries projectives to projectives and L_I carries injectives to injectives.*
(d) *An A-module X is in the category $\operatorname{Im} T_I$ if and only if there is an exact sequence $P_1 \xrightarrow{h} P_0 \longrightarrow X \longrightarrow 0$, where P_1 and P_0 are direct sums of summands of the A-module $e_I A = e_{i_1} A \oplus \cdots \oplus e_{i_s} A$.*

Proof. Apply [**1**, Theorem I.6.8] and [**26**, Section 17.5], and the arguments used there. The details are left to the reader. □

Corollary 2.25. *Suppose that $A = \boldsymbol{M}_n^q(K)$ and $A_I = \boldsymbol{M}_I^{q_I}(K)$ are as above.*

(a) *If A is representation-finite, then A_I is also representation-finite.*
(b) *If $\overline{K} = K$ and A is representation-tame, then A_I is also representation-tame* [**26**, Section 14.4], [**31**, Chapter XIX].
(c) *If $\overline{K} = K$ and A_I is representation-wild, then A is representation-wild* [**26**, Section 14.2], [**31**, Chapter XIX].

Proof.

(a) Assume that A is representation-finite and consider the fully faithful functor $T_I : \operatorname{mod} A_I \longrightarrow \operatorname{mod} A$, see (2.23) and Theorem 2.24. Since T_I carries indecomposable A_I-modules to indecomposable A-modules, and nonisomorphic A_I-modules to nonisomorphic A_I-modules, then (a) follows.
(b) Assume that the field K is algebraically closed and A is representation-tame. Fix a dimension $d \in \boldsymbol{N}$ and consider the functors T_I and res_I presented in (2.23).

First we show that, given a module X in $\operatorname{mod} A_I$ with $\dim_K X = d$, the K-dimension of the A-module $T_I(X)$ is not greater than $\overline{d} = d \cdot p_I$, where $p_I = \max\{\dim_K e_i A; i \in I\}$. To see this we note that the A_I-projective cover of X has the form $\bigoplus_{i\in I}(e_i A_I)^{d_i} \longrightarrow X \longrightarrow 0$, where $d_i = \dim_K(\operatorname{top} X)e_i \le d$. By Theorem 2.24, the functor T_I is right exact and there is an A-module isomorphism $T_I(e_i A_I) \cong e_i A$, for all $i \in I$. It follows that T_I induces an epimorphism $\bigoplus_{i\in I}(e_i A)^{d_i} \longrightarrow T_I(X) \longrightarrow 0$ of right A-modules. Hence we get the inequalies

$$\dim_K T_I(X) \le \dim_K \bigoplus_{i\in I}(e_i A)^{d_i} \le \sum_{i\in I}(d_i \cdot \dim_K e_i A) \le \left(\sum_{i\in I} d_i\right) \cdot p_I \le d \cdot p_I = \overline{d},$$

and our claim follows.

Since the algebra A is representation-tame then, given the K-dimension $\overline{d} = d \cdot p_I$, there exist a non-zero polynomial $h \in K[t]$ and a family of K-linear functors

$$(-) \otimes_S N^{(1)}, \ldots, (-) \otimes_S N^{(r)} : \operatorname{ind}(\operatorname{mod} S) \longrightarrow \operatorname{mod} A$$

where $S = K[t, h^{-1}]$ and $N^{(1)}, \ldots, N^{(r)}$ are S-A-bimodules satisfying the following two conditions:

(T0) The left S-modules ${}_S N^{(1)}, \ldots, {}_S N^{(r)}$ are finitely generated and free.

(T1) All but finitely many indecomposable modules in $\operatorname{mod} A$ of K-dimension $\le \overline{d}$ are isomorphic to modules in $\operatorname{Im}(-) \otimes_S N^{(1)} \cup \cdots \cup \operatorname{Im}(-) \otimes_S N^{(r)}$, see [**26**, Section 14.4] and [**31**, Chapter XIX].

Here $\operatorname{ind}(\operatorname{mod} S)$ is the category of indecomposable S-modules of finite dimension. Consider the restricted S-A_I-bimodules $\operatorname{res}_I N^{(1)} = N^{(1)} e_I, \ldots, \operatorname{res}_I N^{(r)} = N^{(r)} e_I$. It is clear that the S-module $\operatorname{res}_I N^{(j)}$ is finitely generated and free, for each j, because the functor res_I is exact. Now, if X is an indecomposable module in $\operatorname{mod} A_I$ with $\dim_K X = d$ then, according to Theorem 2.24 and our claim above, the A-module $T_I(X)$ is indecomposable and $\dim_K T_I(X) \le \overline{d}$. It follows that there exists an S-module N in $\operatorname{ind}(\operatorname{mod} S)$ such that $T_I(X) \cong N \otimes_S N^{(j)} = N \otimes_S N^{(j)} e_I$, for some $j \le r$. In view of Theorem 2.24(a), we get A_I-module isomorphisms

$$\begin{aligned} X &\cong \operatorname{res}_I(T_I(X)) \cong \operatorname{res}_I \left(N \otimes_S N^{(j)}\right) \\ &= (N \otimes_S N^{(j)}) e_I \cong N \otimes_S (N^{(j)}) e_I \cong N \otimes_S \operatorname{res}_I N^{(j)}. \end{aligned}$$

This shows that the algebra A_I is representation-tame.

(c) Assume that the field K is algebraically closed and that the algebra A_I is representation-wild. By the tame-wild dichotomy [**4**], [**26**, Theorem 14.14], [**31**, Chapter XIX], the algebra A_I is not representation-tame. It follows from (b), that the algebra A is not representation-tame. Hence, A is representation-wild, by the tame-wild dichotomy. □

Corollary 2.26. *Assume that K is a field, $q = [q^{(1)}|\cdots|q^{(n)}]$ is a minor constant structure matrix in $\boldsymbol{ST}_n(K)$ and let $I = \{i_1, \ldots, i_s\}$ be a maximal subset of $\{1, \ldots, n\}$*

such that $q_{jj}^{(r)} = 0$, whenever $j, r \in I$ and $j \neq r$. Then the minor constant matrix q_I in $\boldsymbol{ST}_s(K)$ is basic, the K-algebra $\boldsymbol{M}_s^{q_I}(K)$ is basic and is Morita equivalent to the algebra $\boldsymbol{M}_n^q(K)$.

Proof. Let $A = \boldsymbol{M}_n^q(K)$ and suppose that $I = \{i_1, \ldots, i_s\}$ satisfies the maximality conditions. It follows that the constant matrix q_I is basic and, in view of Theorem 2.9, the K-algebra A is basic and $e_j A \not\cong e_r A$, for all $j, r \in I$ such that $j \neq r$. By the maximality of I, given $r \notin I$ there exists $j \in I$ such that $q_{jj}^{(r)} \neq 0$. Since $q_{jj}^{(r)} = q_{rr}^{(j)}$, by Lemma 2.4, then $e_r A \cong e_j A$, see Theorem 2.9. Consequently, for each $r \in \{1, \ldots, n\}$ there is $i_r \in I$ such that $e_r A \cong e_{i_r} A$ and the modules $e_{i_1} A, \ldots, e_{i_r} A$ are pairwise non-isomorphic. In view of Theorem 2.24(d), it follows that the functor $T_I : \operatorname{Mod} \boldsymbol{M}_s^{q_I}(K) \longrightarrow \operatorname{Mod} \boldsymbol{M}_n^q(K)$ is dense. Since, according to Theorem 2.24(a), the functor T_I is fully faithful then it is an equivalence of categories. This shows that the K-algebras $\boldsymbol{M}_s^{q_I}(K)$ and $\boldsymbol{M}_n^q(K)$ are Morita equivalent. $\square$

3. (0,1)-limits and Nakayama algebras.

Throughout this paper the following definition is of importance.

Definition 3.1. Let $A_q = \boldsymbol{M}_n^q(K)$ be a minor degeneration algebra of $\boldsymbol{M}_n(K)$ with a structure matrix $q = [q^{(1)}| \cdots |q^{(n)}]$, where $q^{(s)} = [q_{ij}^{(s)}]$.

(a) We define a $(0,1)$-limit of q to be the structure $(0,1)$-matrix $\overline{q} = [\overline{q}^{(1)}| \cdots |\overline{q}^{(n)}]$, where the matrix $\overline{q}^{(s)} = [\overline{q}_{ij}^{(s)}]$ is defined by the formulae

$$\overline{q}_{ij}^{(s)} = \begin{cases} 1, & \text{if } q_{ij}^{(s)} \neq 0, \\ 0, & \text{if } q_{ij}^{(s)} = 0. \end{cases}$$

(b) The algebra $\overline{A}_q = A_{\overline{q}} = \boldsymbol{M}_n^{\overline{q}}(K)$ is called the $(0,1)$-limit of $A_q = \boldsymbol{M}_n^q(K)$.

We recall that a finite dimensional K-algebra A is a Frobenius algebra if there exists a K-linear map $\psi : A \longrightarrow K$ such that $\operatorname{Ker} \psi$ does not contain non-zero right (or left) ideals of A, see [**35**]. It is clear that a basic K-algebra A is Frobenius if and only if A is self-injective, see [**33**].

Proposition 3.2. *Assume that K is a field, $A_q = \boldsymbol{M}_n^q(K)$ is a basic minor degeneration of $\boldsymbol{M}_n(K)$ and $\overline{A}_q = \boldsymbol{M}_n^{\overline{q}}(K)$ is the $(0,1)$-limit of A_q.*

(a) *A vector K-subspace $\mathfrak{A}$ of $\boldsymbol{M}_n(K)$ is a two-sided ideal of A_q if and only if $\mathfrak{A}$ is a two-sided ideal of $\overline{A}_q$. In particular, $J(A_q)^s = J(\overline{A}_q)^s$, for each $s \geq 1$.*
(b) *The Gabriel quivers of A_q and $\overline{A}_q$ coincide.*
(c) *Assume that the field K is algebraically closed and $\{A_{q_\mu}\}_{\mu \in K}$ is a 1-parameter algebraic family [**18**] of minor degenerations $A_{q_\mu} = \boldsymbol{M}_n^{q_\mu}(K)$ of $\boldsymbol{M}_n(K)$ such that $A_{q_0} = \overline{A}_q$ and almost all algebras A_{q_μ} are isomorphic. If the algebra $\overline{A}_q$ is representation-finite (resp. representation-tame) then A_{q_μ} is representation-finite (resp. representation-tame), for almost all structure matrices q_μ.*

Proof.

(a) Let $\mathfrak{A}$ be a non-zero vector K-subspace of $\boldsymbol{M}_n(K)$. Suppose that $\mathfrak{A}$ is a two-sided ideal of A_q. It follows from Theorem 2.9(e) that $\mathfrak{A}$ is generated by a finite set of the matrix units e_{ij} of A_q. We show that $\mathfrak{A}$ is a two-sided ideal of $\overline{A}_q$. Denote by $\cdot'$ and $\cdot''$ the multiplication in A_q and in $\overline{A}_q$, respectively.

Since the matrix units form a K-basis of $\overline{A}_q$, it is sufficient to show that $e_{st} \cdot'' e_{ij} \in \mathfrak{A}$ and $e_{ij} \cdot'' e_{rp} \in \mathfrak{A}$, for any $e_{ij} \in \mathfrak{A}$ and any $e_{st}, e_{rp} \in \overline{A}_q$. Recall that $e_{st} \cdot'' e_{ij} = 0$, for $t \neq i$, and $e_{ij} \cdot'' e_{rp} = 0$, for $j \neq r$. Therefore, we can assume that $t = i$ and $j = r$. In this case, we get

$$e_{si} \cdot'' e_{ij} = \overline{q}_{sj}^{(i)} e_{sj} = \begin{cases} e_{sj}; & \text{if } \overline{q}_{sj}^{(i)} \neq 0, \\ 0; & \text{if } \overline{q}_{sj}^{(i)} = 0. \end{cases}$$

Assume that $\overline{q}_{sj}^{(i)} \neq 0$, that is, $\overline{q}_{sj}^{(i)} = 1$. Then $q_{sj}^{(i)} \neq 0$ and the element $e_{si} \cdot' e_{ij} = q_{sj}^{(i)} e_{sj}$ belongs to $\mathfrak{A}$, because $\mathfrak{A}$ is a two-sided ideal of A_q. It follows that $e_{sj} = e_{si} \cdot'' e_{ij} \in \mathfrak{A}$. Similarly, we show that $e_{ij} \cdot'' e_{jp} \in \mathfrak{A}$. Consequently, $\mathfrak{A}$ is a two-sided ideal of $\overline{A}_q$. The same type of arguments shows that $\mathfrak{A}$ is a two-sided ideal of A_q, if $\mathfrak{A}$ is a two-sided ideal of $\overline{A}_q$. This finishes the proof of the first statement in (a). The second one follows from the first one by applying it to $\mathfrak{A} = J(A_q)^s$.

(b) Since $J(A_q) = J(\overline{A}_q)$ and $J(A_q)^2 = J(\overline{A}_q)^2$, then

$$e_i\big[J(A_q)/J(A_q)^2\big]e_j = e_i\big[J(\overline{A}_q)/J(\overline{A}_q)^2\big]e_j,$$

for all i, j, and hence $\mathscr{Q}(A_q) = \mathscr{Q}(\overline{A}_q)$.

(c) Since, according to [**11**], the algebras of finite representation type define an open subset in $\mathscr{A}lg(K^{n^2})$, then almost all algebras A_{q_μ} are of finite representation type if so is $\overline{A}_q = A_{q_0}$, see also [**18**, Chapter III]. Further, according to Geiss [**12**], the tameness of $\overline{A}_q = A_{q_0}$ implies the tameness of A_{q_μ}, for almost all structure matrices q_μ. Hence (c) follows and the proof is complete. $\square$

We recall that a finite dimensional K-algebra A is said to be a Nakayama algebra, if for every primitive idempotent $e \in A$, the left ideal Ae has a unique composition series and the right ideal eA has a unique composition series.

Now we describe the minor degenerations of $\boldsymbol{M}_n(K)$ that are Nakayama algebras.

Theorem 3.3. *Assume that $n \geq 2$ and $q = [q^{(1)}|\cdots|q^{(n)}]$ is a basic structure matrix* (2.2) *of size $n \times n^2$. Let $\overline{q} = [\overline{q}^{(1)}|\cdots|\overline{q}^{(n)}]$ be the $(0,1)$-limit of q, let $A_q = \boldsymbol{M}_n^q(K)$ and $\overline{A}_q = \boldsymbol{M}_n^{\overline{q}}(K)$. The following four conditions are equivalent.*

(a) *A_q is a self-injective Nakayama K-algebra.*

(b) *A_q is a Nakayama K-algebra.*

(c) *There exist a K-algebra isomorphism $A_q \cong \overline{A}_q$ and a permutation $\sigma : \{1,\dots,n\} \longrightarrow \{1,\dots,n\}$ such that the matrix $\sigma * \overline{q} = [\widehat{\overline{q}}^{(1)}|\cdots|\widehat{\overline{q}}^{(n)}]$ has the form*

$$\sigma * \bar{q} = \left[\begin{array}{ccccccc|c|c|c|ccccccc}
1 & 1 & 1 & \cdots & 1 & 1 & 1 & & & & 0 & 0 & 0 & \cdots & 0 & 0 & 1\\
1 & 0 & 0 & \cdots & 0 & 0 & 0 & & & & 1 & 0 & 0 & \cdots & 0 & 0 & 1\\
1 & 1 & 0 & \cdots & 0 & 0 & 0 & & & & 1 & 1 & 0 & \cdots & 0 & 0 & 1\\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \cdots & \widehat{\overline{q}}^{(r)} & \cdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots\\
1 & 1 & 1 & \ddots & 0 & 0 & 0 & & & & 1 & 1 & 1 & \ddots & 0 & 0 & 1\\
1 & 1 & 1 & \cdots & 1 & 0 & 0 & & & & 1 & 1 & 1 & \cdots & 1 & 0 & 1\\
1 & 1 & 1 & \cdots & 1 & 1 & 0 & & & & 1 & 1 & 1 & \cdots & 1 & 1 & 1
\end{array}\right]$$

and, for each $r \in \{1, \ldots, n\}$, $\widehat{\overline{q}}^{(r)}$ *is the matrix*

$$\begin{array}{cc}
 & \begin{array}{ccccccccccc} & & & & & r & & & & & \\ & & & & & \downarrow & & & & & \end{array} \\
\widehat{\overline{q}}^{(r)} = & \left[\begin{array}{ccccccccccc}
0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 & 1\\
1 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 & 1\\
\vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
1 & 1 & \ddots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 & 1\\
1 & 1 & \cdots & 1 & 0 & 1 & 1 & \cdots & 1 & 1 & 1\\
1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1\\
 & & & & & 1 & 0 & \cdots & 0 & 0 & 0\\
 & & & & & 1 & 1 & \ddots & 0 & 0 & 0\\
 & & 0 & & & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots\\
 & & & & & 1 & 1 & \cdots & 1 & 0 & 0\\
 & & & & & 1 & 1 & \cdots & 1 & 1 & 0
\end{array}\right]
\begin{array}{l} \\ \\ \\ \\ \\ \leftarrow r \\ \\ \\ \\ \\ \\ \end{array}
\end{array}.$$

(d) *There exist* K*-algebra isomorphisms* $A_q \cong \overline{A}_q \cong KQ/I$*, where* KQ *is the path* K*-algebra of the quiver*

$$Q: \begin{tikzcd}
1 \arrow[r, "\beta_1"] & 2 \arrow[r, "\beta_2"] & \cdots \arrow[r, "\beta_{n-4}"] & n-3 \arrow[d, "\beta_{n-3}"] \\
n \arrow[u, "\beta_n"] & & n-1 \arrow[ll, "\beta_{n-1}"'] & n-2 \arrow[l, "\beta_{n-2}"']
\end{tikzcd}$$

and $I = (\omega_1, \ldots, \omega_n)$ *is the two-sided ideal of* KQ *generated by* n *zero relations* $\omega_1, \ldots, \omega_n$*, where* $\omega_j = \beta_j \beta_{j+1} \ldots \beta_n \beta_1 \ldots \beta_{j-1}$*, for* $j = 1, \ldots, n$ (see [**1**], [**26**, Chapter 14]).

If any of the conditions (a)–(d) *holds then* $\mathrm{soc}(A_q) = J(A_q)^{n-1}$ *and* $A_q = \boldsymbol{M}_n^q(K)$ *is of finite representation type.*

PROOF. The implication (a)⇒(b) is obvious.

(b)⇒(c) Assume that $A_q = \boldsymbol{M}_n^q(K)$ is a Nakayama algebra. Since A_q is connected, then the Gabriel quiver $Q(A_q)$ of A_q is either an oriented cycle or $Q(A_q)$ is of the form

$$j_1 \longrightarrow j_2 \longrightarrow \cdots \longrightarrow j_n,$$

and has no oriented cycle, see [**2**] and [**1**, Chapter 5]. Since, according to Theorem 2.9(b), there is a non-zero A_q-module homomorphism $e_iA_q \longrightarrow e_jA_q$, for all $i,j \in \{1,\dots,n\}$, then the second form $(*)$ of $Q(A_q)$ is excluded. Consequently, there is a permutation σ of the set $\{1,\dots,n\}$ such that the Gabriel quiver of the algebra $\boldsymbol{M}_n^{\sigma*q}(K)$ is the cycle Q presented in (d). By Corollary 2.20, this implies that $(\sigma * q^{(\sigma^{-1}(r))})_{j\,j+1} = 0$, for all $r = 1,\dots,n$ and $j \neq r$.

It follows from [**1**, Proposition IV.3.8] that $A_q \cong \boldsymbol{M}_n^{\sigma*q}(K) \cong KQ/R_Q^s$, for some $s \geq 2$, where $R_Q = (\beta_1,\dots,\beta_n)$ is the two-sided ideal of the path K-algebra KQ of Q generated by the arrows $\beta_1,\dots,\beta_n$. Since $\dim_K A_q = n^2$, it follows that $s = n$. Similarly, there is a K-algebra isomorphism $\overline{A}_q \cong \boldsymbol{M}_n^{\sigma*\overline{q}}(K) \cong KQ/R_Q^n$. Hence we easily conclude that the matrix $\sigma * \overline{q}$ has the form required in (c).

(c)$\Rightarrow$(d) Assume that $A_q \cong \overline{A}_q$ and σ is a permutation of the set $\{1,\dots,n\}$ such that the matrix $\sigma * \overline{q} = [\widehat{\overline{q}}^{(1)}|\cdots|\widehat{\overline{q}}^{(n)}]$ has the form shown in (c).

By Lemma 2.16, there is a K-algebra isomorphism $\boldsymbol{M}_n^{\overline{q}}(K) \cong \boldsymbol{M}_n^{\sigma*\overline{q}}(K)$. On the other hand, by Corollary 2.20, the Gabriel quiver of the algebra $\boldsymbol{M}_n^{\sigma*\overline{q}}(K)$ is the quiver Q shown in (c). Now we define a K-linear map

$$\varphi : \boldsymbol{M}_n^{\sigma*\overline{q}}(K) \longrightarrow KQ/I$$

as follows. First we note that, by the form of $\sigma * \overline{q}$, each matrix units e_{ij} of $\boldsymbol{M}_n^{\sigma*\overline{q}}(K)$ is the composition of some of the matrix units $e_{1\,2},\dots,e_{n-1\,n},e_{n\,1}$. Consider the correspondences $e_j \mapsto \eta_j$, $e_{n\,1} \mapsto \beta_n$ and $e_{j\,j+1} \mapsto \beta_j$, for $j = 1,\dots,n-1$, where η_j is the stationary path at j. It is easy to see that the correspondences extend to the K-algebra homomorphism $\varphi : \boldsymbol{M}_n^{\sigma*\overline{q}}(K) \longrightarrow KQ/I$. Since $I = (\omega_1,\dots,\omega_n)$, then φ is surjective and $\dim_K KQ/I = \dim_K \boldsymbol{M}_n^{\sigma*\overline{q}}(K) = n^2$. It then follows that φ is bijective.

The implication (d)$\Rightarrow$(a) and the final statement of the corollary are well-known facts and can be found in [**1**, Chapter 5]. This finishes the proof. □

4. Basic minor degenerations of small dimensions.

In this section we study in details basic minor degenerations $A_q = \boldsymbol{M}_n^q(K)$ of $\boldsymbol{M}_n(K)$ for $n = 3$, and some examples of such algebras for $n = 4$, and $n = 6$, by means of their bound quiver presentations of the form $A_q \cong KQ/\Omega$, where Q is the Gabriel quiver of A_q and Ω is an admissible ideal of the path K-algebra KQ of Q. We recall that, up to S_3-action, the constant structure matrices $q = [q^{(1)}|q^{(2)}|q^{(3)}]$ in $\boldsymbol{ST}_3(K)$ are described in Example 2.14.

Theorem 4.1. *Assume that $n = 3$ and let $A_q = \boldsymbol{M}_3^q(K)$ be a basic minor degeneration of $\boldsymbol{M}_3(K)$.*

(a) *The K-algebra $A_q = \boldsymbol{M}_3^q(K)$ is isomorphic to its $(0,1)$-limit $\overline{A}_q = \boldsymbol{M}_3^{\overline{q}}(K)$.*
(b) *Any basic minor degeneration $A_q = \boldsymbol{M}_3^q(K)$ of $\boldsymbol{M}_3(K)$ is isomorphic to one of the five basic minor degeneration K-algebras*

$$A_{q_1} = \boldsymbol{M}_3^{q_1}(K), \quad A_{q_2} = \boldsymbol{M}_3^{q_2}(K), \quad A_{q_3} = \boldsymbol{M}_3^{q_3}(K),$$
$$A_{q_4} = \boldsymbol{M}_3^{q_4}(K), \quad A_{q_5} = \boldsymbol{M}_3^{q_5}(K)$$

defined by the following structure $(0,1)$*-matrices in* $\boldsymbol{ST}_3(K)$

$$q_1 = \begin{bmatrix} 111 & 010 & 001 \\ 100 & 111 & 001 \\ 100 & 010 & 111 \end{bmatrix}, \quad q_2 = \begin{bmatrix} 111 & 010 & 001 \\ 101 & 111 & 001 \\ 100 & 010 & 111 \end{bmatrix}, \quad q_3 = \begin{bmatrix} 111 & 011 & 001 \\ 100 & 111 & 001 \\ 110 & 010 & 111 \end{bmatrix},$$

$$q_4 = \begin{bmatrix} 111 & 010 & 001 \\ 101 & 111 & 001 \\ 110 & 010 & 111 \end{bmatrix}, \quad q_5 = \begin{bmatrix} 111 & 010 & 011 \\ 101 & 111 & 001 \\ 100 & 110 & 111 \end{bmatrix}.$$

(c) *The algebras* A_{q_1}, A_{q_2}, A_{q_3}, A_{q_4} *and* A_{q_5} *are pairwise non-isomorphic, self-dual, and special biserial. The algebra* A_{q_5} *is self-injective, but the algebras* A_{q_1}, A_{q_2}, A_{q_3}, A_{q_4} *are not. The algebra* A_{q_1} *is tame of infinite representation type, and the algebras* A_{q_2}, A_{q_3}, A_{q_4}, A_{q_5} *are of finite representation type, see* [**32**], *compare with* [**27**]. *There exist* K*-algebra isomorphisms*

(c1) $A_{q_1} \cong KQ^{(1)}/\Omega^{(1)}$, *where* Q^1 :
$$\begin{tikzcd} 1 \arrow[r, shift left, "\beta_{12}"] \arrow[rr, bend left, "\beta_{13}"] & 2 \arrow[l, shift left, "\beta_{21}"] \arrow[r, shift left, "\beta_{23}"] & 3 \arrow[l, shift left, "\beta_{32}"] \arrow[ll, bend left, "\beta_{31}"] \end{tikzcd}$$

and the ideal $\Omega^{(1)}$ *of the path algebra* $KQ^{(1)}$ *is generated by all zero relations* $\beta\gamma$, *with* $\beta, \gamma \in Q_1^{(1)}$.

(c2) $A_{q_2} \cong KQ^{(2)}/\Omega^{(2)}$, *where* Q^2 :
$$\begin{tikzcd} 3 \arrow[r, shift left, "\beta_{31}"] \arrow[rr, bend left, "\beta_{32}"] & 1 \arrow[l, shift left, "\beta_{13}"] \arrow[r, shift left, "\beta_{12}"] & 2 \arrow[l, shift left, "\beta_{21}"] \end{tikzcd}$$

and the ideal $\Omega^{(2)}$ *is generated by the zero relations* $\beta_{21}\beta_{12}$, $\beta_{12}\beta_{21}$, $\beta_{13}\beta_{31}$, $\beta_{31}\beta_{13}$, $\beta_{31}\beta_{12}$, $\beta_{32}\beta_{21}$, $\beta_{13}\beta_{32}$.

(c3) $A_{q_3} \cong KQ^{(3)}/\Omega^{(3)}$, *where* $Q^{(3)}$:
$$\begin{tikzcd} 3 \arrow[r, "\beta_{31}"] & 1 \arrow[r, shift left, "\beta_{12}"] & 2 \arrow[l, shift left, "\beta_{21}"] \arrow[ll, bend left, "\beta_{23}"] \end{tikzcd}$$

and the ideal $\Omega^{(3)}$ *is generated by the zero relations* $\beta_{21}\beta_{12}$, $\beta_{12}\beta_{21}$, $\beta_{23}\beta_{31}$, $\beta_{31}\beta_{12}$.

(c4) $A_{q_4} \cong KQ^{(4)}/\Omega^{(4)}$, *where* $Q^{(4)}$:
$$\begin{tikzcd} 3 \arrow[r, shift left, "\beta_{31}"] & 1 \arrow[l, shift left, "\beta_{13}"] \arrow[r, shift left, "\beta_{12}"] & 2 \arrow[l, shift left, "\beta_{21}"] \end{tikzcd}$$
and the ideal

$\Omega^{(4)}$ *is generated by the zero relations* $\beta_{21}\beta_{12}$, $\beta_{12}\beta_{21}$, $\beta_{13}\beta_{31}$, $\beta_{31}\beta_{13}$.

(c5) $A_{q_5} \cong KQ^{(5)}/\Omega^{(5)}$, *where* $Q^{(5)}$:
$$\begin{tikzcd} 3 \arrow[rr, bend left, "\beta_{32}"] & 1 \arrow[l, "\beta_{13}"'] & 2 \arrow[l, "\beta_{21}"'] \end{tikzcd}$$
and the ideal $\Omega^{(5)}$ *is generated by the zero relations* $\beta_{21}\beta_{13}\beta_{32}$, $\beta_{13}\beta_{32}\beta_{21}$, $\beta_{32}\beta_{21}\beta_{13}$.

Proof.

(a) Let $\overline{A}_q$ be the $(0,1)$-limit of A_q. We define a K-linear map $\varphi : \overline{A_q} \longrightarrow A_q$ by setting

$$\varphi(e_{ij}) = \begin{cases} q_{ij}^{(k)} e_{ij}, & \text{if } q_{ij}^{(k)} \neq 0, \text{ for } k \neq i, j, \\ e_{ij}, & \text{otherwise,} \end{cases}$$

for distinct $i, j \in \{1,2,3\}$, and we set $\varphi(e_{ii}) = e_{ii}$, for $i = 1,2,3$. Denote by $\cdot'$ and $\cdot''$ the multiplication in A_q and in $\overline{A}_q$, respectively.

To show that $\varphi : \overline{A}_q \longrightarrow A_q$ is a K-algebra isomorphism, it is sufficient to prove that $\varphi(e_{ir} \cdot'' e_{rj}) = \varphi(e_{ir}) \cdot' \varphi(e_{rj})$, for all $i, r, j \in \{1,2,3\}$.

First, we consider the case when $i, r, j \in \{1,2,3\}$ are pairwise different and $q_{ij}^{(r)} \neq 0$. It follows from Lemma 2.4(c) that $q_{ir}^{(j)} = q_{rj}^{(i)} = 0$, so that $\varphi(e_{ir}) = e_{ir}$ and $\varphi(e_{rj}) = e_{rj}$. Hence

$$\varphi\big(e_{ir} \cdot'' e_{rj}\big) = \varphi(e_{ij}) = q_{ij}^{(r)} e_{ij} = e_{ir} \cdot' e_{rj} = \varphi(e_{ir}) \cdot' \varphi(e_{rj}),$$

and we are done. The proof in remaining cases is analogous and it is left to the reader.

(b) In view of (a), Theorem 2.18 and Corollary 2.19, it is sufficient to classify the S_3-orbits of all basic structure $(0,1)$-matrices in $\boldsymbol{ST}_3(K)$ with respect to the action of the symmetric group S_3 defined in Definition 2.12.

Note that, by Lemma 2.4(c), the product of any successive pair of $q_{23}^{(1)}$, $q_{21}^{(3)}$, $q_{31}^{(2)}$, $q_{32}^{(1)}$, $q_{12}^{(3)}$, $q_{13}^{(2)}$, $q_{23}^{(1)}$ is zero. Hence we conclude that there are precisely five S_3-orbits of basic $(0,1)$-matrices in $\boldsymbol{ST}_3(K)$ and they are represented by the five structure matrices q_1, q_2, q_3, q_4, q_5 listed in (b). The remaining statement in (b) easily follows from the quiver description of the algebras A_{q_1}, A_{q_2}, A_{q_3}, A_{q_4} and A_{q_5} given in (c). On the other hand, this also follows from Theorem 5.5 proved in the next section.

(c) Since the constant matrices q_1, q_2, q_3, q_4, q_5 belongs to different S_3-orbits then, according to Corollary 2.19, the algebras A_{q_1}, A_{q_2}, A_{q_3}, A_{q_4} and A_{q_5} are pairwise non-isomorphic.

Note also that, in the notation of Definition 2.12, we have $q_1^{tr} = q_1$, ${}_{(2,3)}*q_2^{tr} = q_2$, ${}_{(1,3)}*q_3^{tr} = q_3$, $q_4^{tr} = q_4$ and ${}_{(1,3)}*q_5^{tr} = q_5$. It follows from Lemma 2.16(a) that $A_s^{op} \cong A_s$, for $s = 1, \dots, 5$, that is, the algebras A_{q_1}, A_{q_2}, A_{q_3}, A_{q_4} and A_{q_5} are self-dual.

By Corollary 2.20, the Gabriel quivers of the algebras $A_1, \dots, A_5$ are just the quivers listed in (c1)–(c5). It is easy to check that, for each $s \in \{1,2,3,4,5\}$, the correspondences $\varepsilon_j \mapsto e_j$ and $\beta_{ij} \mapsto e_{ij}$ define a K-algebra surjection $KQ^{(s)}/\Omega^{(s)} \longrightarrow A_{q_s}$, where ε_j is the primitive idempotent of the path algebra $KQ^{(s)}$ defined by the stationary path at the vertex j, for every $j \in Q_0^{(s)}$. Since $\dim_K KQ^{(s)}/\Omega^{(s)} = \dim_K A_{q_s} = 9$, the surjection is an isomorphism of K-algebras.

It follows from the shape of $Q^{(s)}$ and $\Omega^{(s)}$ that $KQ^{(s)}/\Omega^{(s)} \cong A_{q_s}$ is a special biserial algebra, that is,

(a) any vertex of $Q^{(s)}$ is a starting point of at most two arrows and is an end point of at most two arrows.

(b) given an arrow $\beta : i \to j$ in $Q^{(s)}$ there is at most one arrow $\alpha : s \to i$ and at most one arrow $\gamma : j \to r$ in $Q^{(s)}$ such that $\alpha\beta \notin \Omega^{(s)}$ and $\beta\gamma \notin \Omega^{(s)}$, see [**32**].

We recall that any special biserial algebra is representation-tame, see [**3**, 5.2]. Note that for $s = 1$, there is a cyclic walk $1 \xrightarrow{\beta_{13}} 3 \xleftarrow{\beta_{23}} 2 \xrightarrow{\beta_{21}} 1 \xleftarrow{\beta_{31}} 3 \xrightarrow{\beta_{32}} 2 \xleftarrow{\beta_{12}} 1$ of the quiver $Q^{(1)}$ and according to the finite representation type criterion in [**32**], the algebra A_{q_1} is of infinite representation type. Similarly, by looking at the walks of each of the quivers $Q^{(2)}$, $Q^{(3)}$, $Q^{(4)}$, $Q^{(5)}$; and by applying the finite representation type criterion in [**32**], we conclude that the algebra A_{q_s} is representation-finite, for $s = 2, 3, 4, 5$. This finishes the proof. $\square$

It follows from Theorem 4.1, that for $n = 3$, each basic minor degeneration $A_q = \boldsymbol{M}_3^q(K)$ of $\boldsymbol{M}_3(K)$ is special biserial and A_q is isomorphic to its $(0,1)$-limit algebra $A_{\bar{q}}$. We show below and in Section 5 that this facts do not hold, for each $n \geq 4$.

Example 4.2. Assume that $n = 4$ and $A_q = \boldsymbol{M}_4^q(K)$ is a basic minor degeneration of $\boldsymbol{M}_4(K)$ given by the following structure matrix

$$q = \begin{bmatrix} 1111 & 0110 & 0010 & 0011 \\ 1001 & 1111 & 0011 & 0001 \\ 1000 & 1100 & 1111 & 1001 \\ 1100 & 0100 & 0110 & 1111 \end{bmatrix} \in \boldsymbol{ST}_4(K).$$

One can show that A_q is isomorphic to the bound quiver K-algebra KQ/Ω (see [**1**]), where Q is the quiver

$$Q : \begin{tikzcd} 1 \arrow[r, shift left, "\beta_1"] \arrow[d, shift left, "\gamma_4"] & 2 \arrow[l, shift left, "\gamma_1"] \arrow[d, shift left, "\beta_2"] \\ 4 \arrow[r, shift left, "\gamma_3"] \arrow[u, shift left, "\beta_4"] & 3 \arrow[l, shift left, "\beta_3"] \arrow[u, shift left, "\gamma_2"] \end{tikzcd}$$

and Ω is the two-sided ideal of the path K-algebra KQ of Q generated by the following relations:

- $\beta_j\gamma_j$ and $\gamma_j\beta_j$, for $j = 1, 2, 3, 4$,
- $\delta_1\delta_2\delta_3$, if the arrows $\delta_1, \delta_2, \delta_3$ form a path of length 3,
- $\beta_1\beta_2 - \gamma_4\gamma_3$, $\beta_2\beta_3 - \gamma_1\gamma_4$,
- $\gamma_2\gamma_1 - \beta_3\beta_4$, $\gamma_3\gamma_2 - \beta_4\beta_1$.

It follows that $A_q \cong KQ/\Omega$ is a special biserial algebra and hence it is representation-tame, see [**3**, 5.2]. Note that there is a cyclic walk $1 \xrightarrow{\delta_1} 4 \xleftarrow{\beta_3} 3 \xrightarrow{\gamma_2} 2 \xleftarrow{\beta_1} 1$ of the quiver Q and, according to the finite representation type criterion in [**32**], the algebra A_q is of infinite representation type, see also [**27**, Proposition 3.7]. Since $_{(2,3)} * q^{tr} = q$ then, by Lemma 2.16, $A_q^{op} \cong A_q$. Note also that $J(A_q)^3 = 0$ and $\operatorname{soc} A_q = J(A_q)^2 = Ke_{13} + Ke_{31} + Ke_{24} + Ke_{42}$.

Example 4.3. Assume that $n = 4$ and $B_q = \boldsymbol{M}_4^q(K)$ is a basic minor degeneration of $\boldsymbol{M}_4(K)$ given by the following structure matrix

$$q = \begin{bmatrix} 1111 & 0110 & 0010 & 0011 \\ 1001 & 1111 & 1011 & 0001 \\ 1101 & 0100 & 1111 & 0001 \\ 1100 & 0100 & 1110 & 1111 \end{bmatrix} \in \boldsymbol{ST}_4(K),$$

see [**7**, (2.4)]. One can show that B_q is isomorphic to the bound quiver K-algebra KQ/Ω (see [**1**]), where Q is the quiver

$$Q: \quad \begin{tikzcd} 1 \arrow[r, "\beta_{12}"] \arrow[d, "\beta_{14}"'] & 2 \arrow[d, "\beta_{23}"] \\ 4 \arrow[r, "\beta_{43}"'] & 3 \arrow[ul, "\beta_{31}"] \end{tikzcd}$$

and Ω is the two-sided ideal of the path K-algebra KQ of Q generated by the following relations:

- $\beta_{12}\beta_{23} = \beta_{14}\beta_{43}$,
- $\beta_{12}\beta_{23}\beta_{31}, \quad \beta_{23}\beta_{31}\beta_{12}, \;\; \beta_{31}\beta_{12}\beta_{23}, \quad \beta_{43}\beta_{31}\beta_{14}, \quad \beta_{14}\beta_{43}\beta_{31}.$

It follows that $J(B_q)^4 = 0$ and $J(B_q)^3 = Ke_{24} \oplus Ke_{42} = K\overline{\beta}_{23}\overline{\beta}_{31}\overline{\beta}_{14} \oplus K\overline{\beta}_{43}\overline{\beta}_{31}\overline{\beta}_{12}$. Since $\overline{\beta}_{31}\overline{\beta}_{12} \neq 0$ and $\overline{\beta}_{31}\overline{\beta}_{14} \neq 0$, then the algebra B_q is not special biserial. Note also that ${}_{(1,3)} * q^{tr} = q$ and Lemma 2.15 yields $B_q^{op} \cong B_q$.

The algebra B_q is not self-injective and the injective dimension $\text{inj.dim}\, B_q$ of B_q equals one. Indeed, there are isomorphisms $e_1 B_q \cong D(B_q e_3)$, $e_2 B_q \cong D(B_q e_4)$, $e_4 B_q \cong D(B_q e_2)$ and that there is a non-split exact sequence $0 \to e_3 B_q \to e_2 B_q \oplus e_4 B_q \to D(B_q e_1) \to 0$, where $D(-) = \mathrm{Hom}_K(-, K)$. Hence we get $\text{inj.dim}\, B_q = 1$. Note also that the algebra B_q is isomorphic to the quotient algebra $\Lambda/\pi\Lambda$ of the tiled R-order

$$\Lambda = \begin{bmatrix} R\, R\, R\, R \\ \pi\;\, R\, R\, \pi \\ \pi\;\, \pi\;\, R\, \pi \\ \pi\;\, \pi\;\, R\, R \end{bmatrix},$$

where $R = K[[t]]$ is the power series K-algebra and $\pi = t \cdot K[[t]]$. We can easily compute that $\text{gl.dim.}\, \Lambda = 2$. Hence we get $\text{inj.dim}\, B_q = \text{inj.dim}\, \Lambda - 1 = \text{gl.dim}\, \Lambda - 1 = 1$, see [**24**, Theorem 2.10]. Finally, we show that B_q is representation-finite.

To prove it, we denote by $R = K\Delta$ the path algebra of the Dynkin subquiver

$$\Delta: \quad \begin{tikzcd} 4 \arrow[dr, "\beta_{43}"] & & \\ & 3 \arrow[r, "\beta_{31}"] & 1 \\ 2 \arrow[ur, "\beta_{23}"'] & & \end{tikzcd}$$

of type $\boldsymbol{D}_4$ of Q. Denote by $\sigma : R \to R$ the K-algebra automorphism of R given by the permutation $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{smallmatrix}\right)$ of the vertices of Δ. Let ${}_\sigma D(R)_R$ be the vector space $D(R) = \operatorname{Hom}_K(R, K)$ viewed as an R-R-bimodule, with the left R-module structure induced by the automorphism $\sigma : R \to R$. It follows from [**29**, Corollary 4 and Remark 2] that the trivial extension $C = R \ltimes {}_\sigma D(R)$ is a non-symmetric selfinjective K-algebra of finite representation type. One can show that $\dim_K C = 18$ and, by applying [**29**, Theorem 2 and Proposition 1], the number of pairwise non-isomorphic indecomposable C-modules equals 24. The Gabriel quiver of C coincides with the quiver Q of the algebra B_q of dimension 16 and there is a K-algebra sujection $\varepsilon : C \to B_q$, with $\operatorname{Ker} \varepsilon = \operatorname{soc} I(1) \oplus \operatorname{soc} I(3)$, where $I(1) = D(B_q e_1)$ and $I(3) = D(B_q e_3)$ are the indecomposable injective C-modules at the vertices 1 and 3 of Q. It follows that the algebra B_q is representation-finite. One can show, as in [**29**, Examples 2 and 3], that the Auslander-Reiten quiver $\Gamma(\operatorname{mod} B_q)$ of B_q has a shape of a Möbius band consisting of 22 indecomposable modules, see also [**29**, Remark 2].

Example 4.4. Assume that $n = 6$ and consider the one-parameter family of basic minor degeneration K-algebras $A_{q_\mu} = \boldsymbol{M}_6^{q_\mu}(K)$, where $\mu \in K$ and

$$q_\mu = \begin{bmatrix} 111111 & 010000 & 011000 & 010100 & 011110 & 011101 \\ 100000 & 111111 & 001000 & 000100 & 001110 & 001101 \\ 100111 & 010111 & 111111 & 000100 & 000110 & 000101 \\ 101011 & 011011 & 001000 & 111111 & 001010 & 001001 \\ 100000 & 010000 & \mu 11000 & 110100 & 111111 & 000001 \\ 100010 & 010010 & 111010 & 110110 & 000010 & 111111 \end{bmatrix}.$$

Note that, if K is infinite, the family $\{A_\mu\}_{\mu \in K \setminus \{0,1\}}$ is infinite, because $A_\mu \cong A_\gamma$ if and only $\mu = \gamma$, for $\mu, \gamma \in K \setminus \{0, 1\}$ (apply Theorem 2.18). One can show that each of the algebras A_μ is representation-wild and not self-injective (the right ideals $e_2 A_\mu$ and $e_5 A_\mu$ are not injective, by [**7**, Proposition 2.3] and [**9**, Lemma 2.3]).

We show in Section 5 that the set of the isomorphism classes of basic self-injective algebras $A_q = \boldsymbol{M}_n^q(K)$ is infinite, for each $n \geq 4$.

Open problem 4.5. Describe all the matrices $q \in \boldsymbol{ST}_n(K)$ such that the algebra $A_q = \boldsymbol{M}_n^q(K)$ has $\operatorname{soc} A_q = J(A_q)^{n-2}$ and $J(A_q)^{n-1} = 0$.

5. Frobenius basic minor degenerations of $\boldsymbol{M}_n(K)$.

In this section we study basic minor q-degenerations of $\boldsymbol{M}_n(K)$ that are Frobenius K-algebras, where K is a field. We start by a description of the socle $\operatorname{soc} A_A$ of such an algebra $A = \boldsymbol{M}_n^q(K)$. In particular we show that $A = \boldsymbol{M}_n^q(K)$ is a Frobenius K-algebra if and only if its $(0,1)$-limit algebra $\overline{A} = \boldsymbol{M}_n^{\overline{q}}(K)$ is a Frobenius K-algebra.

Proposition 5.1. *Assume that $n \geq 2$, q is a basic structure matrix* (2.2) *in $\boldsymbol{ST}_n(K)$ and $\overline{q}$ is the $(0,1)$-limit of q. Let $A = \boldsymbol{M}_n^q(K)$ and $\overline{A} = \boldsymbol{M}_n^{\overline{q}}(K)$ be the corresponding basic minor degenerations of $\boldsymbol{M}_n(K)$, and let $e_1, \ldots, e_n$ be the standard primitive matrix idempotents of A and $\overline{A}$.*

(a) *Given* $j \in \{1, \dots, n\}$, *a right ideal* $S \subseteq e_j A$ *of* A *is simple if and only if* S *has the form* $S = e_{js}K \cong e_s A / e_s J(A)$, *where* e_{js} *is a matrix unit such that* $s \neq j$ *and* $q^{(s)}_{jr} = 0$, *for all* $r \neq s$.

(b) *Given* $j \in \{1, \dots, n\}$, $\mathrm{soc}(e_j A) = \sum_{s \in U_j} e_{js}K$, *where*

$$U_j = \big\{s;\ q^{(s)}_{jr} = 0,\ \textit{for all } r \neq s\big\} = \big\{s;\ s \neq j \textit{ and } e_{is} \cdot_q J(A) = 0\big\} \subseteq \{1, \dots, n\}.$$

(c) *If* S *and* S' *are two different simple submodules of* $e_j A$, *then* $S \not\cong S'$.

(d) *The socle* $\mathrm{soc}(A_A)$ *of the right* A*-module* A *is a two-sided ideal of* A *of the form*

$$\mathrm{soc}(A_A) = \{x \in J(A);\ x \cdot_q J(A) = 0\} = \sum_{j=1}^{n} \sum_{s \in U_j} e_{js}K,$$

that is, the sum runs through all pairs $(j, s) \in \{1, \dots, n\} \times U_j$ *such that* $j \neq s$.

(e) $\mathrm{soc}(A_A) = \mathrm{soc}(\overline{A}_{\overline{A}})$ *and* $\mathrm{soc}(e_j A) = \mathrm{soc}(e_j \overline{A})$, *for all* $j \in \{1, \dots, n\}$.

Proof. Since q is a basic matrix then, according to Theorem 2.9(d), the algebra $A = \boldsymbol{M}^q_n(K)$ is basic and the projective right ideals $e_1 A, \dots, e_n A$ of A are pairwise non-isomorphic.

(a) Assume that $S \subseteq e_j A$ is a simple right ideal of A. Then $S \neq 0$ and S contains a non-zero elelment $a = e_j \cdot_q \sum_{i,r} e_{ir}\lambda_{ir} = \sum_{r=1}^{n} e_{jr}\lambda_{jr}$, where $\lambda_{jr} \in K$ and some λ_{js} is non-zero. It follows that $a \cdot_q e_s = e_{js}\lambda_{js}$ belongs to S, and therefore $S = e_{is}A$. The module S is simple if and only if $S \cdot_q J(A) = 0$, or equivalently, if and only if $e_{js} \cdot_q e_{sr} = q^{(s)}_{jr} e_{jr} = 0$, for all $r \neq s$, because $J(A) = \sum_{s \neq r} e_{sr}K$, by Theorem 2.9. Hence, $S = e_{js}K \cong e_s A / e_s J(A)$ and (a) follows.
The statement (b) is a consequence of (a).

(c) Assume that $S = e_{js}K$ and $S' = e_{js'}K$ are two different simple submodules of $e_j A$ and assume, to the contrary, that there is an R-module isomorphism $\varphi : S \longrightarrow S'$. It follows that $0 \neq \varphi(e_{js}) = \varphi(e_{js} \cdot_q e_s) = \varphi(e_{js}) \cdot_q e_s = \lambda e_{js'} \cdot_q e_s$, for some $\lambda \in K \setminus \{0\}$. Hence, in view of (2.10), we get $s = s'$ and $S = S'$, a contradiction.

(d) Since $\mathrm{soc}(A_A) = \mathrm{soc}(e_1 A) \oplus \cdots \oplus \mathrm{soc}(e_n A)$ then (b) yields $\mathrm{soc}(A_A) = \sum_{s \in U_j} e_{js}K$, that is, $\mathrm{soc}(A_A)$ is spanned by all matrix units $e_{js} \in J(A)$ such that $j \neq s$ and $e_{js} \cdot_q J(A) = 0$. Hence (d) follows.

(e) By Theorem 2.9, $J(A) = J(\overline{A})$. Then (e) immediately follows from (b) and (d); and the proof is complete. □

Remark 5.2. Assume that $A = \boldsymbol{M}^q_n(K)$ is basic. Let $m \geq 1$ be such that $J(A)^m = 0$ and $J(A)^{m-1} \neq 0$. It is clear that $J(A)^{m-1} \subseteq \mathrm{soc}(A_A)$, however the equality does not hold in general. For this consider the algebra $A = A_{q_4} = \boldsymbol{M}^{q_4}_3(K)$ of Theorem 4.1(c4). In this case $m = 3$, $J(A)^2 = e_{32}K + e_{23}K$, $\mathrm{soc}(A_A) = J(A)^2 + e_{13}K + e_{12}K \neq J(A)^2$. Note also that $\mathrm{soc}({}_A A) = J(A)^2 + e_{31}K + e_{21}K \neq J(A)^2$ and hence $\mathrm{soc}({}_A A) \neq \mathrm{soc}(A_A)$.

We recall that a basic finite dimensional K-algebra A, with a complete set of primitive orthogonal idempotents $\{e_1, e_2, \dots, e_n\}$, is a Frobenius algebra if and only if each projective module e_jA has a simple socle and $\mathrm{soc}(e_iA) \ncong \mathrm{soc}(e_jA)$, for all $i \neq j$. In this case, there is a permutation σ of the set $\{1, \ldots, n\}$, called the Nakayama permutation, such that $\mathrm{soc}(e_jA) \cong \mathrm{top}(e_{\sigma(j)}A)$, see [**5**]. If A is a Frobenius algebra then (see [**35**, Theorem 2.4.3] and [**33**])

$$\mathrm{soc}({}_AA) = \mathrm{soc}(A_A) := \mathrm{soc}(A).$$

Now, following Fujita [**7**, Lemma 4.2], we give neccessary and sufficient conditions for a basic structure matrix q in $\boldsymbol{ST}_n(K)$ to be the K-algebra $\boldsymbol{M}_n^q(K)$ Frobenius. In particular, we remove the assumption on $(0,1)$-matrices made in [**7**, Lemma 4.2].

THEOREM 5.3. *Assume that $n \geq 2$, q is a basic structure matrix* (2.2) *in $\boldsymbol{ST}_n(K)$ and $\bar{q}$ is the $(0,1)$-limit of q. Let $A = \boldsymbol{M}_n^q(K)$ and $\bar{A} = \boldsymbol{M}_n^{\bar{q}}(K)$ be the corresponding basic minor degenerations of $\boldsymbol{M}_n(K)$, and let $e_1, \ldots, e_n$ be the standard primitive matrix idempotents of A and $\bar{A}$. The following seven conditions are equivalent.*

- (a) *A is a Frobenius K-algebra.*
- (a′) *$\bar{A}$ is a Frobenius K-algebra.*
- (b) *For each $j \in \{1, \ldots, n\}$, $\dim_K \mathrm{soc}(e_jA) = 1$, and the right simple ideals $\mathrm{soc}(e_1A), \ldots, \mathrm{soc}(e_nA)$ of A are pairwise non-isomorphic.*
- (c) *$\dim_K \mathrm{soc}(A_A) = n$, and the right ideals $e_1(\mathrm{soc}\, A_A), \ldots, e_n(\mathrm{soc}\, A_A)$ of A are pairwise non-isomorphic.*
- (d) *The block matrix $q \in \boldsymbol{ST}_n(K)$ satisfies the following two conditions*:
  - (d1) *For every $j \in \{1, \ldots, n\}$ there exists a unique $s \neq j$ such that $q_{jr}^{(s)} = 0$, for all $r \neq s$.*
  - (d2) *Given $i, j, s \in \{1, \ldots, n\}$ such that $i \neq j$ and $s \notin \{i, j\}$, there exists an $r \in \{1, \ldots, n\}$ such that $r \neq s$ and $q_{ir}^{(s)} \neq 0$ or $q_{jr}^{(s)} \neq 0$.*
- (e) *There exists a permutation σ of the set $\{1, \ldots, n\}$ such that $\sigma(j) \neq j$, for all $j = 1, \ldots, n$, and the block matrix $q \in \boldsymbol{ST}_n(K)$ satisfies the following condition*:
  - (e1) *Given $s, j \in \{1, \ldots, n\}$, the equality $q_{jr}^{(s)} = 0$ holds for all $r \neq s$ if and only if $s = \sigma(j)$.*
- (f) *There exists a permutation σ of the set $\{1, \ldots, n\}$ such that $\sigma(j) \neq j$, for all $j = 1, \ldots, n$, and the matrix q satisfies the following condition*:
  - (f1) *$q_{j\sigma(j)}^{(r)} \neq 0$, for any $j, r \in \{1, \ldots, n\}$.*

In this case σ is the Nakayama permutation of A and $\mathrm{soc}(e_jA) = Ke_{j\sigma(j)}$.

If A is a Frobenius algebra and $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ is as in (f) then:

- (i) *the Frobenius structure of $A = \boldsymbol{M}_n^q(K)$ is given by the K-linear map $\psi : A_q \longrightarrow K$ defined by the formula*

$$\psi(e_{js}) = \begin{cases} 1; & \textit{if } s = \sigma(j) \\ 0; & \textit{otherwise}; \end{cases}$$

- (ii) *any indecomposable module M in* mod A *is projective, or $M \cdot_q \mathrm{soc}(A) = 0$, that is, M is a module over the quotient algebra $A/\mathrm{soc}(A)$.*

Proof. Since q is a basic matrix then the algebra $A = \boldsymbol{M}_n^q(K)$ is basic, by Theorem 2.9(c). Hence, the projective right ideals $e_1A, \dots, e_nA$ of A are pairwise non-isomorphic.

It follows from [**5**] that $A = \boldsymbol{M}_n^q(K)$ is a Frobenius algebra if and only if each projective module e_jA has a simple socle and $\operatorname{soc}(e_iA) \not\cong \operatorname{soc}(e_jA)$, for $i \neq j$. Since simple A-modules are one-dimensional and $e_j(\operatorname{soc} A) = \operatorname{soc}(e_jA)$, then the conditions (a), (b) and (c) are equivalent.

Now we prove that the conditions (b) and (d) are equivalent. We recall from Proposition 5.1, that the module $S_j = \operatorname{soc}(e_jA)$ is simple if and only if there exists a unique s such that $s \neq j$, $S_j = e_{js}K$, $e_{js} \cdot_q J(A) = 0$, and $S_j \cong e_sA/e_sJ(A)$. Since $J(A) = \sum_{s \neq r} e_{sr}K$, then the equality $e_{js} \cdot_q J(A) = 0$ holds if and only if $q_{jr}^{(s)} = 0$, for all $r \neq s$, that is, if (d1) holds.

Assume that (d1) holds and $S_j = e_{js}K \cong e_sA/e_sJ(A)$, $S_i = e_{iu}K \cong e_uA/e_uJ(A)$ are two simple right submodules of A, where $s \neq j$ and $u \neq i$. Then $e_{js} \cdot_q e_{sr} = 0$ and $e_{iu} \cdot_q e_{ur'} = 0$, for all $r \neq s$ and $r' \neq u$, or equivalently, $q_{jr}^{(s)} = 0$ and $q_{jr'}^{(u)} = 0$, for all $r \neq s$ and $r' \neq u$. Hence, we easily conclude that the right simple ideals $\operatorname{soc}(e_1A), \dots, \operatorname{soc}(e_nA)$ of A are pairwise non-isomorphic if and only if the condition (d2) holds.

Since, obviously, the conditions (d) and (e) are equivalent then the conditions (a), (b), (c), (d), and (e) are equivalent. Note that σ is the Nakayama permutation of A.

The conditions (a) and (a$'$) are equivalent, because (d) holds for q if and only if (d) holds for $\overline{q}$.

Now we prove the implication (f)$\Rightarrow$(e) by showing that the condition (f1) implies (e1). To see it, we note that, if the condition (f1) holds and $s, j \in \{1, \dots, n\}$ are such that the equality $q_{jr}^{(s)} = 0$ holds for all $r \neq s$ then $s = \sigma(j)$. Conversely, if $s = \sigma(j)$ then Lemma 2.4(c) yields $q_{j\sigma(j)}^{(r)} q_{jr}^{(\sigma(j))} = 0$, for all $r \neq s = \sigma(j)$. Hence by (f1), we have $q_{jr}^{(\sigma(j))} = 0$, for all $r \neq s$ and and $j \in \{1, \dots, n\}$.

It remains to prove that the implication (e)$\Rightarrow$(f) holds. Assume that $A = \boldsymbol{M}_n^q(K)$ is a Frobenius algebra with Nakayama permutation σ. It follows that, for each $j \in \{1, \dots, n\}$, there is an isomorphism $e_jA \cong D(Ae_{\sigma(j)})$. Since the representation matrix (see [**7**]) of the right ideal e_jA with respect to the K-basis $\{e_{j1}, \dots, e_{jn}\}$ of e_jA is the matrix $(q_{js}^{(r)})_{j,s}$ then, according to [**9**, Lemma 2.3 (ii)], we have $q_{j\sigma(j)}^{(r)} \neq 0$, for all $r \in \{1, \dots, n\}$, and (f) follows.

To finish the proof, assume that $A = \boldsymbol{M}_n^q(K)$ is a Frobenius algebra and let $\sigma : \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$ be as in (f). For the proof of the statement (i), it is enough to show that $\operatorname{Ker}\psi$ does not contain a non-zero right ideal of A. Assume, to the contrary, that $\operatorname{Ker}\psi$ contains a non-zero right ideal aA, where $a = \sum_{i,j=1}^n a_{ij}e_{ij}$ and $a_{ij} \in K$. Since a is non-zero then $a_{rs} \neq 0$, for some $r, s \in \{1, \dots, n\}$. It follows that

$$\psi(a \cdot_q e_{s\sigma(r)}) = \psi\left(\sum_{i=1}^n a_{is}e_{is} \cdot_q e_{s\sigma(r)}\right) = \sum_{i=1}^n a_{is} q_{i\sigma(r)}^{(s)} \psi(e_{i\sigma(r)}) = a_{rs} q_{r\sigma(r)}^{(s)} \neq 0,$$

and we get a contradiction $a \cdot_q e_{s\sigma(r)} \in aA \subseteq \operatorname{Ker}\psi$.

Now we prove (ii) by applying the arguments given in [**15**]. Assume that M is an indecomposable module in $\operatorname{mod} A$ such that $M \cdot_q \operatorname{soc}(A) \neq 0$. Let S be a simple submodule of $M \cdot_q \operatorname{soc}(A)$ and let $P = E(S)$ be the injective envelope of S. Since A is Frobenius then P is indecomposable projective. By the injectivity of P, there is $f \in \operatorname{Hom}_A(M, P)$ such that the restriction of f to S is the embedding $S \hookrightarrow P$. We recall that P has a unique maximal submodule $\operatorname{rad} P = P \cdot_q J(A)$. Note that $\operatorname{Im} f$ is not contained in $\operatorname{rad} P$, because the inclusions $S \subseteq P$, $S \subseteq M \cdot_q \operatorname{soc}(A)$ and $\operatorname{Im} f \subseteq \operatorname{rad} P$ imply $0 \neq f(S) \subseteq f(M \cdot_q \operatorname{soc}(A)) = f(M) \cdot_q \operatorname{soc}(A) \subseteq P \cdot_q J(A) \cdot_q \operatorname{soc}(A) = 0$; and we get a contradiction. It follows that $\operatorname{Im} f + \operatorname{rad} P = P$, and the Nakayama lemma yields $\operatorname{Im} f = P$. By the projectivity of P, the homomorphism f is bijective, because M is indecomposable. Consequently, the module M is projective. This finishes the proof. $\square$

Now we give a simple description of all basic structure matrices q in $\boldsymbol{ST}_n(K)$ such that the K-algebra $A_q = \boldsymbol{M}_n^q(K)$ is Frobenius and $J(A_q)^3 = 0$. To present it, we associate to a given $n \geq 3$ and a permutation $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ such that $\sigma(i) \neq i$, for all $i \in \{1, \ldots, n\}$, the block matrix

$$q(\sigma) = \left[q(\sigma)^{(1)} | \cdots | q(\sigma)^{(n)}\right] \tag{5.4}$$

defined in [**7**, Lemma 4.4] by the formulae

$$q(\sigma)_{ij}^{(r)} = \begin{cases} 1, & \text{if } r \in \{i, j\}, \text{ or } j = \sigma(i), \\ 0, & \text{otherwise}, \end{cases}$$

for all $i, j, r \in \{1, \ldots, n\}$. It is easy to check that the block matrix $q(\sigma)$ is a basic structure $(0,1)$-matrix in $\boldsymbol{ST}_n(K)$, see [**7**, Theorem 4.4] and [**8**, Corollary 1.8].

THEOREM 5.5. *Assume that $n \geq 2$, q is a basic structure matrix* (2.2) *in $\boldsymbol{ST}_n(K)$ and $\overline{q}$ is the $(0,1)$-limit of q. Let $A = \boldsymbol{M}_n^q(K)$ and $\overline{A} = \boldsymbol{M}_n^{\overline{q}}(K)$ be the corresponding basic minor degenerations of $\boldsymbol{M}_n(K)$, and let $e_1, \ldots, e_n$ be the standard primitive matrix idempotents of A and of $\overline{A}$. The following conditions are equivalent.*

(a) *A is a Frobenius K-algebra and $J(A)^3 = 0$.*
(a′) *$\overline{A}$ is a Frobenius K-algebra and $J(\overline{A})^3 = 0$.*
(b) *Either $n = 2$ and $A = \boldsymbol{M}_2^q(K)$ is the Nakayama algebra $A(0)$ of Example* 2.8, *or $n \geq 3$ and A is a Frobenius K-algebra such that $J(A)^2 = \operatorname{soc}(A)$.*
(c) *Either $n = 2$ and $q = q(0) = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix} \middle| \begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, or $n \geq 3$ and there exists a permutation σ of the set $\{1, \ldots, n\}$ such that $\sigma(j) \neq j$, for all $j = 1, \ldots, n$, and the block matrix $q \in \boldsymbol{ST}_n(K)$ satisfies the following condition*:

$$q_{ij}^{(r)} \neq 0 \quad \textit{if and only if} \quad r \in \{i, j\} \textit{ or } j = \sigma(i).$$

(d) *Either $n = 2$ and $q = q(0) = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix} \middle| \begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, or $n \geq 3$ and there exists a permutation σ of the set $\{1, \ldots, n\}$ such that $\sigma(j) \neq j$, for all $j = 1, \ldots, n$ and the $(0,1)$-limit $\overline{q} \in \boldsymbol{ST}_n(K)$ of the block matrix q has the form $\overline{q} = q(\sigma)$* (5.4).

In this case σ is the Nakayama permutation of A and of $\overline{A}$. Moreover, $A/J(A)^2 \cong \overline{A}/J(\overline{A})^2$.

Proof. Since q is a basic matrix and $n \geq 2$ then the algebra $A = \boldsymbol{M}_n^q(K)$ is basic, non-semisimple, and the projective right ideals $e_1A, \dots, e_nA$ of A are pairwise non-isomorphic, by Theorem 2.9(d).

(a)$\Rightarrow$(b) Assume that $J(A)^3 = 0$ and that the algebra $A = \boldsymbol{M}_n^q(K)$ is Frobenius. It follows from Proposition 5.1 that, for each $j \in \{1, \dots, n\}$, the simple module $S_j = \mathrm{soc}(e_jA)$ has the form $S_j = e_{j\sigma(j)}K$, where $\sigma \in S_n$ is the Nakayama permutation of A. Note that if $e_{j\sigma(j)} \in J(A) \setminus J(A)^2$ then, by the description of the simple ideals given in Theorem 5.3, $e_jA = e_jK + e_{j\sigma(j)}K$ is of dimension two. It follows that $n = \dim_K e_jA = 2$. Consequently, if $J(A)^2 = 0$ then $n = 2$ and $A = A(0)$ is the Nakayama algebra of Example 2.8. Moreover, if $n \geq 3$ then $J(A)^2 \neq 0$ and $e_{j\sigma(j)} \in J(A)^2$, for every j. It follows that $\mathrm{soc}(A) = \mathrm{soc}(e_1A) \oplus \cdots \oplus \mathrm{soc}(e_nA) \subseteq J(A)^2$. Since $J(A)^3 = 0$, then $\mathrm{soc}(A) \supseteq J(A)^2$ and we get the equality $\mathrm{soc}(A) = J(A)^2$.

(b)$\Rightarrow$(a) If $n = 2$ and $A = A(0)$ is the Nakayama algebra of Example 2.8, then A is a non-semisimple Frobenius algebra such that $J(A)^2 = 0$. If $n \geq 3$ and $J(A)^2 = \mathrm{soc}(A)$ then $J(A)^3 = J(A)\,\mathrm{soc}(A) = 0$, and (a) follows.

(b)$\Rightarrow$(c) In case $n = 2$, the matrix q has the form $q(0) = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix} \middle| \begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, see Example 2.8.

Assume that $n \geq 3$, $J(A)^2 = \mathrm{soc}(A)$ and that the algebra $A = \boldsymbol{M}_n^q(K)$ is Frobenius. Take for $\sigma \in S_n$ the Nakayama permutation of A. It follows from Theorem 5.3 that, for each $j \in \{1, \dots, n\}$, the simple submodule $S_j = \mathrm{soc}(e_jA)$ of e_jA has the form $S_j = e_{j\sigma(j)}K$, where $e_{j\sigma(j)} \in e_jJ(A)^2$. Since $J(A)^3 = 0$ then the condition (d1) of Proposition 5.1 (with $s = \sigma(j)$), together with the condition (d2), implies the condition required in (c) for $n \geq 3$.

The implication (c)$\Rightarrow$(d) easily follows from the definition of the $(0,1)$-limit $\bar{q}$ of q and of the block matrix $q(\sigma)$ associated to σ.

(d)$\Rightarrow$(a) If $n = 2$ and $q = q\left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix} \middle| \begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, then $A = \boldsymbol{M}_2^q(K)$ is the Nakayama algebra of Example 2.8. Hence A is a Frobenius algebra such that $J(A)^2 = 0$.

Assume that $n \geq 3$ and there exists a permutation $\sigma \in S_n$ such that $\bar{q} = q(\sigma)$ and $\sigma(j) \neq j$, for all $j = 1, \dots, n$. Let $\bar{A} = \boldsymbol{M}_n^{\bar{q}}(K)$ be the $(0,1)$-limit of A.

It is clear that, for each $j \in \{1, \dots, n\}$, the module $S_j = \mathrm{soc}(e_j\bar{A}) = e_{j\sigma(j)}K$ is simple and $S_j \cong S_i$ if and only if $j = i$. It follows that $\bar{A}$ is a Frobenius algebra and, according to Theorem 5.3, the algebra A is Frobenius. Since $n \geq 3$ and $q_{j\,s}^{(r)} = 0$ if and only if $r \notin \{s, j\}$ and $s \neq \sigma(j)$, then $J(\bar{A})^2 = \sum_{j=1}^{n} e_{j\sigma(j)}K$ and $J(\bar{A})^3 = J(A)^3 = 0$, see Proposition 3.2. Hence (a) follows.

Since the conditions (a) and (a$'$) are equivalent, by Theorem 5.3 and Proposition 3.2, then the proof is complete. $\square$

Following Gabriel [**10**] we associate to a basic algebra $A = e_1A \oplus \cdots \oplus e_nA$ the separated quiver $\mathscr{Q}^s(A) = (\mathscr{Q}^s(A)_0, \mathscr{Q}^s(A)_1)$ of A with the set of points $\mathscr{Q}^s(A)_0 = \{1, \dots, n, 1', \dots, n'\}$. There is an arrow $\beta'_{ij} : i \to j'$ in $\mathscr{Q}^s(A)_1$ if and only if there is an arrow $\beta_{ij} : i \to j$ in the quiver $\mathscr{Q}(A)$ of A, see 2.20.

Corollary 5.6. *Assume that $n \geq 3$, q is a basic structure matrix* (2.2) *in $\boldsymbol{ST}_n(K)$ such that $A_q = \boldsymbol{M}_n^q(K)$ is a Frobenius algebra and $J(A_q)^3 = 0$.*

(a) *The algebra A_q is of finite representation type if and only if $n = 3$.*
(b) *Assume that the field K is algebraically closed. Then A_q is tame of infinite repre-*

sentation type if and only if $n = 4$.

(c) *If the field* K *is algebraically closed then the algebra* A_q *is of wild representation type if and only if* $n \geq 5$.

Proof. Since A_q is a Frobenius algebra and $J(A_q)^3 = 0$ then, according to Theorem 5.5, either $n = 2$ and A_q is the Nakayama algebra of Example 2.8, or $n \geq 3$ and $\operatorname{soc}(A_q) = J(A_q)^2$. Assume that $n \geq 3$. By Theorem 5.3, any indecomposable non-projective A_q-module is a module over the quotient algebra $B_q = A_q/J(A_q)^2$. It follows that A_q is representation-finite (resp. representation-tame) if and only if so is B_q.

Since $J(B_q)^2 = 0$, then by Gabriel [**10**], B_q is representation-finite if and only if the separated quiver $\mathscr{Q}^s(B_q)$ is a disjoint union of Dynkin quivers, and B_q is representation-tame if and only if the separated quiver $\mathscr{Q}^s(B_q)$ is a disjoint union of Dynkin quivers and Euclidean quivers. Moreover, B_q is representation-infinite if and only if $\mathscr{Q}^s(A_q)$ contains a subquiver isomorphic to an Euclidean quiver.

It follows from Theorem 4.1 that in case $n = 3$, up to isomorphism, the only Frobenius algebra A_q is the Nakayama algebra A_{q_5} of 4.1. Obviously, A_{q_5} is of finite representation type.

Assume that $n \geq 4$. Since A_q is a Frobenius algebra then $\overline{A}_q = \boldsymbol{M}_n^{\bar{q}}(K)$ is also a Frobenius algebra and, according to Theorem 5.5, the $(0,1)$-limit $\bar{q}$ of q has the form $\bar{q} = q(\sigma)$, where $\sigma \in S_n$ is the Nakayama permutation of A_q. It follows from Corollary 2.20 and Theorem 5.5(c) that there is an arrow $i \to j$ in $\mathscr{Q}(A_q)$ if and only if $i \neq j$ and $j \neq \sigma(i)$.

Now assume that $n = 4$. By the observation made above and the definition of the separated quiver $\mathscr{Q}^s(A_q) = \mathscr{Q}^s(\overline{A}_q)$, we conclude that $\mathscr{Q}^s(A_q)$ is the Euclidean quiver

$$\begin{tikzcd}
1 \arrow[dr] \arrow[drrrrrrr] & & \sigma^3(1) \arrow[dl] \arrow[dr] & & \sigma^2(1) \arrow[dl] \arrow[dr] & & \sigma(1) \arrow[dl] \arrow[dr] & \\
& \sigma^2(1)' & & \sigma(1)' & & 1' & & \sigma^3(1)'
\end{tikzcd}$$

of type $\widetilde{\boldsymbol{A}}_7$. It follows that the algebra A_q is of infinite representation type, and A_q is tame if K is algebraically closed.

Finally assume that $n \geq 5$. It is easy to see that $\mathscr{Q}^s(A_q)$ contains the wild quiver

$$\begin{tikzcd}
\sigma(1) \arrow[d] \arrow[dr] & \sigma^2(1) \arrow[dl] \arrow[d] & \sigma^3(1) \arrow[dl] \\
\sigma^4(1)' & 1' &
\end{tikzcd}$$

It follows that A_q is representation-wild, and the proof is complete. $\square$

Theorem 5.7. *Assume that* K *is a field and* $n \geq 4$. *Given* $\mu \in K^* = K \setminus \{0\}$, *we define the matrix* $q_\mu = [q_\mu^{(1)}| \cdots |q_\mu^{(n)}] \in \boldsymbol{ST}_n(K)$ *of the form* (2.2) *by the formulae*

$$(q_\mu)^{(r)}_{ij} = \begin{cases} \mu, & \textit{if } r=1,\ i=2,\ j=3, \\ 1, & \textit{if } (i,r,j) \neq (2,1,3) \quad \textit{and} \quad r \in \{i,j\},\ \textit{or } j = i+1\ (\textit{modulo } n), \\ 0, & \textit{otherwise}, \end{cases}$$

for all $i, j, r \in \{1, \dots, n\}$.

(a) *For each* $\mu \in K^*$, q_μ *is a basic matrix in* $\boldsymbol{ST}_n(K)$ *such that* $C_{q_\mu} = \boldsymbol{M}_n^{q_\mu}(K)$ *is a basic Frobenius* K*-algebra with* $J(C_\mu)^3 = 0$ *and with the Nakayama permutation* $\sigma = (1, 2, \dots, n)$.

(b) *If* $\mu, \nu \in K^*$ *are such that* $\mu \neq \nu$ *and* $\mu \neq \nu^{-1}$, *then* $C_\mu \ncong C_\nu$.

(c) *If the field* K *is algebraically closed and* $n = 4$, *each of the algebras* C_μ *is tame of infinite representation type.*

(d) *If the field* K *is algebraically closed and* $n \geq 5$, *each of the algebras* C_μ *is of wild representation type.*

Proof.

(a) Fix $n \geq 4$ and set $q^{(r)}_{ij} = (q_\mu)^{(r)}_{ij}$, for simplicity of the notation. It is clear that the matrix $q_\mu = [q_\mu^{(1)}| \cdots |q_\mu^{(n)}]$ satisfies the conditions (C1) and (C3) of Definition 2.1. To prove that q_μ satisfies the condition (C2), we denote by $\mathscr{I}$ the set of all triples (i,r,j) such that $1 \leq i,r,j \leq n$, and $r \in \{i,j\}$ or $j = i+1$ modulo n. First we recall from [**8**, Proposition 1.7 (1)] that $(i,r,j), (i,j,s) \in \mathscr{I}$ if and only if $(i,r,s), (r,j,s) \in \mathscr{I}$. It follows that $q^{(r)}_{ij} q^{(j)}_{is} \neq 0$ if and only if $q^{(r)}_{is} q^{(j)}_{rs} \neq 0$, whenever $1 \leq i,j,r,s \leq n$. The verification of (C2) splits into several cases.

1° Assume that $(i,r,j,s) = (2,1,3,s)$ and $q^{(r)}_{ij} q^{(j)}_{is} \neq 0$. Then $(2,3,s) \in \mathscr{I}$. It follows that $s = 3$ and we get $q^{(1)}_{23} q^{(3)}_{23} = \mu = q^{(1)}_{23} q^{(3)}_{13}$.

2° Assume that $(i,r,j,s) = (2,r,1,3)$ and $q^{(r)}_{ij} q^{(j)}_{is} \neq 0$. Then $(2,r,1) \in \mathscr{I}$ and therefore $r = 1$ or $r = 2$. In either case we have $q^{(r)}_{21} q^{(1)}_{23} = \mu = q^{(r)}_{23} q^{(1)}_{r3}$.

3° Assume that $(i,r,j,s) = (2,1,j,3)$ and $q^{(r)}_{ij} q^{(j)}_{is} \neq 0$. Then $(1,j,3) \in \mathscr{I}$ and therefore $j = 1$ or $j = 2$. In either case we have $q^{(1)}_{2j} q^{(j)}_{23} = \mu = q^{(1)}_{23} q^{(j)}_{13}$.

4° Assume that $(i,r,j,s) = (i,2,1,3)$ and $q^{(r)}_{ij} q^{(j)}_{is} \neq 0$. Then $(i,2,3) \in \mathscr{I}$ and therefore $i = 2$. Then we get $q^{(2)}_{21} q^{(1)}_{23} = \mu = q^{(2)}_{23} q^{(1)}_{23}$.

5° Assume that $(2,1,3) \notin \{(i,r,j), (i,j,s), (i,r,s), (r,j,s)\}$ and $q^{(r)}_{ij} q^{(j)}_{is} \neq 0$. Then $q^{(r)}_{ij} q^{(j)}_{is} = 1 = q^{(r)}_{is} q^{(j)}_{rs}$.

This shows that the matrix $q_\mu = [q_\mu^{(1)}| \cdots |q_\mu^{(n)}]$ satisfies the conditions (C2) and, consequently, q_μ is a basic matrix in $\boldsymbol{ST}_n(K)$. By Theorem 5.3, the minor q_μ-deformation $C_{q_\mu} = \boldsymbol{M}_n^{q_\mu}(K)$ is a basic Frobenius K-algebra with Nakayama permutation $\sigma = (1, 2, \dots, n)$.

(b) Assume that $\mu, \nu \in K^*$ are such that $\mu \neq \nu$ and $\mu \neq \nu^{-1}$. Without loss of generality, we may suppose that $\nu \neq 1$. For simplicity of the notation, we set $q^{(r)}_{ij} = (q_\mu)^{(r)}_{ij}$ and $p^{(r)}_{ij} = (q_\nu)^{(r)}_{ij}$.

Suppose, to the contrary, that there is a K-algebra isomorphism $C_\mu \cong C_\nu$. By Theorem 2.18, the matrices q_μ and q_ν belong to the same $\boldsymbol{G}_n(K)$-orbit, that is, there exist a permutation $\tau : \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$ and a square matrix $T = [t_{ij}] \in \boldsymbol{M}_n(K)$

such that

- $t_{11} = \cdots = t_{nn} = 1$,
- $t_{ij} \neq 0$, for all $i, j \in \{1, \dots, n\}$, and
- $t_{ir} p_{ij}^{(r)} t_{rj} = q_{\tau(i)\,\tau(j)}^{(\tau(r))} t_{ij}$, for all $i, r, j \in \{1, \dots, n\}$.

We set $d_{ij}^{(r)} := q_{\tau(i)\,\tau(j)}^{(\tau(r))}$, for short, and let $\sigma = (1, 2, \dots, n)$ be the cyclic permutation of $\{1, 2, \dots, n\}$. Then

$$\prod_{i=1}^{n} \Big(d_{i\,\sigma(i)}^{(\sigma^2(i))} t_{i\,\sigma(i)} \Big) \Big(p_{i\,\sigma(i)}^{(\sigma^{-1}(i))} t_{i\,\sigma^{-1}(i)} t_{\sigma^{-1}(i)\,\sigma(i)} \Big)$$
$$= \prod_{i=1}^{n} \Big(p_{i\,\sigma(i)}^{(\sigma^2(i))} t_{i\,\sigma^2(i)} t_{\sigma^2(i)\,\sigma(i)} \Big) \Big(d_{i\,\sigma(i)}^{(\sigma^{-1}(i))} t_{i\,\sigma(i)} \Big)$$

and hence we get

$$\prod_{i=1}^{n} d_{i\,\sigma(i)}^{(\sigma^2(i))} \cdot \prod_{i=1}^{n} p_{i\,\sigma(i)}^{(\sigma^{-1}(i))} = \prod_{i=1}^{n} p_{i\,\sigma(i)}^{(\sigma^2(i))} \cdot \prod_{i=1}^{n} d_{i\,\sigma(i)}^{(\sigma^{-1}(i))}.$$

Since $n \geq 4$ and $\sigma = (1, 2, \dots, n)$, then $p_{i\,\sigma(i)}^{(\sigma^2(i))} = 1$ for all $i = 1, \dots, n$. Hence, in view of the equality $\nu = \prod_{i=1}^{n} p_{i\,\sigma(i)}^{(\sigma^{-1}(i))}$, we get

$$\nu \cdot \prod_{i=1}^{n} d_{i\,\sigma(i)}^{(\sigma^2(i))} = \prod_{i=1}^{n} d_{i\,\sigma(i)}^{(\sigma^{-1}(i))}.$$

Since $t_{ir} p_{i\,\sigma(i)}^{(r)} t_{r\,\sigma(i)} \neq 0$ then $d_{i\,\sigma(i)}^{(r)} = q_{\tau(i)\,\tau(\sigma(i))}^{(\tau(r))} \in \{1, \mu\}$, for $1 \leq r \leq n$. Note that $\mu \neq 1$, because the equality $\mu = 1$ yields $\nu = 1$, contrary to our assumption $\nu \neq 1$. Further, note that there is at most one $i \in \{1, \dots, n\}$ such that $\mu = d_{i\,\sigma(i)}^{(\sigma^2(i))}$ or $\mu = d_{i\,\sigma(i)}^{(\sigma^{-1}(i))}$. On the other hand, since $n \geq 4$ and $\sigma = (1, 2, \dots, n)$, there is no such an i such that $\mu = d_{i\,\sigma(i)}^{(\sigma^2(i))} = d_{i\,\sigma(i)}^{(\sigma^{-1}(i))}$. Then $\nu \neq 1$ and the equality yield $\mu\nu = 1$ or $\mu = \nu$, contrary to the assumption that $\mu \neq \nu$ and $\mu \neq \nu^{-1}$.

Since the statements (c) and (d) follow from Corollary 5.6, the proof is complete. □

Corollary 5.8. *Assume that K is an infinite field. Then for each $n \geq 4$ there exists a one-parameter K-algebraic family $\{C_\mu\}_{\mu \in K^*}$ of basic Frobenius K-algebras of the form $C_\mu = \boldsymbol{M}_n^{q_\mu}(K)$ such that $\sigma = (1, 2, \dots, n)$ is the Nakayama permutation of C_μ and $C_\mu \ncong C_\nu$, if $\mu \neq \nu$ and $\mu \neq \nu^{-1}$.*

Proof. Apply Theorem 5.7. □

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